Coherent groups of units of integral group rings and direct products of free groups†

By Ángel del Río and Pavel Zalesski

(*Received*)

Abstract

We classify the finite groups G for which $\mathcal{U}(\mathbb{Z}G)$, the group of units of the integral group ring of G, does not contain a direct product of two non-abelian free groups. These list of groups contains all the groups for which $\mathcal{U}(\mathbb{Z}G)$ is coherent. This reduces the problem to classify the finite groups G for which $\mathcal{U}(\mathbb{Z}G)$ is coherent to decide about the coherency of a finite list of groups of the form $\mathrm{SL}_n(R)$, with R an order in a finite dimensional rational division algebra.

1. *Introduction*

A group is called coherent if all its finitely generated subgroups are finitely presented. This notion has been studied within various classes of groups and has a long history. For instance Scott proved that the fundamental group of any 3-manifold is also coherent [**Sco73**]. On the other hand, it has been known for a long time that the direct product of two non-abelian free groups is not coherent, see [**Ger81**]. This implies for instance that $SL_n(\mathbb{Z})$ is not coherent if $n = 4$ whereas the coherence of $SL_3(\mathbb{Z})$ is an old open problem suggested by J.-P. Serre, see [**Ser79**, page 129]. For other examples of incoherent groups the reader can consult Wise's article [**Wis11**].

The main aim of this paper is looking for a criteria to decide if the group of units $\mathcal{U}(\mathbb{Z}G)$ of the integral group ring of a finite group G is coherent. For that we first classify the finite groups G for which $\mathcal{U}(\mathbb{Z}G)$ does not contain a direct product of two non-abelian free groups.

In order to state our main results we need to introduce some notation. All rings and algebras are suppose to be associative and unital. If R is a ring then $\mathcal{U}(R)$ denotes the group of units of R. If moreover G is a group then RG denotes the group ring of G with coefficients in R. If n is a positive integer then $M_n(R)$ denotes the ring of $n \times n$ matrices with entries in R and $\mathrm{GL}_n(R) = \mathcal{U}(M_n(R))$. If R is a subring of finite dimensional division rational algebra then $\mathrm{SL}_n(R)$ denotes the subgroup of $\mathrm{GL}_n(R)$ formed by the elements of reduced norm 1. If F is a field and $a, b \in \mathcal{U}(F)$ then $\left(\frac{a,b}{F}\right)$ denotes the *quaternion algebra* $F[i, j | i^2 = a, j^2 = b, ji = -ij]$. If R is a subring of F containing a

† The first author has been partially supported by Ministerio de Economía y Competitividad project MTM2012-35240 and Fondos FEDER, Proyecto Hispano-Brasileño de Cooperación Interuniversitaria PHB-2012-0135 and Fundación Séneca of the Region of Murcia 19880/GERM/15. The second author has been partially supported by Capes, bilateral Spanish-Brazilian project DGU

and b then $\left(\frac{a,b}{R}\right) = R[i,j]$. The standard quaternion algebra is $\mathbb{H}(F) = \left(\frac{-1,-1}{F}\right)$ and $\mathbb{H}(R) = \left(\frac{-1,-1}{R}\right)$. and $\mathbb{H}(R) = \left(\frac{-1,-1}{R}\right)$. A quaternion algebra $\left(\frac{a,b}{F}\right)$ is said to be totally definite if F is a totally real number field and a, b are totally negative, i.e. for every embedding σ of F in the complex numbers, $\sigma(F) \subseteq \mathbb{R}$ and $\sigma(a), \sigma(b) < 0$.

We use the standard notation for conjugation $g^h = h^{-1}gh$ and commutators $(g,h) = g^{-1}h^{-1}gh$. We also use ζ_n to denote a complex primitive n-th root of unity.

A cyclic group of order n is usually denoted by C_n. To emphasise that a is a generator of C_n, we write C_n either as $\langle a \rangle$ or $\langle a \rangle_n$. Recall that a group G is metabelian if G has an abelian normal subgroup N such that $A = G/N$ is abelian. We simply denote this information as $G = N : A$. To give a concrete presentation of G we will write N and A as direct products of cyclic groups and give the necessary extra information on the relations between the generators. By $\overline{x}$ we denote the coset xN. For example, the dihedral group of order $2n$ and the quaternion group of order $4n$ can be described as

$$\begin{array}{lcll} D_{2n} & = & \langle a \rangle_n : \langle \overline{b} \rangle_2, & b^2 = 1, a^b = a^{-1}. \\ Q_{4n} & = & \langle a \rangle_{2n} : \langle \overline{b} \rangle_2, & a^b = a^{-1},\ b^2 = a^n. \end{array}$$

If N has a complement in G then A can be identified with this complement and we write $G = N \rtimes A$. For example, the dihedral group also can be given by $D_{2n} = \langle a \rangle_n \rtimes \langle b \rangle_2$ with $a^b = a^{-1}$. Some other groups that are going to have a role in the paper are:

$$\begin{array}{lcll} D^{\pm}_{2^{n+2}} & = & \langle a \rangle_{2^{n+1}} \rtimes \langle b \rangle_2, & a^b = a^{2^n \pm 1}. \\ G_{32} & = & (\langle a_1 \rangle_4 \times \langle a_2 \rangle_4) : \langle \overline{b} \rangle_2, & a_1^b = a_1^{-1}a_2^2, (b, a_2) = 1, b^2 = a_1^2. \\ H_{2^{n+2}} & = & (\langle a_1 \rangle_2 \times \langle a_2 \rangle_{2^n}) \rtimes \langle b \rangle_2, & a_1^b = a_1 a_2^{2^{n-1}}, a_2^b = a_2\ (n \geq 2). \\ K_{3^{n+2}} & = & (\langle z \rangle_3 \times \langle a \rangle_{3^n}) \rtimes \langle b \rangle_3, & z \text{ central}, a^b = za. \\ L_{3^{n+2}} & = & \langle a \rangle_{3^n} \rtimes \langle b \rangle_9, & a^b = ab^3. \end{array}$$

Observe that $D^{\pm}_{2^{n+2}}$ represent two groups: $D^{+}_{2^{n+2}}$, with $a_2^b = a_2^{2^{n-1}+1}$, and the quasidihedral group $D^{-}_{2^{n+2}}$, with $a_2^b = a_2^{2^{n-1}-1}$.

When we write $N \rtimes K$ we are implicitly assuming that the action of K on N is not trivial. If the action is trivial we simply write $N \times K$. In some cases there is only one non-trivial action or all the non-trivial actions define isomorphic groups. For example, if q is either 4 or a power of an odd prime then the only possible action on $C_q \rtimes C_2$ is the action by inversion. Another example with unique non-trivial action is $C_4 \rtimes C_4$. On the other hand, if q is a prime power and p is prime divisor of $q-1$ then $C_q \rtimes C_p$ can be given by $p-1$ possible non-trivial actions, but all of them define isomorphic groups. In this cases we simply omit the action.

If G and H are groups with isomorphic centre then GYH denotes the central product of G and H.

We are ready to state our main results:

THEOREM 1.1. *The following conditions are equivalent for a finite group G.*

(i) *$\mathcal{U}(\mathbb{Z}G)$ does not contain a direct product of two non-abelian free groups.*

(ii) *G is either abelian or isomorphic to one of the following groups:*

  (a) *D_8, Q_{16}, $C_4 \rtimes C_4$, $D_8 Y Q_8$, A_4, $C_9 \rtimes C_3$, G_{32};*

  (b) *$Q_8 \times C_2^n$ with $n \geq 0$;*

  (c) *$D^{+}_{2^{n+2}}$ or $H_{2^{n+2}}$ with $n \geq 2$;*

  (d) *$K_{3^{n+2}}$ or $L_{3^{n+2}}$, with $n \geq 1$;*

  (e) *D_{2p}, Q_{4p}, $Q_8 \times C_p$ or $C_p \rtimes C_3$ with p an odd prime.*

COROLLARY 1.2. *Let G be a finite non-abelian group.*

(i) *If $\mathcal{U}(\mathbb{Z}G)$ is coherent then G is isomorphic to one of the groups listed in Theorem 1.1.(ii).*

(ii) *If G is D_8, Q_{16}, $C_4 \rtimes C_4$, G_{32}, D_{16}^-, H_{16}, H_{32}, D_6, Q_{12}, $Q_8 \times C_3$ or $Q_8 \times C_2^n$ with $n \geq 0$ then $\mathcal{U}(\mathbb{Z}G)$ is coherent:*

(iii) *$\mathcal{U}(\mathbb{Z}(D_8 Y Q_8))$ is coherent if and only if $\mathrm{SL}_2(\mathbb{H}(\mathbb{Z}))$ is coherent.*

(iv) *$\mathcal{U}(\mathbb{Z}A_4)$ is coherent if and only if $\mathrm{SL}_3(\mathbb{Z})$ is coherent.*

(v) *$\mathcal{U}(\mathbb{Z}(C_9 \rtimes C_3))$ is coherent if and only if $\mathrm{SL}_3(\mathbb{Z}[\zeta_3])$ is coherent.*

(vi) *if $n \geq 4$ then $\mathcal{U}(\mathbb{Z}D^+_{2^{n+1}})$ is coherent if and only if $\mathcal{U}(\mathbb{Z}H_{2^{n+2}})$ is coherent if and only if $\mathrm{SL}_2(\mathbb{Z}[\zeta_{2^{n-1}}])$ is coherent.*

(vii) *$\mathcal{U}(\mathbb{Z}K_{3^{n+2}})$ is coherent if and only if $\mathcal{U}(\mathbb{Z}L_{3^{n+2}})$ is coherent if and only if $\mathrm{SL}_3(\mathbb{Z}[\zeta_{3^n}])$ is coherent.*

(viii) *If p is prime and $p \geq 5$ then*

- *$\mathcal{U}(\mathbb{Z}D_{2p})$ is coherent if and only if $\mathcal{U}(\mathbb{Z}Q_{4p})$ is coherent if and only if $\mathrm{SL}_2(\mathbb{Z}[\zeta_p + \zeta_p^{-1}])$ is coherent;*
- *$\mathcal{U}(\mathbb{Z}(Q_8 \times C_p))$ is coherent if and only if $\mathrm{SL}_1(\mathbb{H}(\mathbb{Z}[\zeta_p]))$ is coherent; and*
- *if $p \equiv 1 \mod 3$, F is a subfield of index 3 in $\mathbb{Q}(\zeta_p)$ and $\mathcal{O}_p$ is an order in F then $\mathcal{U}(\mathbb{Z}(C_p \rtimes C_3))$ is coherent if and only if $\mathrm{SL}_3(\mathcal{O}_p)$ is coherent.*

Hence to obtain a complete classification of the finite groups G for which $\mathcal{U}(\mathbb{Z}G)$ is coherent one has to answer the question of which of the following groups is coherent:

$$\mathrm{SL}_1(\mathbb{H}(\mathbb{Z}[\zeta_p]), \mathrm{SL}_2(\mathbb{H}(\mathbb{Z})), \mathrm{SL}_2(\mathbb{Z}[\zeta_{2^{n+2}}]), \mathrm{SL}_2(\mathbb{Z}[\zeta_p+\zeta_p^{-1}]), \mathrm{SL}_3(\mathbb{Z}), \mathrm{SL}_3(\mathbb{Z}[\zeta_{3^n}]), \mathrm{SL}_3(\mathcal{O}_p),$$

with $n \geq 1$, p prime with $p \geq 5$ and $\mathcal{O}_p$ an order in a field of index 3 in $\mathbb{Q}(\zeta_p)$, if $p \equiv 1 \mod 3$.

