Corrections and Addenda to "The Isomorphism Problem for Incidence Rings"

G. Abrams, J. Haefner and Á. del Río *

Abstract

In this note the authors correct and extend results presented in their article "The Isomorphism problem for incidence rings", Pacific J. Math **187**(2), 1999, 201-214. Specifically we show that for a large class of rings (including those with finite right Goldie dimension, semilocal, and many commutative rings), if P and P' are finite preordered sets for which there is an isomorphism of incidence rings $I(P, R) \simeq I(P', R)$, then $P \simeq P'$.

In [1, Theorem 1.6], the authors prove that for a unital ring R having finite $D(R)$ (notation below), and for finite partially ordered sets P and P', an isomorphism of the incidence rings $I(P, R)$ and $I(P', R)$ yields an isomorphism of P and P'. In Lemma 1.10, Proposition 1.11 and Theorem 1.12 of [1] the authors attempt to extend this result to finite preordered sets by restricting the class of rings from which R can be chosen. Although Theorem 1.6 is valid, each of these three subsequent results is false (owing to an error in the proof of Lemma 1.10). The purpose of this note is to provide a corrected version of [1, Theorem 1.12]. This is achieved by providing a modified version of [1, Lemma 1.10] which will render the existing proofs of [1, Proposition 1.11] and [1, Theorem 1.12] valid in this slightly more restrictive setting. To wit, we show that [1, Theorem 1.12] remains true for various classes of rings (including rings with finite right Goldie dimension, semilocal rings, and a large class of commutative rings). In particular, we are able to provide a positive solution to the isomorphism problem for right Noetherian rings, the question which originally sparked the investigation in [1].

A counterexample to Theorem 1.12 can be constructed using the techniques of [2], [5], or [6], as follows. (We thank A. Facchini for suggesting this approach.) Consider the abelian monoid S generated by a single element I, with one relation $2I = 3I$. Then by [2, Theorem 6.2], for every field k there is a k-algebra R such that the monoid of isomorphism classes of finitely generated projective right modules over R is isomorphic to S, with the isomorphism class of R corresponding to I. As I is not the sum of two nonzero elements in S, the module R_R is indecomposable, and thus

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$D(R)$ is finite. But $R_R^2 \simeq R_R^3$, which induces an isomorphism of matrix rings $M_2(R) \simeq M_3(R)$. For each integer n we let P_n denote the preordered set having n elements, for which $x \leq y$ for every pair $x, y \in P_n$. Then R is a ring which satisfies the original hypotheses of Theorem 1.12, for which $I(P_2, R) \simeq M_2(R) \simeq M_3(R) \simeq I(P_3, R)$. But obviously $P_2 \not\simeq P_3$.

(In fact, using the original notation of [1], since R is indecomposable we have $D(R) = \{0, R\}$, so that $D(R)_1 = \{R\} = D^*(R)$. Then $A = I(P_2, R) \simeq M_2(R)$ and $D(A)_1 = \{A\}$. However $D^*(A)$ is infinite because A does not have finite summand length. Indeed, from $R_R^2 \simeq R_R^3$ one deduces that $R_R^2 \simeq R_R^n$ for every $n \geq 2$ and hence $A \simeq M_n(R)$ for every $n \geq 2$. This contradicts both Lemma 1.10 and Proposition 1.11 of [1]. For completeness, we note that the error in the original proof of Lemma 1.10 occurs in the paragraph that starts with "If $L = \oplus_{i=1}^n L_i$, then..." Specifically, the authors incorrectly assumed that the set of right modules $\{e_x L_i : x \in X\}$ is independent, which need not be the case in general.)

We show below how to modify the arguments given in [1] in two distinct ways (the 'Module Approach' and 'Ring Approach', respectively) to obtain the aforementioned positive results. Before doing so, we remind the reader of some notation, and make some general observations pertaining to the partially ordered result [1, Theorem 1.6].