2. *Preliminaries and technical results*

We start introducing the basic notation. Let R be a ring and G a group. If $X = R$ or G then the centre of X is denoted $Z(X)$. If, moreover, $Y, Z \subseteq X$ then $C_Y(Z)$ denotes the centraliser of Z in Y. The derived subgroup of G is denoted G' and c.d.(G) stands for the set of the degrees of the irreducible characters of G.

The *crossed product* with *action* $\alpha : G \to \mathrm{Aut}(R)$ and *twisting* $\tau : G \times G \to \mathcal{U}(R)$ is the ring $R *_\tau^\alpha G = \bigoplus_{g \in G} Ru_g$ with multiplication determined by the following rules: $u_g a = \alpha_g(a) u_g$ and $u_g u_h = \tau(g,h) u_{gh}$, for $a \in R$ and $g, h \in G$ (see [**Pas89**, Lemma 1.1] for necessary and sufficient conditions for this to define a ring). Recall that a classical crossed product is a crossed product $L *_\tau^\alpha G$, where L/F is a finite Galois extension, $G = \mathrm{Gal}(L/F)$ is the Galois group of L/F and α is the natural action of G on L. The *classical crossed product* $L *_\tau^\alpha G$ is denoted by $(L/F, \tau)$. If L/F is a cyclic extension of degree n, $\mathrm{Gal}(L/F)$ is generated by g, and $a = u_g^n \in F$, then the classical crossed product $(L/F, \tau)$ is completely determined by a and g. Namely $(L/F, \tau)$ is isomorphic to the algebra given by the presentation $L[u_g : u_g^n = a, u_g^{-1} x u_g = g(x), \text{ for } x \in L]$. This algebra is then denoted by $(L/F, g, a)$, or simply $(L/F, a)$. For example, if $a \in F \setminus F^2$ and $b \in F$ then the quaternion algebra $\left(\frac{a,b}{F}\right)$ is the cyclic algebra $(F(\sqrt{a})/F, b)$. The cyclic algebra $(L/F, a)$ is split (i.e. isomorphic to a matrix algebra over F) if and only if a belongs to the image of the Galois norm of L over F [**Rei75**, Theorem 30.4].

We adopt the notation of [**Pie82**] for central simple algebras. For example, if A is a central simple algebra over F then $\mathrm{DEG}(A)$ and $\mathrm{IND}(A)$ denote the degree and index of A. Then $A = M_n(D)$ for D a division ring and $n = \frac{\mathrm{DEG}(A)}{\mathrm{IND}(A)}$ is called the *reduced degree* of A [**Pie82**, Sections 13.1 and 13.4]. If moreover F is a number field and v is a place in F then $\mathrm{INV}_v(A)$ is the invariant of A at v and $\mathrm{INV}(A)$ is the list of invariants of A [**Pie82**, Chapter 18].

Assume that A is a finite dimensional semisimple rational algebra. An order in A is a subring of A containing a basis of A over $\mathbb{Q}$ and whose underlying additive group is finitely generated. It is well known that if $\mathcal{O}_1$ and $\mathcal{O}_2$ are two orders in A then $\mathcal{U}(\mathcal{O}_1 \cap \mathcal{O}_2)$ has finite index in $\mathcal{U}(\mathcal{O}_1)$ and in $\mathcal{U}(\mathcal{O}_2)$. This implies that $\mathcal{U}(\mathcal{O}_1)$ contains a direct product of two non-abelian free groups if and only if so does $\mathcal{U}(\mathcal{O}_2)$. Similarly $\mathcal{U}(\mathcal{O}_1)$ is coherent if and only if so is $\mathcal{U}(\mathcal{O}_2)$. Our first aim is to classify the finite dimensional simple algebras A over $\mathbb{Q}$ such that the group of units of an order in A does not contain a direct product of non-abelian free groups. For that aim the following result will be of great use.

THEOREM 2.1. [**Kle00**] *The following conditions are equivalent for an order $\mathcal{O}$ in a finite dimensional simple algebra A over $\mathbb{Q}$.*

(i) *$\mathcal{U}(\mathcal{O})$ has an abelian subgroup of finite index.*
(ii) *$\mathcal{U}(\mathcal{O})$ does not contain a non-abelian free group.*
(iii) *A is either a field or a totally definite quaternion algebra.*

The following two propositions gives respectively some necessary conditions and some sufficient conditions for the units of an order in a finite dimensional simple algebra to not contain a direct product of two non-abelian free groups. See Remark 2.4 for the gap between the necessary and sufficient conditions.

PROPOSITION 2.2. *Let D be a finite dimensional division rational algebra and k a positive integer. Assume that $\mathrm{GL}_k(\mathcal{O})$ does not contain a direct product of two non-abelian free-groups for some (any) order $\mathcal{O}$ in D. Then one of the following conditions holds:*

(i) *$k \leq 3$ and D is a field or a totally definite quaternion algebra over $\mathbb{Q}$.*
(ii) *$k = 1$ and either the degree of D is a prime power or $D = D_1 \otimes D_2$ where D_1 is a totally definite quaternion algebra and the degree of D_2 is an odd prime power.*

Proof. As $\mathrm{GL}_2(\mathcal{O})$ contains a non-abelian free group and, if $n \geq 4$ then $\mathrm{GL}_n(\mathcal{O})$ contains $\mathrm{GL}_2(\mathcal{O}) \times \mathrm{GL}_2(\mathcal{O})$ we deduce that $k < 4$. Moreover, if $k > 1$ and D is neither a field nor a totally definite quaternion algebra then, by Theorem 2.1, $\mathcal{U}(\mathcal{O})$ contains a non-abelian free group and hence, in such case, $\mathrm{GL}_2(\mathcal{O})$ and $\mathrm{GL}_3(\mathcal{O})$ contains a direct product of free groups. Therefore, if $k = 2$ or 3 then D is either a field or a totally definite quaternion algebra.

To complete the proof in case $k \neq 1$ we have to prove that if D is a totally definite quaternion algebra over a field $F \neq \mathbb{Q}$ then $\mathrm{GL}_2(\mathcal{O})$ contains a direct product of non-abelian free groups. Let $v_1, \dots, v_n$ be the Archimedean places of D and let $w_1, \dots, w_m$ be the non-Archimedean places of F at which D ramifies. Then $\mathrm{INV}_{v_i} = \mathrm{INV}_{w_i} = \frac{1}{2}$ for every i and $n+m$ is even, by [**Pie82**, Theorem 18.5]. By assumption, $n \geq 2$. If $m = 0$ then let w_1 be an arbitrary non-Archimedean place of F. We construct two division algebras D_1 and D_2 satisfying the following conditions for v any place of F, where if $m = 0$ then we let w_1 denote an arbitrary non-Archimedean place of F.

$$\mathrm{INV}_v(D_1) = \begin{cases} \frac{1}{2}, & \text{if } v \in \{v_1, w_1\}; \\ 0, & \text{otherwise.} \end{cases}$$

If $m \geq 1$ then

$$\mathrm{INV}_v(D_2) = \begin{cases} \frac{1}{2}, & \text{if } v \in \{v_2, \dots, v_n, w_2, \dots, w_m\}; \\ 0, & \text{otherwise.} \end{cases}$$

If $m = 0$ then

$$\mathrm{INV}_v(D_2) = \begin{cases} \frac{1}{2}, & \text{if } v \in \{v_2, \dots, v_n, w_1\}; \\ 0, & \text{otherwise.} \end{cases}$$

The existence of D_1 and D_2 is a consequence of [**Pie82**, Theorem 18.5]. Moreover $\mathrm{INV}(D) = \mathrm{INV}(D_1) + \mathrm{INV}(D_2)$ and $\mathrm{DEG}(M_2(D)) = 4 = \mathrm{DEG}(D_1 \otimes D_2)$. Therefore $M_2(D) \cong D_1 \otimes D_2$. On the other hand D_1 are not totally definite because $\mathrm{INV}_{v_2}(D_1) = \mathrm{INV}_{v_1}(D_2) = 0$ and hence if $\mathcal{O}_i$ is an order in D_i then $\mathrm{GL}_1(\mathcal{O}_i)$ contains a non-abelian free group. Thus $\mathrm{GL}_2(\mathcal{O})$ contains a direct product of free groups.

It remains to consider the case when $k = 1$. Let $\mathrm{DEG}(D) = m_1 \cdots m_n$ with $m_1, \dots, m_n$ relatively coprime prime powers. Then $D = D_1 \otimes \cdots \otimes D_n$ with D_i an algebra of degree m_i. Each D_i which is not a totally definite quaternion algebra has an order $\mathcal{O}_i$ such that $\mathrm{GL}_1(\mathcal{O}_i)$ contains a free group. Thus, the assumption implies that either $n = 1$, or, $n = 2$ and up to a permutation, D_1 is a totally definite quaternion algebra. Thus condition (2) holds. □

PROPOSITION 2.3. *Let D be a finite dimensional division algebra over $\mathbb{Q}$, let $\mathcal{O}$ be an order D and let k be a positive integer. If one of the following conditions holds then $\mathrm{GL}_k(\mathcal{O})$ does not contain a direct product of two non-abelian free groups.*

(i) *$k \leq 3$ and D is a either a field or a totally definite quaternion algebra over $\mathbb{Q}$.*

(ii) *$k = 1$ and either $\mathrm{DEG}(D)$ is prime or $D = D_1 \otimes D_2$ where D_1 is a totally definite quaternion algebra over $\mathbb{Q}$ and $\mathrm{DEG}(D_2)$ is prime.*

Proof. (1) We can suppose that $k = 3$. First suppose that D is a field. In this case it is enough to show that $\mathrm{GL}_3(\mathbb{C})$ does not contain a direct product of two non-abelian free groups. Indeed, let F be a non-abelian free subgroup of $\mathrm{GL}_3(\mathbb{C})$ and let A be the complex algebra generated by F. Then the semisimple part of A is not commutative. This implies that A contains a non-commutative simple complex subalgebra B. By the Double Centralizer Theorem, $\dim_{\mathbb{C}}(B)$ is a prime power which divides 9 and hence $B = M_3(\mathbb{C})$. Thus $\mathrm{C}_{M_3(\mathbb{C})}(F) = \mathbb{C}$ and therefore $\mathrm{GL}_3(\mathbb{C})$ does not contain a direct product of two non-abelian free groups.

Assume that D is a totally definite quaternion algebra over $\mathbb{Q}$. Let F be a non-abelian free subgroup of $\mathrm{GL}_3(\mathcal{O})$ and let A be the $\mathbb{Q}$-algebra generated by F and $B = \mathrm{C}_{M_3(D)}(A)$. By Theorem 2.1, it is enough to prove that B is either commutative or a totally definite quaternion algebra over $\mathbb{Q}$. As in the previous paragraph A contains a non-commutative simple algebra A_1. So assume that B is non-commutative. Let $A_2 = \mathrm{C}_{M_3(D)}(A_1)$. Then $4 \leq \dim_{\mathbb{Q}}(B) \leq \dim_{\mathbb{Q}}(A_1)$. By [**Pie82**, Theorem 12.7], if $d = \mathrm{DEG}(A_1)$ and $E = Z(A_1)$ then $\dim_{\mathbb{Q}}(A_1) = [E : \mathbb{Q}]d^2$ and $[E : \mathbb{Q}]d^2 \dim_{\mathbb{Q}}(A_2) = \dim_{\mathbb{Q}}(M_3(D)) = 36$. As A_1 is non-commutative $d = 2, 3$ or 6 and hence $M_3(D) = A \otimes_{\mathbb{Q}} B$ and A and B are central simple rational algebras of degrees 2 and 3 (or viceversa). Moreover $2 = \mathrm{IND}(D) = \mathrm{lcm}(\mathrm{IND}(A), \mathrm{IND}(B))$. Therefore, the algebra, A or B, of degree 3 is split and hence the other is Brauer equivalent to D, i.e. a totally definite quaternion algebra. As A contains a free group, it is not a totally definite quaternion algebra. Thus B is a totally definite quaternion algebra.

(2) Follows by similar arguments. □

REMARK 2.4. It is not clear to us whether $\mathcal{U}(\mathcal{O})$ contains a direct product of two non-abelian free group if $\mathcal{O}$ is an order in a division algebra satisfying condition (ii) of Proposition 2.2 but not condition (ii) of Proposition 2.3.

For the sake of our objectives we are only interested in Schur algebras, i.e. simple algebras generated by finite groups over its centre. The Benard-Schacher Theorem states that if such an algebra has index n then its centre has a primitive n-th root of unit [**BS72**]. In particular, if D is a central division F-algebra of the form $D = D_1 \otimes_F D_2$ with D_1 a totally definite quaternion algebra and D_2 of degree a power of an odd prime then D is not a Schur algebra.

We now recall the main ingredients of a method to calculate the Wedderburn decomposition of some rational group algebras introduced in [**OdRS04**].

For a subgroup H of G, let $\widehat{H} = \frac{1}{|H|}\sum_{h\in H} h$. Clearly, $\widehat{H}$ is an idempotent of $\mathbb{Q}G$ which is central if and only if H is normal in G. If $K \lhd H \leq G$ then let

$$\varepsilon(H,K) = \prod_{M/K\in\mathcal{M}(H/K)} (\widehat{K} - \widehat{M}) = \widehat{K} \prod_{M/K\in\mathcal{M}(H/K)} (1 - \widehat{M}),$$

where $\mathcal{M}(H/K)$ denotes the set of all minimal normal subgroups of H/K. We extend this notation by setting $\varepsilon(H,H) = \widehat{H}$. Let $e(G,H,K)$ be the sum of the distinct G-conjugates of $\varepsilon(H,K)$, that is, if T is a right transversal of $C_G(\varepsilon(H,K))$ in G, then

$$e(G,H,K) = \sum_{t\in T} \varepsilon(H,K)^t.$$

Clearly, $\varepsilon(H,K)$ is an idempotent of the group algebra $\mathbb{Q}H$, $e(G,H,K)$ is a central element of $\mathbb{Q}G$ and if the G-conjugates of $\varepsilon(H,K)$ are orthogonal, then $e(G,H,K)$ is a central idempotent of $\mathbb{Q}G$.

A *strong Shoda pair* of G is a pair (H,K) of subgroups of G with the following properties: $K \leq H \trianglelefteq N_G(K)$, H/K is cyclic and a maximal abelian subgroup of $N_G(K)/K$ and the different conjugates of $\varepsilon(H,K)$ are orthogonal.

The following proposition collects the main properties that we need from strong Shoda pairs (see [**OdRS04**] and [**OdRS06**]). All throughout ζ_n denotes a complex primitive n-th root of unity.

PROPOSITION 2.5. *Let G be a finite group.*

(i) *Let (H,K) be a strong Shoda pair of G. Then*
  (a) *$e(G,H,K)$ is a primitive central idempotent of $\mathbb{Q}G$.*
  (b) *Let $k = [H:K]$, $N = N_G(K)$, $n = [G:N]$, yK a generator of H/K and $\phi : N/H \to N/K$ a left inverse of the canonical projection $N/K \to N/H$. Then $\mathbb{Q}Ge(G,H,K)$ is isomorphic to $M_n(\mathbb{Q}(\zeta_k) *_\tau^\alpha N/H)$ and the action and twisting are given by*

$$\alpha_{nH}(\zeta_k) = \zeta_k^i, \text{ if } yK^{\phi(nH)} = y^iK,$$
$$\tau(nH,n'H) = \zeta_k^j, \text{ if } \phi(nn'H)^{-1}\phi(nH)\phi(n'H) = y^jK,$$

  for $nH, n'H \in N/H$ and integers i and j.

(ii) *If (H_1,K_1) and (H_2,K_2) are strong Shoda pairs then $e(G,H_1,K_1) = e(G,H_2,K_2)$ if and only if $H_1^g \cap K_2 = K_1 \cap H_2^g$ for some $g \in G$.*

(iii) *Suppose that G is metabelian and let W be a maximal abelian subgroup of G containing G'. Then*
   (a) *Every pair (H,K) of subgroups of G such that $W \subseteq H \subseteq N_G(K)$ and H/K is cyclic and maximal abelian in $N_G(K)/K$ is a strong Shoda pair of G.*
   (b) *Every primitive central idempotent of G is of the form $e(G,H,K)$ with H and K satisfying the conditions of (iii)(a).*

Note that the action α of the crossed product $\mathbb{Q}(\zeta_k) *_\tau^\alpha N/H$ in Proposition 2.5 is faithful. Therefore the crossed product $\mathbb{Q}(\zeta_k) *_\tau^\alpha N/H$ can be described as a classical crossed product $(\mathbb{Q}(\zeta_k)/F,\tau)$, where F is the centre of the algebra, which is determined by the action of N/H on $\mathbb{Q}(\zeta_n)$.

The Wedderburn decomposition of $\mathbb{Q}G$ for all the group in Theorem 1.1.(ii) can be calculated using Proposition 2.5, because all these groups are metabelian. This is done in the following proposition:

PROPOSITION 2.6. *Let G be one of the groups in Theorem 1.1.(ii). Then $\mathbb{Q}G = A \times B$ where B is a direct product of fields and totally definite quaternion algebras and A is as given in the following table:*

G	A	Comment
$D_8, Q_{16}, C_4 \rtimes C_4$	$M_2(\mathbb{Q})$	
$D_8 Y Q_8$	$M_2(\mathbb{H}(\mathbb{Q}))$	
A_4	$M_3(\mathbb{Q})$	
$C_9 \rtimes C_3$	$M_3(\mathbb{Q}(\zeta_3))$	
G_{32}	$M_2(\mathbb{Q}(i))$	
$Q_8 \times C_2^n$	$\mathbb{H}(\mathbb{Q})$	$n \geq 0$
$D^+_{2^{n+2}}$	$M_2(\mathbb{Q}(\zeta_{2^n}))$	$n \geq 2$
$H_{2^{n+2}}$	$M_2(\mathbb{Q}(\zeta_{2^{n-1}}))$	$n \geq 2$
$K_{3^{n+2}}, L_{3^{n+2}}$	$M_3(\mathbb{Q}(\zeta_{3^n}))$	$n \geq 1$
D_{2p}, Q_{4p}	$M_2(\mathbb{Q}(\zeta_p + \zeta_p^{-1}))$	p odd prime
$Q_8 \times C_p$	$\mathbb{H}(\mathbb{Q}(\zeta_p))$	p odd prime
$C_p \rtimes C_3$	$M_3(F)$	p odd prime, $p \equiv 1 \mod 3$, F subfield of index 3 in $\mathbb{Q}(\zeta_p)$

(2.1)

Proof. We have to show that one of the non-commutative simple component of $\mathbb{Q}G$ is the algebra given in the table and all the other are totally definite quaternion algebras. In all the cases G has an abelian normal subgroup W with G/W abelian and hence we can use Proposition 2.5 to calculate the non-commutative simple components of $\mathbb{Q}G$. For example, in D_8, W is the only cyclic subgroup of order 4 of D_8 and the only non-commutative simple component of $\mathbb{Q}D_8$ is $\mathbb{Q}D_8 e(D_8,W,1) \cong M_2(\mathbb{Q})$.

If $G = Q_{16} = \langle a\rangle_8 : \langle \bar{b}\rangle_2$, then $\langle a\rangle$ is maximal abelian subgroup of G and $G' = \langle a^2\rangle$. Thus $\mathbb{Q}G$ has two non-commutative simple components, namely $\mathbb{Q}Ge(G,W,1) = (\mathbb{Q}(\zeta_8)/\mathbb{Q}(\zeta_8+\zeta_8^{-1}),-1)$ and $\mathbb{Q}Ge(G,W,\langle a^2\rangle) \cong (\mathbb{Q}(i)/\mathbb{Q},1)$. The first one is isomorphic to $\left(\frac{-2,-1}{\mathbb{Q}(\sqrt{2})}\right)$, because $(\zeta_8+\zeta_8^{-1})^2 = 2$ and $(\zeta_8-\zeta_8^{-1})^2 = -2$, and the second one is isomorphic to $M_2(\mathbb{Q})$.

Suppose $G = \langle a\rangle_4 \rtimes \langle b\rangle_4$. Then $W = \langle a\rangle_4 \times \langle b^2\rangle_2$ is a maximal abelian subgroup of G and $G' = \langle a^2\rangle$. Hence the non-commutative simple components of $\mathbb{Q}G$ are $\mathbb{Q}Ge(G,W,\langle b^2\rangle) \cong (\mathbb{Q}(i)/\mathbb{Q},1) \cong M_2(\mathbb{Q})$ and $\mathbb{Q}Ge(G,W,\langle b^2a^2\rangle) \cong (\mathbb{Q}(i)/\mathbb{Q},-1) \cong \mathbb{H}(\mathbb{Q})$.

Let now $G = D_8 Y Q_8$, with $D_8 = \langle a\rangle_4 \rtimes \langle b\rangle_2$ and $Q_8 = \langle x\rangle_4 : \langle \overline{y}\rangle_2$. Then G has a maximal abelian subgroup $W = \langle a, x\rangle = \langle a, x | a^4 = a^2x^2 = 1, (a, x) = 1\rangle$ of index 2 and $G' = \langle a^2 = x^2\rangle$. Then $(W, K = \langle ax\rangle)$ is a strong Shoda pair of G and $N_G(K) = \langle W, by\rangle$. Hence $B = \mathbb{Q}Ge(G, W, K) \cong M_2(\mathbb{Q}Ne(N, W, K))$ and $\mathbb{Q}Ne(N, W, K) \cong (\mathbb{Q}(i)/\mathbb{Q}, -1) \cong \mathbb{H}(\mathbb{Q})$. Thus $B \cong M_2(\mathbb{H}(\mathbb{Q}))$ is one of the simple components of $\mathbb{Q}G(1 - \widehat{G'})$ and it has dimension $16 = |G| - [G : G'] = \dim \mathbb{Q}G - \dim \mathbb{Q}G\widehat{G'}$. Therefore B is the only non-commutative simple component of $\mathbb{Q}G$.

The unique non-commutative simple component of $\mathbb{Q}A_4$ is $\mathbb{Q}Ge(A_4, W, 1) \cong M_3(\mathbb{Q})$, with W the 2-Sylow subgroup of A_4.

If $G = C_9 \rtimes C_3$ then the only non-commutative simple component of G is $\mathbb{Q}Ge(G, C_9, 1) \cong M_2(\mathbb{Q}(\zeta_3))$.

If W is a cyclic subgroup of order 4 of Q_8 then the only non-commutative simple component of $\mathbb{Q}Q_8$ is $\mathbb{Q}Q_8e(Q_8, W, 1) \cong \mathbb{H}(\mathbb{Q})$. Therefore every non-commutative simple component of $\mathbb{Q}(Q_8 \times C_2^n) \cong \mathbb{Q}Q_8 \otimes_{\mathbb{Q}} \mathbb{Q}C_2^n$ is isomorphic to $\mathbb{H}(\mathbb{Q})$.