The set of positive integers will be denoted by $\mathbf{N}$. For any ring R we define

$$D(R) = \{K : K \text{ is a two sided ideal of } R, \text{ and } K \text{ is a direct summand of } R_R\}.$$

$D(R)$ is partially ordered by inclusion. For any preordered set $(P, \leq)$ and any $x \in P$ we define

$$\hat{x} = \{y \in P : x \leq y \leq x\} \quad \text{and} \quad \widehat{P} = \{\hat{y} : y \in P\}.$$

$\widehat{P}$ is a partially ordered set with order inherited from P. Any finite preordered set P may be viewed as the pair $(\widehat{P}, \alpha_P)$, where $\alpha_P : \widehat{P} \to \mathbf{N}$ is defined by setting $\alpha_P(X) = |X|$ for every $X \in \widehat{P}$. Two finite preordered sets P and P' are isomorphic if and only if there exists an isomorphism $f : \widehat{P} \to \widehat{P'}$ of partially ordered sets such that $\alpha_{P'} \circ f = \alpha_P$. We will pass between the classical and this alternate viewpoint of preordered sets without specific mention.

For any ring R and finite preordered set P we denote by $A = I(P, R)$ the incidence ring of P with coefficients in R. For each right ideal K of R and element X of $\widehat{P}$ we denote by $H(X, K)$ the right ideal of A consisting of those matrices M for which the only nonzero entries are in rows indexed by $x \in X$, and are taken from K. For $x \in P$ the element $e_x \in A$ is defined by setting $e_x(y, z) = 1$ if $x = y = z$, zero otherwise.

For any partially ordered set Q we define the partially ordered subset Q_1 of Q by setting

$$Q_1 = \{x \in Q : \text{ there exists } x' < x \text{ with the property that } y \leq x' \text{ for every } y < x\}.$$

In particular, for any ring R the subset $D(R)_1$ of $D(R)$ will play an important role in the sequel. We note that since $D(R)$ always contains the minimal element $\{0\}$, if $D(R)$ is finite then $D(R)_1$ is necessarily nonempty.

By [1, Proposition 1.5] there is an isomorphism of partially ordered sets $H : \widehat{P} \times D(R)_1 \to D(A)_1$ given by $(X, K) \mapsto H(X, K)$. For any ring R we set

$$E_1(R) = \{f = f^2 \in R : eR \in D(R)_1, (eR)' = e'R, \text{ and } f = e - e'e\}.$$

Note that in this case, $eR/(eR)' \simeq fR$. In particular, for $K \in D(R)_1$ we can construct the right ideal $H(X, K/K')$ of A.

Lemma 1 *Let $A = I(P, R)$. Then*

$$\mathrm{End}(H(X,K)/H(X,K)'_A) \simeq \mathrm{End}(H(X, K/K')_A) \simeq M_{|X|}(\mathrm{End}(K/K'_R))$$

for every $(X, K) \in \widehat{P} \times D(R)_1$. In particular, if $f \in E_1(A)$, then there exists $g \in E_1(R)$ and $n \in \mathbf{N}$ for which $fAf \simeq M_n(gRg)$ as rings.

Proof. The first isomorphism follows from the right A-module isomorphism $H(X,K)/H(X,K)' \simeq H(X, K/K')$ which was correctly established in the first part of the proof of [1, Lemma 1.10]. In addition, the first part of the proof of [1, Lemma 1.10] also yields the right A-module isomorphism $H(X, K/K') \simeq \oplus_{x \in X} e_x H(X, K/K')$. Since $e_x H(X, K/K') \simeq e_y H(X, K/K')$ for $x, y \in X$, the second isomorphism of the Lemma will follow by establishing the ring isomorphism

$$\mathrm{End}(e_x H(X, K/K')_A) \simeq \mathrm{End}(K/K'_R).$$

But since $K/K'_R = fR$ we have $e_x H(X, K/K')_A = e_x fA$, and hence $\mathrm{End}(e_x H(X, K/K')_A) \simeq e_x fAe_x f \simeq fRf \simeq \mathrm{End}(fR_R) \simeq \mathrm{End}(K/K'_R)$. ∎

The proof provided for [1, Theorem 1.6] can be used verbatim to prove the following generalization.