Let $G = G_{32}$ or $G = H_{2^{n+2}}$. Then $W = \langle a_1, a_2\rangle$ is a maximal abelian subgroup of G. Let $S = \{K \le W : W/K \text{ is cyclic and } G' \not\subseteq K\}$. If $G = G_{32}$ then $G' = \langle a_1^2a_2^2\rangle$ and $S = \{K = \langle a_1\rangle, K^g = \langle a_1a_2^2\rangle, K_1 = \langle a_2\rangle, K_2 = \langle a_1^2a_2\rangle\}$. As K_1 and K_2 are normal in G, $\mathbb{Q}G$ has three non-commutative simple components, namely $\mathbb{Q}G(G, W, K) \cong M_2(\mathbb{Q}(i))$, $\mathbb{Q}Ge(G, W, K_1)$ and $\mathbb{Q}Ge(G, W, K_2)$ and the last two are isomorphic to $(\mathbb{Q}(i)/\mathbb{Q}, -1) \cong \mathbb{H}(\mathbb{Q})$. If $G = H_{2^{n+2}}$ then $G' = \langle a_2^{2^{n-1}}\rangle$ and $S = \{K = \langle a_1\rangle, K^g = \langle a_1a_2^{2^{n-1}}\rangle\}$. Then the only non-commutative simple component of $\mathbb{Q}G$ is $\mathbb{Q}Ge(G, W, K) \cong M_2(\mathbb{Q}(\zeta_{2^{n-1}}))$.

Suppose $G = K_{3^{k+2}}$. Then $G' = \langle z\rangle$ and $W = \langle z, a\rangle$ is maximal abelian in G. Therefore, the non-commutative simple components of $\mathbb{Q}G$ are of the form $\mathbb{Q}Ge(G, W, K)$ with K a subgroup of W such that W/K is cyclic and $G' \not\subseteq K$. There are three such subgroups, namely $\langle a\rangle, \langle za\rangle$ and $\langle z^2a\rangle$. Moreover they are conjugate in G. Thus $\mathbb{Q}Ge(G, W, \langle a\rangle)$ is the unique non-abelian simple component of $\mathbb{Q}G$ and it is isomorphic to $M_3(\mathbb{Q}(\zeta_{p^k}))$. The same argument is valid for $L_{3^{k+2}}$ with b^3 playing the role of z.

Let p be an odd prime. Assume that $G = Q_{4p} = \langle a\rangle_{2p} : \langle \bar{b}\rangle_2$ with p and odd prime. Then $\langle a\rangle$ is a maximal abelian subgroup of G and $G' = \langle a^2\rangle_p$. Thus, the only non-commutative simple components of $\mathbb{Q}G$ are $\mathbb{Q}Ge(G, \langle a\rangle, 1) \cong (\mathbb{Q}(\zeta_p), \mathbb{Q}(\zeta_p + \zeta_p^{-1}), -1) = \mathbb{H}(\mathbb{Q}(\zeta_p + \zeta_p^{-1}))$, a totally definite quaternion algebra over $\mathbb{Q}$, and $\mathbb{Q}Ge(G, \langle a\rangle, a^p) = (\mathbb{Q}(\zeta_p)/\mathbb{Q}(\zeta_p + \zeta_p^{-1}), 1) \cong M_2(\mathbb{Q}(\zeta_p + \zeta_p^{-1}))$. Similarly, the only non-commutative simple component of $D_{2p} = \langle a\rangle_p \rtimes \langle b\rangle_2$ is $\mathbb{Q}D_{2p}e(D_{2p}, \langle a\rangle, 1) \cong M_2(\mathbb{Q}(\zeta_p + \zeta_p^{-1}))$.

If $G = Q_8 \times C_p$ with p odd then $\mathbb{Q}G \cong \mathbb{Q}Q_8 \otimes_{\mathbb{Q}} \mathbb{Q}C_p$ has two simple components. One isomorphic to $\mathbb{H}(\mathbb{Q})$ and another isomorphic to $\mathbb{H}(\mathbb{Q}(\zeta_p))$.

Finally, suppose that $G = C_p \rtimes C_3$, with p an odd prime (and $p \equiv 1 \mod 3$ for the action not to be trivial). Then the only non-commutative simple component of $\mathbb{Q}G$ is $\mathbb{Q}Ge(G, C_p, 1) \cong (\mathbb{Q}(\zeta_p)/F, 1) \cong M_3(F)$, where F is the only subfield of $\mathbb{Q}(\zeta_p)$ of index 3. □

We shall need the following technical

LEMMA 2.7. *Let G be a finite 2-group having an abelian subgroup W of index 2 satisfying the following conditions:*

(i) *W has a direct factor which is not normal in G.*

(ii) *The set $T = \{K \le W : W/K$ is cyclic and K is not normal in $G\}$ is a conjugacy class of G.*

Then $W = \langle a_1\rangle \times \langle a_2\rangle$ with $a_1^b = a_1^r a_2^y$, where $b \in G \setminus W$, and a_2^y has order 2.

Proof. Let T be the set of subgroups K of W with W/K cyclic and K not normal in G. Writing W as a direct product $\langle a_1\rangle \times \cdots \times \langle a_k\rangle$ of cyclic groups we may assume without loss of generality that $\langle a_1\rangle$ is not normal in G. For each $j = 1,\dots,k$ let $K_j = \prod_{i\neq j}\langle a_i\rangle$. As $\langle a_1\rangle = \cap_{j=2}^k K_j$, K_j is not normal in G for some $j \neq 1$. So we may assume that K_2 is not normal in G and hence $T = \{K_2, K_2^b\}$, with $b \in G\setminus W$. This implies that $a_i^b \notin K_2$ for some $i \neq 2$. In particular, $a_i^b \notin \langle a_i\rangle$ and reordering the a_i's we may assume that $a_1^b \notin K_2$. As b^2 is central in G, we deduce that $a_1 \notin K_2^b$ and hence $K_2^b \neq K_i$ for every $i \geq 2$. Hence $K_3,\dots,K_k$ are normal in G and hence so is $\cap_{i=r}^k K_i = \langle a_1, a_2\rangle$. In particular $a_1^b \in \langle a_1, a_2\rangle$. Write $a_1^b = a_1^{r_1}a_2^y$. As $a_1^b \notin K_2$ we have $a_2^y \neq 1$. We claim that $k = 2$ and $|a_2^y| = 2$. Indeed, let $|a_2^y| = 2^m$ and assume that $k \geq 3$ or $m \neq 1$. Then $K = \langle a_1, a_2^{2^{m-1}y}a_3, a_4,\dots,a_k\rangle$ is a subgroup of W different from K_2 and K_2^b because $a_3 \in K_2 \setminus K$ and $a_2^y \notin K$ and hence $a_1^b = a_1^{r_1}a_2^y \in K_2^b \setminus K$. Moreover $K \in T$, i.e. W/K is cyclic, but $a_1^b \notin K$, contradicting the uniqueness of K_2, K_2^b above. Therefore $k = |a_2^y| = 2$, as desired. □

3. *Proof of the main results*

Theorem 1.1 is an obvious consequence of the following theorem.

Theorem 3.1. *The following conditions are equivalent for a finite group G.*

(i) *$\mathcal{U}(\mathbb{Z}G)$ does not contain a direct product of two non-abelian free groups.*

(ii) *$\mathbb{Q}G \cong A\times B$ where B is a product of fields and totally definite quaternion algebras and either A is a division algebra or $A = M_k(D)$ with $k = 1, 2$ and D is either a field or a totally definite quaternion algebra over $\mathbb{Q}$.*

(iii) *$\mathbb{Q}G \cong A\times B$ where B is a product of fields and totally definite quaternion algebras and either A is quaternion division algebra or $A = M_k(D)$ with $k = 1, 2$ and D is either a field or a totally definite quaternion algebra over $\mathbb{Q}$.*

(iv) *G is either abelian or isomorphic to one of the following groups:*
   (a) *D_8, Q_{16}, $C_4 \rtimes C_4$, $D_8 Y Q_8$, G_{32}, $C_9 \rtimes C_3$, A_4;*
   (b) *$Q_8 \times C_2^n$ with $n \geq 0$;*
   (c) *$D^+_{2^{n+2}}$ or $H_{2^{n+2}}$ with $n \geq 2$;*
   (d) *$K_{3^{n+2}}$ or $L_{3^{n+2}}$, with $n \geq 1$;*
   (e) *D_{2p}, Q_{4p}, $C_p \rtimes C_3$ or $Q_8 \times C_p$, with p an odd prime.*

Proof. Let $\mathbb{Q}G = A_1 \times \cdots \times A_n$ be the Wedderburn decomposition of $\mathbb{Q}G$. Let $\mathcal{O}_i$ denote an order of A_i for every $i = 1,\dots,n$. Then $\mathbb{Z}G$ and $\mathcal{O} = \mathcal{O}_1 \times \cdots \times \mathcal{O}_n$ are orders in $\mathbb{Q}G$ and hence $\mathcal{U}(\mathbb{Z}G \cap \mathcal{O})$ has finite index in $\mathcal{U}(\mathbb{Z}G)$ and $\mathcal{U}(\mathcal{O})$.

(iii) implies (ii) is obvious and (iv) implies (iii) is a consequence of Proposition 2.6.

(i) implies (ii). Suppose that $\mathcal{U}(\mathbb{Z}G)$ does not contain a direct product of two non-abelian free groups. Thus $\mathcal{U}(\mathcal{O})$ does not contain a direct product of two non-abelian free groups. By Theorem 2.1, if A_i's is neither a field nor a totally definite quaternion algebras then $\mathcal{U}(\mathcal{O}_i)$ contains a non-abelian free group. Therefore after reordering the A_i's we may assume that each A_i, with $i \geq 2$, is either a field or a totally definite quaternion algebra. If A_1 is not a division algebra, then by Proposition 2.2, $A = M_k(D)$ with $k = 2$ or 3 and D is either a field or a totally definite quaternion algebra over $\mathbb{Q}$.

(ii) implies (i). Suppose that $\mathbb{Q}G = A \times B$ with A and B as in (ii). Let $\mathcal{O}_1$ be an order in A and $\mathcal{O}_2$ an order in B. By Proposition 2.1, $\mathcal{U}(\mathcal{O}_2)$ has an abelian subgroup U of finite index. By means of contradiction, suppose that $\mathcal{U}(\mathbb{Z}G)$ contains a direct product of two non-abelian free groups. Then so does $\mathcal{U}(\mathcal{O}_1) \times U$, i.e. $F_1 \times F_2 \subseteq \mathcal{U}(\mathcal{O}) \times U$ for F_1 and F_2 non-abelian free groups. Then $(F_1 \times F_2) \cap U$ is a central subgroup of

$F_1 \times F_2$ and hence it is trivial. This implies that $\mathcal{U}(\mathcal{O}_1)$, contains a subgroup isomorphic to $F_1 \times F_2$. By Proposition 2.3, A is a division algebra which is not a quaternion algebra. By Roquette Theorem [**Roq58**], G is not nilpotent. However $\mathbb{Q}G$ is a direct product of division algebras and hence every idempotent of $\mathbb{Q}G$ is central. This implies that G is Dedekind (i.e. every subgroup of G is normal). This yields a contradiction, because every Dedekind group is nilpotent.

(ii) implies (iv). Suppose that G is non-abelian and $\mathcal{U}(\mathbb{Z}G)$ satisfy (ii). Since we have already proved that (i) and (ii) are equivalent and the class of finite groups G satisfying (i) is closed under subgroups and epimorphic images, every subgroup and every epimorphic image of G satisfies (ii).

Suppose first that G is Hamiltonian (i.e. Dedekind but non-abelian). Hence $G = Q_8 \times C_2^n \times W$ with $n \geq 0$ and W is an abelian group of odd order. We claim that either $n = 0$ or $W = 1$. Otherwise G contains $H = Q_8 \times C_2 \times C_p$ for p an odd prime and $\mathbb{Q}H$ has two simple components isomorphic to $\mathbb{H}(\mathbb{Q}(\zeta_p))$ and this is neither a field nor a totally definite quaternion algebra, yielding a contradiction. If $W = 1$ then G is as in (iv)(b). Assume that $n = 0$. If W is not a cyclic p-group for some prime p then G contains $H = Q_8 \times C_p \times C_q$ for some primes p and q, possibly equal. Again $\mathbb{Q}H$ contains two copies of $\mathbb{H}(\mathbb{Q}(\zeta_p))$, contradicting the hypothesis. Hence W is a cyclic p-group for some prime p. If W is not of order p then G contains $H = Q_8 \times C_{p^2}$ and then $\mathbb{Q}H$ has one simple component isomorphic to $\mathbb{H}(\mathbb{Q}(\zeta_p))$ and another component isomorphic to $\mathbb{H}(\mathbb{Q}(\zeta_{p^2}))$, again a contradiction. Thus W has order p and we conclude that G is of the last group of (iv)(e).

In the remainder of the proof we assume that G is not Hamiltonian. Then $\mathbb{Q}G = M_q(D) \times D_1 \times \cdots \times D_m$ with $q = 2$ or 3 and D and each D_i is a field or a totally definite quaternion algebra. Thus the list of reduced degrees of $\mathbb{Q}G$ only contains 1 and q. This implies that $G = N \rtimes P$ for P a Sylow q-subgroup of G, by [**GH87**, Theorem 3.6 (b)]. Moreover, by [**GH87**, Theorem 1.5 (b)], all the reduced degrees of $\mathbb{Q}N$ are 1 and hence N is Dedekind. Moreover G is metabelian by [**GH88**, Theorem 4.1]. Let W be a maximal normal abelian subgroup of G containing G'. Let

$$S = \left\{ (H,K) : \begin{array}{l} W \leq H < G, K \leq H \leq N_G(K), H/K \text{ cyclic and} \\ H/K \text{ maximal abelian in } N_G(K)/K \end{array} \right\}.$$

Then $(H,K) \mapsto \mathbb{Q}Ge(G,H,K)$ defines a surjective map from S to the set of non-commutative simple components of $\mathbb{Q}G$. Furthermore, by Proposition 2.5.(iii)(b), if (H_1,K_1) and (H_2,K_2) are elements of S then $\mathbb{Q}Ge(G,H_1,K_1) = \mathbb{Q}Ge(G,H_2,K_2)$ if and only if $H_1 \cap K_2 = H_2 \cap K_1^g$, for some $g \in G$. In particular, if $H_1 = H_2$ then $e(G,H_1,K_1) = e(G,H_1,K_2)$ if and only if K_1 and K_2 are conjugate in G.

To complete the proof we prove the following statements for G a non-Dedekind group satisfying (ii):

(A) If G is a 3-group then G is isomorphic to either $C_9 \rtimes C_3$, K_3^{n+2} or L_3^{n+2} for some $n \geq 1$.

(B) Assume that G is a 2-group.

  (B1) If G has a cyclic subgroup of index 2 then G is isomorphic to either D_8, Q_{16} or $D^+_{2^{n+2}}$ with $n \geq 2$.

  (B2) If G has not a cyclic subgroup of index 2 but it has an abelian subgroup of index 2 then G is isomorphic to either $C_4 \rtimes C_4$, G_{32} or $H_{2^{n+2}}$ with $n \geq 2$.

(B3) If G does not have an abelian subgroup of index 2 then $G \cong D_8 Y Q_8$.

(C) If G nilpotent then G is either a 2-group or a 3-group.

(D) If G is not nilpotent then is isomorphic to A_4 or one of the first three groups of (iv)(e).

(**A**) Suppose that G is a 3-group. Then, by Roquette Theorem [**Roq58**], D and each D_i are number fields and hence c.d.$(G) = \{1,3\}$. Let H be a normal subgroup of G contained in G'. If $H \neq G'$ then $\mathbb{Q}G\widehat{H} \cong \mathbb{Q}(G/H)$ contains the unique non-commutative simple component of $\mathbb{Q}G$ and contains $\mathbb{Q}G\widehat{G'}$. This implies that $H = 1$. Therefore the only normal subgroups of G contained in G' are 1 and G'. Hence $G' \cap Z(G) = G'$, i.e. $G' \subseteq Z(G)$ and G' has order 3.

If G has a cyclic subgroup W of index 3 then $G = C_9 \rtimes C_3$ because the derived subgroup of $C_{3^k} \rtimes C_3$ with non-trivial action has order 3^{k-1}.

Suppose that G has an abelian non-cyclic subgroup W of index 3. Then $T = \{K \leq W : (W,K) \in S\} = \{K \leq W : G' \not\subseteq K \leq W, W/K \text{ cyclic}\}$ and this set has a unique G-conjugacy class. As W is not cyclic and G' has order 3, T has cardinality greater than 1 and hence it has exactly 3 elements and W/G' is cyclic. Therefore $W = \langle a \rangle \times \langle z \rangle$ with $G' = \langle z \rangle$ and then $a^b = az$, for some $b \in G \setminus W$. Moreover $b^3 \in Z(G)$, i.e. $b^3 = a^{3x} z^y$ for some integers x and y. Then $(a^{-x}b)^3 = z^y$ and hence, we may assume that $b^3 = \langle z \rangle$. Suppose that $|a| = 3^n$. If $b^3 = 1$ then $G \cong K_{3^{n+2}}$ and otherwise $G \cong L_{3^{n+2}}$.

Finally suppose that G does not have an abelian subgroup of index 3. Then, by [**Isa76**, Theorem 12.11], $[G : Z(G)] = 27$. Moreover, the exponent of $G/Z(G)$ is 3 for otherwise G has an element a of order 9 modulo $Z(G)$ and hence $W = \langle Z(G), a \rangle$ is abelian of index 3 in G. Thus, if $a \in G \setminus Z(G)$, then $W = \langle Z(G), a \rangle$ is a maximal abelian subgroup of G and $G/W \cong C_3 \times C_3$. If $(W,K) \in S$ for some $K \leq W$ then $\mathbb{Q}Ge(G,W,K)$ is a simple component of $\mathbb{Q}G$ of degree 9 in contradiction with the hypothesis. Thus for every $K \leq W$ such that W/K is cyclic there is $W < B$ such that $B' \subseteq K$. Thus, as $|G'| = 3$, if $G' \not\subseteq K$ we have that B is abelian, in contradiction with the hypothesis. This implies that W is cyclic, say of order 3^k. However G/W is a non-cyclic group isomorphic to a subgroup of Aut(W), while this group is cyclic. This yields a contradiction and finishes the proof of (A).

(**B**) Suppose that G is a 2-group, and remember that G is not Hamiltonian and satisfies (ii).

(**B1**) We assume first that G contains a cyclic subgroup $W = \langle a \rangle_{2^k}$ of index 2. This implies that c.d.$(G) = \{1,2\}$ and in particular, D is a field. Fix $b \in G \setminus W$. Then $a^b = a^i$ with $i \in \{-1, -1+2^{k-1}, 1-2^{k-1}\}$ and $b^2 \in Z(G)$. If $k \geq 3$ and $i = 1 - 2^{k-1}$ then $G \cong D^+_{2^{k+1}}$. Otherwise, $i \equiv -1 \mod 2^{k-1}$, $b^2 \in Z(G) = \langle a^2 \rangle$ and $G/\langle a^2 \rangle \cong D_{2^k}$. As $\mathbb{Q}D_{2^k}$ has at most one simple component which is not a division algebra, necessarily $k \leq 3$. Thus, in this case, $G \cong D_8, Q_8, D^-_{16}$ or Q_{16}. However, $G \not\cong Q_8$ because we are assuming that G is not Hamiltonian and it is not isomorphic to D^-_{16} because $\mathbb{Q}D^-_{16}$ has one simple component isomorphic to $M_2(\mathbb{Q})$ and another isomorphic to $M_2(\mathbb{Q}(i))$. This finishes the proof of (B1).

(**B2**) Suppose that G has no cyclic subgroups of index 2 but has an abelian subgroup W of index 2. Let $S = \{(W,K) : W/K \text{ is cyclic and } G' \not\subseteq K\}$. Let $T = \{K \leq W :$

$(W,K) \in S\}$ and $T_1 = \{K \in T : K$ is not normal in $G\}$. All the elements of T_1 are conjugate because if $K \in T_1$ then $\mathbb{Q}Ge(G,H,K)$ is the unique non-division algebra in the Wedderburn component of $\mathbb{Q}G$. This implies that T_1 is either empty or it has 2 elements and they are conjugate in G. In the latter case $\mathbb{Q}Ge(G,W,K)$ is a division algebra for every $K \in T \setminus T_1$.

Write

$$W = \langle a_1\rangle_{2^{k_1}} \times \cdots \times \langle a_n\rangle_{2^{k_n}},$$

let $b \in G \setminus W$ and suppose

$$b^2 = a_1^{s_1} \dots a_n^{s_n} \text{ with } -2^{k_i-1} < s_i \le 2^{k_i-1}.$$

Moreover, if $\langle a_i\rangle$ is normal in G then, we write $a_i^b = a_i^{r_i}$.

We consider two cases separately.

(**B2.1**) Suppose that <u>every direct factor of W is normal in G</u>. Let

$$K_i = \prod_{j\neq i} \langle a_i\rangle.$$

By (B1), G/K_j is either abelian or isomorphic to Q_8, D_8, Q_{16} or $D^+_{2^{k_n+1}}$ and hence one of the following conditions hold (maybe after a change of b):

(i) $r_i = 1$;
(ii) $k_i = 2$, $r_i = \pm 1$ and $s_i \in \{0, 2\}$;
(iii) $k_i = 3$, $r_i = -1$ and $s_i = 4$; or
(iv) $k_i \ge 3$, $r_i = 1 - 2^{k_i-1}$ and $s_i = 0$.

Each K_i with $r_i \ne 1$ belongs to T. Moreover, if $s_i = 0$ then $\mathbb{Q}Ge(G,W,K_i)$ is not a division algebra. Furthermore, if $r_i \ne 1$ and $k_i \ge 3$ then $K = \langle K_i, a_i^{2^{k_i-1}}\rangle$ is an element of T such that $\mathbb{Q}Ge(G,W,K)$ is not a division algebra. This implies that

$$X = \{i : \text{ either } r_i \ne 1 \text{ and } s_i = 0, \text{ or } r_i \ne 1 \text{ and } k_i \ge 3\}$$

has at most one element, since $K_1, \dots, K_n$ and K are pairwise non-conjugate in G. Let

$$Y = \{i : r_i \ne 1 \text{ and } i \notin X\}.$$

If Y has two different elements then G contains a group isomorphic to $H = \langle a_1, a_2, b | a_i^4 = (a_1, a_2) = 1, b^2 = a_1^2 a_2^2, a_i^b = a_i^{-1}\rangle$. Let $B = \langle a_1, a_2\rangle$, $K_1 = \langle a_1 a_2\rangle$ and $K_2 = \langle a_1 a_2^{-1}\rangle$. Then (B, K_1) and (B, K_2) are strong Shoda pairs of H such that $\mathbb{Q}He(H,B,K_1)$ and $\mathbb{Q}He(H,B,K_2)$ are two different simple components of $\mathbb{Q}H$ isomorphic to $M_2(\mathbb{Q})$, in contradiction with the hypothesis. Thus $|Y| \le 1$. Suppose that both X and Y are not empty and let $X = \{i\}$ and $Y = \{j\}$. Then K_j and $K = \langle K_i \cap K_j, a_i a_j^{2^{k-1}}\rangle$ belong to S and both $\mathbb{Q}Ge(G,W,K_j)$ and $\mathbb{Q}Ge(G,W,K)$ are not division algebras, yielding a contradiction. Thus either X or Y is empty. Therefore we may assume that $a_2, \dots, a_n \in Z(G)$ and $\langle a_1, b\rangle$ is isomorphic to either Q_8, D_8 or $D^+_{2^{k_1+1}}$. If $s_i = 0$ for some $i \ge 2$ then $G = H \times C_2$ for some subgroup H of G and hence in the Wedderburn decomposition of $\mathbb{Q}G$, every simple component of $\mathbb{Q}G$ appears an even number of times (up to isomorphisms). This yields a contradiction. Thus $s_i \ne 0$ for every $i \ge 2$.