Theorem 2 *Let P and P' be finite partially ordered sets, and let R be a ring for which $D(R)_1$ is finite and nonempty. If there is an isomorphism of incidence rings $I(P, R) \simeq I(P', R)$, then $P \simeq P'$.*

To show that Theorem 2 is indeed more general than the original version (in which we assume that $D(R)$ is finite, and thus necessarily that $D(R)_1$ is finite and nonempty), we provide a ring S for which $D(S)$ is infinite, but $|D(S)_1| = 1$. Specifically, the ring R described in [7, Example 2] (where the coefficient ring is taken to be a field) is a commutative unital ring having no nonzero indecomposable idempotents. Thus $D(R)$ is infinite, but $D(R)_1$ is empty. Then is straightforward to check that the ring $S = R \oplus \mathbf{Z}$ is a ring of the desired type.

We complete the introduction by reminding the reader that the ring R has *finite summand length* in case the number of nonzero summands in a direct decomposition of R_R is bounded. By [1, Lemma 1.8], if R is a ring with finite summand length then $D(R)$ is finite.

The 'Ring Approach'

We show here that by modifying the properties of the 'dimension function' used in [1], we can obtain a positive solution to the Isomorphism Problem for finite preordered sets for many types of rings, including right Noetherian, semilocal, and a large class of commutative rings. This 'Ring Approach' requires us to consider various classes of rings for which natural dimension functions exist. Let $\mathcal{C}$ be a class of unital rings with the property that for each $R \in \mathcal{C}$,

$$M_n(R) \in \mathcal{C} \text{ for every } n \geq 1, \text{ and } fRf \in \mathcal{C} \text{ for every } f \in E_1(R). \tag{1}$$

Suppose there exists a function $w : \mathcal{C} \to \mathbf{N}$ with the property that for each $n \in \mathbf{N}$,

$$w(M_n(R)) = n \cdot w(R). \tag{2}$$

We in turn define the class of unital rings $\widehat{\mathcal{C}}$ as follows:

$$R \in \widehat{\mathcal{C}} \quad \text{in case } fRf \in \mathcal{C} \text{ for every } f \in E_1(R).$$

In particular, $\mathcal{C} \subseteq \widehat{\mathcal{C}}$. Using Lemma 1 it is easy to verify

Lemma 3 *Suppose $\mathcal{C}$ is a class satisfying (1). If $R \in \widehat{\mathcal{C}}$ then $A = I(P, R) \in \widehat{\mathcal{C}}$ for any finite preordered set P.*

Theorem 4 *Suppose the pair $(\mathcal{C}, w)$ satisfies (1) and (2). Let P and P' be two finite preordered sets, and suppose $R \in \widehat{\mathcal{C}}$ is a ring for which $D(R)_1$ is finite and nonempty. If $I(P, R) \simeq I(P', R)$, then $P \simeq P'$.*

Proof. For $R \in \mathcal{C}$ we define the preordered set $D^*(R)$ as follows:

$$D^*(R) = (D(R)_1, \alpha) \quad \text{where } \alpha(K) = w(\mathrm{End}(K/K'_R)) \text{ for each } K \in D(R)_1.$$

(This definition is a modification of the corresponding preordered set given in [1]). Note that if $K \in D(R)_1$, then $K = eR$ and $K' = e'R$ for two idempotents e and e' of R, and $f = e - e'e \in E_1(R)$. Then $\mathrm{End}(K/K'_R) \simeq fRf \in \mathcal{C}$ and hence $w(\mathrm{End}(K/K'_R)) \in \mathbf{N}$.

Now let $A = I(P, R)$. The proof of Theorem 4 relies on the following equality for every $K \in D(R)_1$ and $X \in \widehat{P}$:

$$w(\mathrm{End}(H(X,K)/H(X,K)'_A)) = |X| \cdot w(\mathrm{End}(K/K'_R)). \tag{3}$$

The validity of this equation follows directly from Lemma 1 and Equation (2). With (3) in hand, the isomorphism of preordered sets

$$D^*(I(P,R)) \simeq P \times D^*(R)$$

follows as in the proof given in [1, Proposition 1.11]. This isomorphism in turn can be used to prove Theorem 4, arguing exactly as indicated in the proof of [1, Theorem 1.12]. ∎

Corollary 5 *Let R be a right Noetherian ring, and let P and P' be finite preordered sets. If there is an isomorphism of incidence rings $I(P,R) \simeq I(P',R)$, then $P \simeq P'$.*