Suppose $G/K_j \cong Q_8$ or D_8. Using that $\langle a_1 a_i\rangle$ is a direct factor of W, by assumption it is normal in G. Hence we deduce that $k_i = 1$ for every $i \ne 1$. Hence $b^2 = a_1^2 a_2 \cdots a_n$, if $G \cong Q_8$ and otherwise $b^2 = a_2 \dots a_n$. Then, if $n \ge 3$ then $G = \langle a_1, a_2, \dots, a_{n-2}, b\rangle \times \langle a_n\rangle$, yielding again a contradiction as in the previous case. Thus $n = 2$ and an easy argument shows that $G \cong C_4 \rtimes C_4$.

Assume otherwise that $\langle a_1, b\rangle \cong D^+_{2^{k_1+1}}$ with $k_1 \geq 3$. If s_i is even for some $2 \leq i \leq k$ then an epimorphic image of a subgroup of G is isomorphic to $D_{2^{k_1+1}} \times C_2$ and the rational group algebra of this group has two components which are not division rings. This implies that we may assume that $s_i = 1$ for every $i \neq 1$. Arguing as in the previous paragraph we deduce that $n = 2$. Hence $G = \langle a\rangle_{2^{k_1}} \rtimes \langle b\rangle_{2^{k_2+1}}$ with $a^b = a^{1-2^{k-1}}$. Then G has an epimorphic image isomorphic to $H = \langle a\rangle_8 \rtimes \langle b\rangle_4$ with $a^b = a^5$. If $B = \langle a, b\rangle$, $K_1 = \langle b^2\rangle$ and $K_2 = \langle a^4 b^2\rangle$. Then (W, K_1) and (W, K_2) are strong Shoda pairs of H such that $\mathbb{Q}He(H, B, K_i) \cong M_2(\mathbb{Q}(\sqrt{2}))$ for each $i = 1, 2$. This yields a contradiction and finishes the case when every direct factor of W is normal in G.

(**B2.2**) Suppose that <u>some direct factor of W is not normal in G</u>. Then, by Lemma 2.7, $W = \langle a_1\rangle_{2^{k_1}} \times \langle a_2\rangle_{2^{k_2}}$ with $ba_1b^{-1} = a_1^{r_1} a_2^y$, where $b \in G \setminus W$ and a_2^y has order 2. Then $T_1 = \{\langle a_1\rangle, \langle a_1^b\rangle\}$. If $a_1^{r_1} = 1$ then $\langle a_1^b\rangle$ is a direct factor of W included in $\langle a_2\rangle$ and hence it is equal to $\langle a_2\rangle$. Then $|a_1| = |a_2| = 2$ and $b^2 \in W \cap Z(G) = \langle a_1 a_2\rangle$. Hence $G \cong D_8$, contradicting the assumption that G does not have a cyclic subgroup of index 2. Thus $a_1^{r_1} \neq 1$. In particular, a_2 does not belong to any element of T_1 and therefore $\langle a_2\rangle$ is normal in G. Write $a_2^b = a_2^{r_2}$. Hence $G' = \langle a_1^{r_1-1} a_2, a_2^{r_2-1}\rangle$. By (B1), both $G/\langle a_2\rangle$ is either abelian or isomorphic to either Q_8, D_8, Q_{16} or $D^-_{2^{k+2}}$ with $k \geq 2$. If $G/\langle a_2\rangle$ is one of the last three groups then T contains an element K containing $\langle a_2\rangle$ such that $\mathbb{Q}Ge(G, W, K)$ is a matrix algebra. Then $K \in T_1$, and this yields a contradiction because no element of T_1 contains a_2. Thus $G/\langle a_2\rangle$ is either abelian or isomorphic to Q_8. In other words, <u>either $r_1 = 1$ or $k_1 = 2$, $r_1 = -1$ and $b^2 a_1^2 \in \langle a_2\rangle$</u>. We have to split the proof again in subcases:

(**B2.2.a**) Suppose that <u>$k_2 = 1$</u>. We claim that $k_1 \leq 2$. If not then $K = \langle a_1^4 a_2\rangle \in T \setminus T_1$, because $a_1^4 a_2 \notin \langle a_1\rangle \cup \langle a_1^{r_1} a_2\rangle$. Moreover, $\mathbb{Q}Ge(G, W, K) \cong M_2(\mathbb{Q}(i))$ because $G/K \cong D^+_{16}$. This yields a contradiction. If $k_1 = 1$ then $G \cong D_8$. If $k_1 = 2$ then G is given by a presentation of one of the following two forms:

$$\langle a_1, a_2, b \mid a_1^4 = a_1^2 = (a_2, b) = (a_1, a_2) = 1, a_1^b = a_1 a_2,\ b^2 \in \langle a_1^2, a_2\rangle\rangle,$$
$$\langle a_1, a_2, b \mid a_1^4 = a_1^2 = (a_2, b) = (a_1, a_2) = 1, a_1^b = a_1^{-1} a_2,\ b^2 \in \{a_1^2, a_1^2 a_2\}\rangle$$

Replacing, b by $a_1 b$, if necessarily, one may assume that in the first case $b^2 \in \langle a_2\rangle$ and in the second case $b^2 = a_1^2$. If, in the first case, $b^2 = 1$ then $K = \langle a_1^2 a_2\rangle \in T \setminus T_1$ and $\mathbb{Q}Ge(G, W, K) \cong M_2(\mathbb{Q})$, yielding a contradiction. Thus, in the first case $b^2 = a_2$ and $b^{a_1} = b^{-1}$ and hence $G = \langle b\rangle \rtimes \langle a_1\rangle \cong C_4 \rtimes C_4$. In the second case, $(a_1 b)^2 = a_1^2 a_2$ and $(a_1 b)^{a_1} = (a_1 b)^{-1}$ and hence $G = \langle a_1 b\rangle \rtimes \langle a_1\rangle \cong C_4 \rtimes C_4$.

(**B2.2.b**) Assume <u>$k_2 \geq 2$ and $r_1 = 1$</u>. Then $G' = \langle a_2^{r_2-1}, a_2^{2^{k_2-1}}\rangle$.

(**B2.2.b.1**) Suppose that <u>$r_2 \equiv 1 \mod 2^{k_2-1}$</u>. Then $G' = \langle a_2^{2^{k_2-1}}\rangle$, a subgroup of order 2. Therefore $T = \{K \leq W : W/K \text{ is cyclic and } a^{2^{k_2-1}} \notin K\}$. An obvious calculation shows that

$$T = \{\langle a_1 a_2^{2^i x}\rangle : \max\{0, k_2 - k_1\} \leq i \leq k_2, 2 \nmid x, 1 \leq x < 2^{k_2-i}\} \cup$$
$$\{\langle a_1^{2^i x} a_2\rangle : 1 \leq i \leq k_1 - k_2, 2 \nmid x, 1 \leq x \leq 2^{k_1-i}\}$$

and $T_1 = \{\langle a_1\rangle, \langle a_1 a_2^{2^{k_2-1}}\rangle\}$. Hence, if $K \in T \setminus T_1$, then $\mathbb{Q}Ge(G, W, K)$ is a totally definite quaternion algebra, K is normal in G and hence $(G, K) \subseteq K \cap G' = 1$, i.e., $K \subseteq Z(G)$. Therefore, if $\max\{0, k_2 - k_1\} \leq i \leq k_2$ and x is odd then $1 = (a_1 a_2^{2^i x}, b) = a_2^{2^{k_2-1} + 2^i x(r_2-1)}$. This implies that $r_2 \neq 1$ and $i = 0$. This shows that either, $k_1 = 1$, or $k_2 = 1$ or $k_1 = k_2 = 2$ and $r_2 = -1$. On the other hand, if $1 \leq i \leq k_1 - k_2$

and x is odd then $1 = (a_1^{2^i x} a_2, b) = a_2^{r_2-1}$. Thus, if $k_1 > k_2$ then $r_2 = k_2 = 1$. In this case, $K = \langle a_1^{2^{k_1-1}} a_2 \rangle \in T$ and $\mathbb{Q}Ge(G,W,K) \cong M_2(\mathbb{Q}(\zeta_{2^{k_1}} + \zeta_{2^{k_1}}^{-1}))$. This yields a contradiction. Therefore $k_1 \le k_2$ and hence either $k_1 = 1 < k_2$ or $k_1 = k_2 = 2$ and $r_2 = -1$. However, in the second case $\langle a_1 a_2 \rangle$ is a non-normal subgroup of G in $T \setminus \{\langle a_1 \rangle, \langle a_1 a_2^2 \rangle\}$, yielding a contradiction. Thus $k_1 = 1 < k_2$ and $b^2 \in Z(G) \subseteq \langle a_2 \rangle$. If $r_2 \ne 1$ then $r_2 = 1 - 2^{k_2-1}$. Then replacing a_2 by $a_1 a_2$ we may assume that $r_2 = 1$. As we are assuming that G has not a cyclic subgroup of index 2 then $b^2 \in \langle a_2^2 \rangle$. If $b^2 = a_2^{2i}$ then replacing b by ba_2^{-i}, we may assume that $b^2 = 1$. Then $G \cong H^+_{2^{k_2+2}}$.

(**B2.2.b.2**) Suppose now that $\underline{r_2 \not\equiv 1 \mod 2^{k_2-1}}$ (and still $r_1 = 1$). Then $k_3 \ge 3$ and $r_2 \in \{-1, -1 + 2^{k_2-1}\}$. We claim that $k_1 = 1$. Otherwise $r_2 \equiv -1 \mod 4$. Moreover $\langle a_1 a_2 \rangle \trianglelefteq G$ and hence there is an integer x such that $a_1^x a_2^x = (a_1 a_2)^b = a_1 a_2^{2^{k_2-1}+r_2}$. Then $x \equiv 1 \mod 4$ and $x \equiv 2^{k_2-1} + r_2 \mod 2^{k_2}$. Therefore $r_2 \equiv 1 \mod 4$, yielding a contradiction. Hence $k_1 = 1$ and so $b^2 \in Z(G) \subseteq \langle a_2 \rangle$. In particular, $\langle a_2, b \rangle$ has a cyclic subgroup of index 2. Since $r_2 \not\equiv 1 \mod 2^{k_2-1}$, by B1, we deduce that $\langle a_2, b \rangle \cong Q_{16}$. Therefore $\langle a_2, a_1 b \rangle \cong D_{16}$, contradicting (B1).