Proof. Let $\mathcal{C}$ denote the class of right Noetherian rings, and let w denote the right Goldie dimension function. That w is finite on $\mathcal{C}$ is well known. The fact that $\mathcal{C}$ is closed under matrix rings is clear, and by [8, Lemma 2.7.12] we have that $eRe \in \mathcal{C}$ for any $e = e^2 \in R$ and $R \in \mathcal{C}$. The formula $w(M_n(R)) = nw(R)$ is a direct consequence of the additivity of Goldie dimension over direct sums of modules. Finally, any right Noetherian ring has finite summand length, so that $D(R)_1$ is finite (and necessarily nonempty). We now apply Theorem 4, keeping in mind that $\mathcal{C} \subseteq \widehat{\mathcal{C}}$ for any class $\mathcal{C}$. ∎

Corollary 5 is not the most general which can be derived using the class $\mathcal{C}$ of Noetherian rings. In fact, the proof of Corollary 5 yields

Corollary 6 *Let R be a ring with the property that fRf is Noetherian for each $f \in E_1(R)$. If there is an isomorphism of incidence rings $I(P,R) \simeq I(P',R)$, then $P \simeq P'$.*

To show that Corollary 6 is indeed more general than Corollary 5, we remind the reader of the ring given subsequent to Theorem 2; specifically, this non-Noetherian ring S has $|D(S)_1| = 1$, and $fSf \simeq \mathbf{Z}$ for $f \in E_1(S)$.

Recall that the ring R is called *semilocal* in case $R/J(R)$ is semisimple Artinian. This condition is equivalent to R having finite dual Goldie dimension by [3, Proposition 2.4.3].

Corollary 7 *Let R be a semilocal ring, and let P and P' be finite preordered sets. If there is an isomorphism of incidence rings $I(P,R) \simeq I(P',R)$, then $P \simeq P'$.*

Proof. Let $\mathcal{C}$ denote the class of semilocal rings, and let w denote the right dual Goldie dimension function. That the class of semilocal rings is closed under matrix rings is obvious, and that it is closed under subrings of the form eRe is shown in [8, Lemma 2.7.12]. The equation $w(M_n(R)) = nw(R)$ is a consequence of the additivity of the dual Goldie dimension [3, Prop. 2.42]. Finally, it is obvious that semilocal rings have finite summand length. The proof now proceeds in a manner virtually identical to that of the proof of Corollary 5. ∎

We note that there are examples which show that the class of rings having finite right Goldie dimension is not closed under taking subrings of the form eRe, so that Theorem 4 does not automatically apply to this class. Nonetheless, we will present below an approach which *will* allow us to deduce a positive solution to the isomorphism problem for these rings. We now give a class of rings which satisfy the conditions of Theorem 4, for which the 'Module Approach' of the next section will not apply. For a ring R which is a finitely generated free module over its center $Z(R)$ we denote by $\dim(R_{Z(R)})$ the rank of this module. (The uniqueness of this rank is a consequence of the commutativity of $Z(R)$.)

Corollary 8 *Let R be a ring such that there exists a commutative ring S having $D(S)_1$ finite and nonempty, and such that there exists a finite preordered set Q for which $R \simeq I(Q,S)$. Let P and P' be finite preordered sets. If there is an isomorphism of incidence rings $I(P,R) \simeq I(P',R)$, then $P \simeq P'$.*

Proof. Let $\mathcal{C}$ be the class of rings of the form $M_n(S)$ for S a commutative ring. (We note that for $R \simeq M_n(S)$, both n and S are uniquely determined.) Since $Z(M_n(S)) = S$ we have that R is a finitely generated free module over its center, and $\dim(R_{Z(R)}) = n^2$. Thus for the elements of $\mathcal{C}$ the function $w(R) = \sqrt{\dim(R_{Z(R)})}$ takes on values in $\mathbf{N}$. $\mathcal{C}$ is obviously closed under matrix rings, and Equation (2) holds by the previous observation. If $e \in E_1(M_n(S))$, then by Lemma 1 there exists $f \in E_1(S)$ and $t \in \mathbf{N}$ such that $eM_n(S)e \simeq M_t(fSf) \in \mathcal{C}$. Thus the pair $(\mathcal{C}, w)$ satisfies the hypotheses of Theorem 4. All that remains is to show that any ring of the indicated type is in $\widehat{\mathcal{C}}$. But this follows directly from Lemma 3. ∎

A special case of Corollary 8 bears mentioning.