(**B2.2.c**) Now suppose that $\underline{k_2 \ge 2 \text{ and } r_1 \ne -1 \text{ (so that } k_1 = 2 \text{ and } b^2 a_1^2 \in \langle a_2 \rangle)}$. Then $a_1^2 \in Z(G)$ and the projection of W on $\overline{G} = G/\langle a_1^2 \rangle$ is $\langle \overline{a_1} \rangle_2 \times \langle \overline{a_2} \rangle_{2^{k_2}}$, with $\langle \overline{a_1} \rangle$ not normal in $\overline{G}$. So, $\overline{G}$ satisfies the conditions of (B2.2.b), with $k_2 \ne 1$. Thus it satisfies the conditions of (B2.2.b.1). Applying the arguments of (B2.2.b.1) to $\overline{G}$ we deduce that we may assume (after a change of generators) that $r_2 = 1$ and $b^2 \in \langle a_1^2 \rangle$. Then $G' = \langle a_1^2 a_2^{2^{k_2-1}} \rangle$. If $k_2 > 2$ then $\langle a_1 a_2 \rangle \in T \setminus T_1$ and hence this group is normal in G. Therefore $a_1^x a_2^x = (a_1 a_2)^b = a_1^{-1} a_2^{2^{k_2-1}+1}$ for some integer x. Then $x \equiv -1 \mod 4$ and $x \equiv 2^{k_2-1} + 1 \mod 2^{k_2}$. The latter implies $x \equiv 1 \mod 4$, a contradiction. Therefore $k_2 = 2$. If $b^2 = 1$ then $G/\langle a_2^2 \rangle \cong D_8 \times C_2$, a contradiction because $\mathbb{Q}(D_8 \times C_2)$ has two simple components isomorphic to $M_2(\mathbb{Q})$. Therefore $b^2 = a_1^2$ and we conclude that $G \cong G_{32}$.

(**B3**) Now assume that $\underline{G \text{ has no abelian subgroups of index } 2}$. We claim that $[G : Z(G)] \ne 8$. Otherwise, $G/Z(G)$ is elementary abelian, for if not G has an element g of order 4 modulo $Z(G)$ and hence $\langle Z(G), g \rangle$ is abelian of index 2 in G. Hence $G = \langle Z(G), b_1, b_2, b_3 \rangle$ with $b_i^2 \in Z(G)$ and $z_{ij} = (b_i, b_j) \in Z(G)$. Moreover, $G' = \langle z_{12}, z_{13}, z_{23} \rangle \cong C_2^3$ and $H_{ij} = G/\langle z_{ij} \rangle$ is non-abelian and has an abelian subgroup of index 2, namely $\langle Z(G), b_i, b_j \rangle / \langle z_{ij} \rangle$, and $H'_{ij} \cong C_2 \times C_2$. By (B2), H_{ij} is either a Hamiltonian 2-group or it is isomorphic to either D_8, Q_{16}, $D^+_{2^{n+2}}$, $C_4 \rtimes C_4$, G_{32} or $H_{2^{n+2}}$. However all these groups have a cyclic commutator while H_{ij} does not. This proves the claim.

Thus G has no abelian subgroups of index 2 and $[G : Z(G)] \ne 8$. Then, by [**Isa76**, Theorem 12.11], c.d.$(G) \ne \{1, 2\}$. This implies that D is not a field and hence it is a totally definite quaternion algebra over $\mathbb{Q}$. By [**EKVG15**] $D = \mathbb{H}_1 = \mathbb{H}(\mathbb{Q})$ and the identification of the projection of G in $M_2(D)$ in the GAP library is one of the following (see [**EKVG15**, Table 2]):

[32,8], [32,44], [32,50], [64,37], [64,137], [128,937]

However the last three have more than one component isomorphic to $M_2(\mathbb{H}_1)$, as it is displayed in [**EKVG15**, Table 2]. The group with GAP identification $[32, 8]$ is given by the following presentation

$$H = \langle a, b, c | a^8 = c^2 = (b, c) = 1, a^4 = b^2, a^b = ac, a^c = a^5 \rangle$$

Then $H/\langle a^4, a^2c\rangle \cong D_8$ and therefore $\mathbb{Q}H$ has a simple component isomorphic to $M_2(\mathbb{Q})$. As it also has a simple component isomorphic to $M_2(\mathbb{H}_1)$, this option should be excluded. The group with GAP identification $[32, 44]$ is given by the following presentation

$$H = (\langle a, b\rangle_{Q_8} \times \langle c\rangle_2) \rtimes \langle x\rangle_2$$

with $c^x = a^2c, a^x = b$ and $b^x = a$. Then $H/\langle a^2, c\rangle \cong D_8$ and again this option should be excluded. Therefore the projection of G on the only simple component of $\mathbb{Q}G$ which is not a division algebra is the group with GAP identification [32,50] which is the central product $D_8 Y Q_8$. Thus $G/N \cong D_8 Y Q_8$ for some normal subgroup N of G. We claim that $N = 1$. Otherwise, N has a subgroup H of index 2 which is normal in G. Hence G/H is a group of order 64 having $D_8 Y Q_8$ as an epimorphic image and such that $M_2(\mathbb{H}_1)$ is the only non-commutative simple component of $\mathbb{Q}G$ which is not a totally definite quaternion algebra. In particular, $M_2(\mathbb{Q})$ is not a simple component of $\mathbb{Q}(G/H)$ and hence D_8 is not an epimorphic image of G and G has exactly one irreducible character of degree 4. A computer search using GAP shows that there are only three groups G of order 64 up to isomorphism satisfying the following conditions: G has an epimorphic image isomorphic to $D_8 Y Q_8$, G has exactly one irreducible character of degree 4 and D_8 is not an epimorphic image of G. They are the groups number 233, 237 and 238 in the GAP list of groups of order 64. Computing the Wedderburn decomposition of $\mathbb{Q}G$ for these groups we observe that all of them have a simple component isomorphic to $M_2(\mathbb{Q}(i))$, a contradiction. This finishes this case, concluding that G is as in (4(a)). The proof of (B) is concluded.

(**C**) Suppose that G is nilpotent but it is not a q-group. Then $G = N \times P$ with P a q-group, for $q = 2$ or 3 and N a Dedekind q'-group with $N \neq 1$. By assumption G is not Dedekind and hence P is not Dedekind. This implies that $\mathbb{Q}P$ has a simple component of the form $M_k(D)$, with $k \neq 1$. Then N contains Q_8 or C_p for some odd prime and we may assume without loss of generality that N is one of these groups. In this case either $\mathbb{Q}N$ has two simple factors isomorphic to $\mathbb{Q}$ or $\mathbb{Q}N = \mathbb{Q} \otimes \mathbb{Q}(\zeta_p)$. Then $\mathbb{Q}G$ has either two simple components isomorphic to $M_k(D)$ or one component isomorphic to $M_k(D)$ and a direct factor isomorphic to $\mathbb{Q}(\zeta_p) \otimes M_k(D) \cong M_k(\mathbb{Q}(\zeta_p) \otimes D)$. As the former is not a product of division algebras, $\mathbb{Q}G$ has more than one simple component which is not a division algebra, contradicting the hypothesis. This proves (C).

(**D**). Suppose that G is not nilpotent. Then, by [**GH88**, Lemma 5.1], $G = N \rtimes P$ with P a q-group, for $q = 2$ or 3 and N a Dedekind q'-group with $N \neq 1$. If N is a p-group then, as $N \rtimes P$ is not Hamiltonian, then $\mathbb{Q}G$ has a 2×2 matrix algebra as simple component. If N is not a p-group and P act trivially on a Sylow subgroup of N then $\mathbb{Q}G$ has more than one simple component which is not a division algebra. Thus P acts non-trivially on every Sylow subgroup of N.

We claim that if N is abelian then a Sylow subgroup of N is either cyclic or $C_2 \times C_2$. To prove this, we may assume without loss of generality that N is a p-group of the form $\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_k}$ for prime $p \neq q$, P is cyclic of order q and the action of P on N is non-trivial. The action of P on N induces an action on $\overline{N} = N/N^p$. By [**Hup67**, Satz III.3.18] this action of P on $\overline{N}$ is non-trivial too. Hence we may assume that N is an elementary abelian p-group, so that we can consider N as a $\mathbb{Z}_p C_q$-module.

By Maschke Theorem $\mathbb{Z}_p C_q$ is semisimple. Assume moreover that $p \equiv 1 \mod q$. Then $\mathbb{Z}_p C_q$ is split. Therefore, we may assume without loss of generality that $N = \langle a_1\rangle \times$

$\cdots \times \langle a_n \rangle$ and each $\langle a_i \rangle$ is invariant under the action of P and reordering the factors we may assume that the action of P on $\langle a_1 \rangle$ is non-trivial. In particular $a_1 \in G'$. For very subgroup K of index p in N, such that N/K is cyclic and $G' \not\subseteq K$, (N, K) is a Shoda pair and $\mathbb{Q}Ge(G, N, K) \cong M_2(\mathbb{Q}(\zeta_p + \zeta_p^{-1}))$. If K_1 is another subgroup with the same property then $\mathbb{Q}Ge(G, N, K) = \mathbb{Q}Ge(G, N, K_1)$ if and only if K and K_1 are conjugate in G. If $n > 1$ there are more than two conjugacy classes of such subgroups (at least $\langle a_2 \rangle$ and $\langle a_1 a_2 \rangle$ are not conjugate because the first is normal and different from the second). Thus $\mathbb{Q}G$ has more than one simple component which is not a division algebra, contradicting the hypothesis.

Suppose that $p \not\equiv 1 \mod q$. Then $q = 3$, $p \equiv -1 \mod q$, and $\mathbb{Z}_p C_3$ has two simple modules up to isomorphisms. One of them is the trivial module. As the action of P on N is non-trivial, $N = M \times K$, as $\mathbb{Z}_p C_q$-module, with M the non-trivial simple $\mathbb{Z}_p C_3$-module. Hence $M = \langle a_1 \rangle \times \langle a_2 \rangle = C_p \times C_p$ and the action of $P = \langle b \rangle_3$ on N is given by $a_1^b = a_2$ and $a_2^b = a_1^{-1} a_2^{-1}$. As in the previous case every subgroup K of M of order p is not normal in G and parametrizes a simple component $\mathbb{Q}Ge(G, N, K)$ isomorphic to $M_3(\mathbb{Q}(\zeta_p))$. There are in total $p + 1$ such subgroups dividing in $\frac{p+1}{3}$ conjugacy classes. Thus this group algebra have $\frac{p+1}{3}$ simple components which are not division algebras. If $p \geq 5$, then $\frac{p+1}{3} \geq 2$ and this yields a contradiction. Otherwise $p = 2$ and $\langle M, P \rangle \cong A_4$. If $M \neq N$ then G contains a subgroup H isomorphic to either $C_2 \rtimes A_4$ or isomorphic to $(M_1 \times M_2) \cong C_3$ with each M_i the non-trivial simple $\mathbb{Z}_2 C_3$-module. In both cases we find more than one simple component in $\mathbb{Q}H$ which is not a division algebra. Hence $N = C_2 \times C_2$. This finishes the proof that if N is abelian then it is cyclic or $C_2 \times C_2$.

We claim that N is abelian. As N is Dedekind, to prove this one may assume without loss of generality that $N = Q_8 \times W$ with W an elementary abelian 2-group and $P = C_3$. If $W = 1$ then $G = \mathrm{SL}(2, 3) = Q_8 \rtimes C_3$, which is not possible because this group is not metabelian. Thus $W \neq 1$ and the centre of G is non-cyclic. Then by the previous paragraph, the action on W is trivial, for otherwise $Z(N)$ contains a subgroup K such that $\langle K, P \rangle = A_4$. As $N' \subseteq Z(G)$, we deduce that G contains a subgroup isomorphic to $C_2 \times A_4$, which is not possible because $\mathbb{Q}(C_2 \times A_4)$ has two simple components isomorphic to $M_3(\mathbb{Q})$. Hence the action of P on N induces an action on $N/W \cong Q_8$. By the first part of this paragraph this action is trivial. Thus, if $x \in Q_8$ then $x^b = xa$ with $a \in W$. Then $x^{b^2} = xa^2 = x$. As b has order 3 we deduce that the action of P on N is trivial, contradicting the hypothesis.

We now prove that if N is cyclic then $|N|$ is prime. Suppose N is cyclic, say generated by a. Let $b \in P$. We claim that $(a, b^q) = 1$ for every $b \in P$. As the kernel of the natural homomorphism $\mathrm{Aut}(\langle a \rangle) \to \mathrm{Aut}(\langle a \rangle / \langle a^p \rangle)$ is a p-group, to prove the claim we may assume that a is of order p. Thus, if the claim fails then $H = \langle a, b \rangle / \mathrm{C}_{\langle b \rangle}(a) \cong C_p \rtimes C_{q^k}$ with faithful action and $k \geq 2$. Then $(C_p, 1)$ is a strong Shoda pair of H and $\mathbb{Q}He(H, C_p, 1) \cong (\mathbb{Q}(\zeta_p)/F, 1) \cong M_{q^k}(F)$ for some field F. This contradicts the hypothesis, and hence proves the claim. By [**Hup67**, Satz III.3.18] if $b \in P$ acts non-trivially on the p-Sylow subgroup N_p of N then it induces a non-trivial action on N_p/N_p^p. Let p' be another prime dividing the order N and let $H = \langle N_p, N_{p'}, b \rangle$. If b acts trivially on $N_{p'}$ then $H = N_{p'} \times (N_p \rtimes \langle b \rangle)$. As $N_p \rtimes \langle b \rangle$ is not Dedekind, one of the simple components of $N_p \rtimes \langle b \rangle$ is not a division algebra and hence $\mathbb{Q}H$ has two simple components which are not division algebras. Otherwise H has an epimorphic image isomorphic to $(C_p \times C_{p'}) \rtimes C_q = (\langle a_1 \rangle_p \times \langle a_2 \rangle_{p'}) \rtimes \langle b \rangle_q$, with non-trivial action on C_p and $C_{p'}$. Then $(\langle a_1, a_2 \rangle, \langle a_1 \rangle)$ and $(\langle a_1, a_2 \rangle, \langle a_2 \rangle)$ are two strong Shoda pairs of G parametrizing two simple components which are not division algebras. This proves that N is a p-group. If p^2 divides the order of N then an epimorphic image of a subgroup of G is isomorphic to $H = \langle a \rangle_{p^2} \rtimes \langle a \rangle_q$ and $(H, 1)$ and $(H, \langle a^2 \rangle)$ are two strong Shoda pairs of H parametrizing two simple components of $\mathbb{Q}H$ which are not division algebras. This shows that N is of order p.

We now prove that if N is cyclic then the order of every element of P acting non-trivially on N is either $2, 4$ or 3. If P has an element b of order q^k acting non-trivially on N with $k \geq 3$ and $\langle a, b \rangle$ has an epimorphic image isomorphic to $H = \langle a \rangle_p \rtimes \langle b \rangle_{q^3}$. Then $(W, \langle b^q \rangle)$ and $(W = \langle a, b^q \rangle, 1)$ are strong Shoda pair of H and $\mathbb{Q}He(H, W, \langle b^q \rangle) \cong M_q(\mathbb{Q}(\zeta_{pq^2}))$ and $\mathbb{Q}G(H, W, 1) = (\mathbb{Q}(\zeta_{pq^2})/\mathbb{Q}(\zeta_{q^2}), \zeta_{q^2})$. This yields a contradiction because the second algebra is not a totally definite quaternion algebra. This shows that if $b \in Q$ acts non-trivially on N then $b^{q^2} = 1$. Furthermore, if $q = 3$ and b is a element of Q of order 9 then $(W, \langle b^3 \rangle)$ is a strong Shoda pair of H such that $\mathbb{Q}Ge(W, H, 1) \cong (\mathbb{Q}(\zeta_{3p})/\mathbb{Q}(\zeta_3), \zeta_3)$ which is not a totally definite quaternion algebra too. Thus if $q = 3$ and $b \in P$ acts non-trivially on N then $|b| = 3$.

If N is cyclic and P is abelian then after a change of generators we may assume that $P = \langle b \rangle \times Q$ with $(N, b) \neq 1$ and $(N, Q) = 1$. Then $G = Q \times (N \rtimes \langle b \rangle)$. As $N \rtimes \langle b \rangle$ is not Hamiltonian we deduce that $Q = 1$, i.e. P is cyclic of order 2, 4 or 3. Then G is isomorphic to either D_{2p}, Q_{4p} or $C_p \rtimes C_3$.

We claim that if N is cyclic then P is abelian. Otherwise P is one of the q-groups in (iv). For all these group P/P' is not cyclic. As P' is contained in the kernel of the action of P on N we deduce that $G/P' \cong C_p \rtimes W$ with W abelian non-cyclic and acting non-trivially on C_p. This contradicts the previous paragraph and finishes the proof for the case when N is cyclic.

Finally, suppose that N is non-cyclic. Then the Sylow 2-subgroup of G is isomorphic to $C_2 \times C_2$ and the Sylow $2'$-subgroups are cyclic. If N is not a 2-group then G has a quotient isomorphic to $(C_2 \times C_2 \times C_p) \rtimes P$, with P a 3-group and it is easy to prove that the rational group algebra of this group has two simple components which are not division algebras, in contradiction with the hypothesis. Thus $N = C_2 \times C_2$. We claim that $G \cong A_4$. If P an element of order 9 acting non-trivially on N then G contains a group isomorphic to $H = (\langle a_1 \rangle_2 \times \langle a_2 \rangle_2) \rtimes \langle b \rangle_9$. Then $(H = \langle a_1, a_2, b^3 \rangle, K_1 = \langle a_1 \rangle)$ and $(H, K_2 = \langle a_1, b^3 \rangle)$ are strong Shoda pair of G with K_1 and K_2 non-normal and non-conjugate in G. Then $\mathbb{Q}Ge(G, H, K_i)$ is not a division algebra. This shows that if $b \in P$ acts non-trivially on G then $b^3 = 1$. If P is not cyclic then it contains a subgroup $Q \cong C_3 \times C_3$ such that $\langle N, Q \rangle \cong A_4 \rtimes C_3$. As the rational group algebra of this group has two simple components which are not division rings, we deduce that P is cyclic of order 3. Thus $G \cong A_4$. This finishes the proof of the theorem. $\square$

We finish with the

Proof of Corollary 1.2 (i) is a direct consequence of Theorem 1.1 and the well known fact that a direct product of two non-abelian free groups is not coherent.

To prove statement (ii) we use Proposition 2.6. For any of the groups G considered $\mathbb{Q}G = A \times B$ with B a direct product of fields and division rings and A the algebra displayed (2.1). Let $\mathcal{O}$ be an order in A and $\mathcal{O}_1$ an order in G. As $\mathbb{Z}G$ and $\mathcal{O} \times \mathcal{O}_1$ are orders in $\mathbb{Q}G$, their groups of units are commensurable (see e.g. [**Seh93**, Lemma 4.5]). By Theorem 2.1, $\mathcal{U}(\mathcal{O}_1)$ has an abelian subgroup of finite index. Moreover, $\mathcal{U}(\mathcal{O})$ is commensurable with $\mathrm{SL}_1(\mathcal{O}) \times Z(\mathcal{O}_1)$. Therefore $\mathcal{U}(\mathbb{Z}G)$ is commensurable with $\mathrm{SL}_1(\mathcal{O}) \times$

U, for an abelian group U. This implies that $\mathcal{U}(\mathbb{Z}G)$ is coherent if and only if so is $\mathrm{SL}_1(\mathcal{O})$. Then the result is a consequence of Proposition 2.6 and the fact that if d is a non-negative integer then $\mathrm{SL}_2(\mathbb{Z}[-d])$ is coherent by [**Sco73**], since it is a 3-manifold group.

REFERENCES

[**BS72**] M. Benard and M. M. Schacher, *The Schur subgroup. II*, J. Algebra **22** (1972), 378–385. MR 0302747 (46 #1890)

[**EKVG15**] F. Eisele, A. Kiefer, and I. Van Gelder, *Describing units of integral group rings up to commensurability*, J. Pure Appl. Algebra **219** (2015), no. 7, 2901–2916. MR 3313511

[**Ger81**] S. M. Gersten, *Coherence in doubled groups*, Comm. Algebra **9** (1981), no. 18, 1893–1900. MR 638240 (82m:20032)

[**GH87**] R. Gow and B. Huppert, *Degree problems of representation theory over arbitrary fields of characteristic* 0. *On theorems of N. Itô and J. G. Thompson*, J. Reine Angew. Math. **381** (1987), 136–147. MR 918845 (89b:20029)

[**GH88**] ———, *Degree problems of representation theory over arbitrary fields of characteristic* 0. *II. Groups which have only two reduced degrees*, J. Reine Angew. Math. **389** (1988), 122–132. MR 953668 (89i:20021)

[**Hup67**] B. Huppert, *Endliche Gruppen. I*, Die Grundlehren der Mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin, 1967. MR 0224703 (37 #302)

[**Isa76**] I. M. Isaacs, *Character theory of finite groups*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1976, Pure and Applied Mathematics, No. 69. MR 0460423 (57 #417)

[**Kle00**] E. Kleinert, *Two theorems on units of orders*, Abh. Math. Sem. Univ. Hamburg **70** (2000), 355–358. MR 1809557

[**OdRS04**] A. Olivieri, Á. del Río, and J. J. Simón, *On monomial characters and central idempotents of rational group algebras*, Comm. Algebra **32** (2004), no. 4, 1531–1550. MR 2100373 (2005i:16054)

[**OdRS06**] A. Olivieri, Á. del Río, and J. J. Simón, *The group of automorphisms of the rational group algebra of a finite metacyclic group*, Comm. Algebra **34** (2006), no. 10, 3543–3567. MR 2262368 (2007g:20037)

[**Pas89**] D. S. Passman, *Infinite crossed products*, Pure and Applied Mathematics, vol. 135, Academic Press Inc., Boston, MA, 1989. MR 979094 (90g:16002)

[**Pie82**] R. S. Pierce, *Associative algebras*, Graduate Texts in Mathematics, vol. 88, Springer-Verlag, New York, 1982, Studies in the History of Modern Science, 9. MR 674652 (84c:16001)

[**Rei75**] I. Reiner, *Maximal orders*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], London-New York, 1975, London Mathematical Society Monographs, No. 5. MR 0393100 (52 #13910)

[**Roq58**] P. Roquette, *Realisierung von Darstellungen endlicher nilpotenter Gruppen*, Arch. Math. (Basel) **9** (1958), 241–250. MR 0097452 (20 #3921)

[**Sco73**] G. P. Scott, *Finitely generated 3-manifold groups are finitely presented*, J. London Math. Soc. **6** (1973), 437–440. MR 0380763 (9,224e)

[**Seh93**] S. K. Sehgal, *Units in integral group rings*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 69, Longman Scientific & Technical, Harlow, 1993. MR 1242557 (94m:16039)

[**Ser79**] J.-P. Serre, *Arithmetic groups*, Homological group theory (Proc. Sympos., Durham, 1977), London Math. Soc. Lecture Note Ser., vol. 36, Cambridge Univ. Press, Cambridge-New York, 1979, pp. 105–136. MR 564421 (82b:22021)

[**Wis11**] D. T. Wise, *Morse theory, random subgraphs, and incoherent groups*, Bull. Lond. Math. Soc. **43** (2011), no. 5, 840–848. MR 2854555

Ángel del Río,
Departamento de Matemáticas, Universidad de Murcia, 30100,
Murcia, Spain, adelrio@um.es

Pavel Zalesski,
Departamento de Matemática, Universidade de Brasília,
70.910-900, Brasilia-DF, Brasil, pz@mat.unb.br.