Corollary 9 *Let R be a commutative ring such that $D(R)_1$ finite and nonempty. Let P and P' be finite preordered sets. If there is an isomorphism of incidence rings $I(P,R) \simeq I(P',R)$, then $P \simeq P'$.*

We note that the hypothesis that $D(R)_1$ is finite cannot be dropped in the previous result. For instance, it is well known that for k a field, the isomorphism problem has a negative solution for the coefficient ring $R = \Pi_{i=1}^{\infty} k$.

The 'Module Approach'

We give a second approach to obtaining a positive solution to the isomorphism problem for preordered sets. This alternate approach will allow us not only to re-obtain the results of Corollaries 5 and 7, but will allow us also to obtain a positive solution to the isomorphism problem for rings of finite Goldie dimension. For notational clarity, and to avoid cumbersome generality, we begin our presentation by giving the result and proof for rings with finite Goldie dimension, then subsequently teasing out the germane properties and stating a more general result.

Prior to doing so, we present a recovery result for preordered sets which will form the basis of the argument. If $P = (\widehat{P}, \alpha_P)$ is a finite preordered set, we denote by $P_{max} = (\widehat{P}, \alpha_{P_{max}})$ the preordered set given by

$$\alpha_{P_{max}}(a) = k(a) \cdot \alpha_P(a)$$

where for $a \in \widehat{P}$, $k(a)$ is the number of maximal elements in the partially ordered subset $[a, \infty) = \{x \in \widehat{P} : a \leq x\}$ of $\widehat{P}$.

Lemma 10 *If P and P' are two finite preordered sets, then $P \simeq P'$ if and only if $P_{max} \simeq P'_{max}$.*

Proof. The 'only if' statement is obvious. Now suppose $P_{max} \simeq P'_{max}$. Then there exists an isomorphism of preordered sets $f : \widehat{P} = \widehat{P_{max}} \to \widehat{P'_{max}} = \widehat{P'}$ with the property that $\alpha_{P'_{max}}(f(a)) = \alpha_{P_{max}}(a)$ for all $a \in \widehat{P_{max}}$. In particular we have $k(f(a)) \cdot \alpha_{P'}(f(a)) = k(a) \cdot \alpha_P(a)$. But $k(f(a)) =$

$k(a)$ for each $a \in \widehat{P_{max}}$ (as k is clearly an isomorphism invariant). Then $k(a) \neq 0$ yields $\alpha_{P'}(f(a)) = \alpha_P(a)$, so that f is in fact an isomorphism between P and P' as desired. ∎

Proposition 11 *For any ring R, and any right R-module M, let $w(M_R)$ denote the Goldie dimension of M_R. Let R be a ring such that $w(R_R)$ is finite, and let P and P' be two finite preordered sets. If there is an isomorphism of incidence rings $I(P,R) \simeq I(P',R)$, then $P \simeq P'$.*

Proof. As $w(R_R)$ is finite, then $w(K/K'_R)$ is also finite for every $K \in D(R)_1$ because K/K' is isomorphic to a direct summand of R_R. We define a preordered set which will play a role analogous to that of $D^*(R)$ defined previously; for consistency we use the same notation. Let

$$D^*(R) = (D(R)_1, \alpha_R), \quad \text{where } \alpha_R(K) = w(K/K'_R) \text{ for each } K \in D(R)_1.$$

Note that $D^*(R)$ is finite by [1, Lemma 1.8]. For every $x \in P$, $Y \in [\hat{x}, \infty)$, and I a right ideal of R let $\Phi = \Phi_{x,Y}(I) : P \times P \to \{\text{right ideals of } R\}$ be given by $\Phi(a,b) = I$ if $a = x$ and $Y \leq \hat{b}$, and $\Phi(a,b) = 0$ otherwise. The map Φ satisfies the conditions of Lemma 1.1(1) of [1] and hence $h_{x,Y}(I) = I(\Phi)$ is a right ideal of $A = I(P,R)$. We analyze the right A-ideal $h_{x,\hat{x}}(I)$. It is straightforward to show that for every maximal element Y of $[\hat{x}, \infty)$, the lattice of submodules of $h_{x,Y}(I)_A$ is isomorphic to the lattice of submodules of I_R. Therefore $w(h_{x,Y}(I)_A) = w(I_R)$. Moreover if $Y_1, \ldots, Y_{k(\hat{x})}$ are the maximal elements of $[\hat{x}, \infty)$, then the sum $\sum_{i=1}^{k(\hat{x})} h_{x,Y_i}(I)_A$ is direct, and is an essential (and superfluous) submodule of $h_{x,\hat{x}}(I)$. Thus we have $w(h_{x,\hat{x}}(I)_A) = w(\oplus_{i=1}^{k(\hat{x})} h_{x,Y_i}(I)_A) = k(\hat{x}) \cdot w(I_R)$ by [4, Proposition 3.20] and the previous observation. So we have just proved

$$w(h_{x,\hat{x}}(I)_A) = k(\hat{x}) w(I). \tag{4}$$

We note that one consequence of (4) is that $w(A)$ is finite, as $A = \oplus_{x \in P} h_{x,\hat{x}}(R)_A$. Another consequence of (4) is the following formula for every $X \in \hat{P}$ and $K \in D(R)_1$:

$$w(H(X, K/K')_A) = |X| k(X) w(K/K'_R).$$

Using this formula with the arguments given in the proof of Proposition 1.11 of [1] implies

$$D^*(A) \simeq P_{max} \times D^*(R). \tag{5}$$

If $A = I(P,R)$ and $A' = I(P',R)$ are isomorphic, then by (5), the finiteness of $D^*(R)$, and the well-known cancellation property of Lovász, we deduce that $P_{max} \simeq P'_{max}$. Finally, $P \simeq P'$ by Lemma 10. ∎

It is interesting to note that the positive answer to the isomorphism problem for semilocal rings which was established in Corollary 7 using the 'Ring Approach' described above can be re-established by simply replacing 'Goldie dimension' by 'dual Goldie dimension' throughout the proof of Proposition 11. (Here one uses the fact that $\sum_{i=1}^{k(\hat{x})} h_{x,Y_i}(I)_A$ is superfluous in $h_{x,\hat{x}}(I)$, rather than it being essential as used above.)

For completeness, we write down a generalized version of the previous proposition. Its proof is established by simply tracing through the proof of Proposition 11 and using the hypotheses as necessary.

Theorem 12 *Let $\mathcal{C}$ be a class of rings. Let $\mathcal{D}$ denote the class $\{M : M$ is a right R-module for some $R \in \mathcal{C}\}$. Suppose w is a function $w : \mathcal{D} \to \mathbf{Z}^+ \cup \{\infty\}$ which satisfies the following conditions:*

1. *$w(M)$ depends only on the lattice of submodules of M. That is, if the lattices of submodules of M and N are isomorphic, then $w(M) = w(N)$.*
2. *$w(M) = 0$ if and only if $M = 0$.*
3. *If M, N are R-modules for some fixed R, then $w(M \oplus N) = w(M) + w(N)$.*
4. *If N is an essential and superfluous submodule of M, then $w(M) = w(N)$.*

Let R be a ring such that $w(R_R)$ is finite, and let P and P' be two finite preordered sets. If there is an isomorphism of incidence rings $I(P, R) \simeq I(P', R)$, then $P \simeq P'$.

We conclude this article with some open questions:

(1) Both Goldie dimension and dual Goldie dimension satisfy the hypotheses of Theorem 12. Are there any other 'natural' dimension functions defined on classes of modules which also do so? (We thank the referee for pointing out that any positive linear combination of such dimension functions yields a function with these properties.)

(2) Suppose R is a ring for which there is a positive solution to the isomorphism problem for finite partially ordered sets. Suppose further that R has *invariant matrix type*; that is, if the matrix rings $M_m(R)$ and $M_n(R)$ are isomorphic, then $n = m$. Must R then have a positive solution to the isomorphism problem for finite preordered sets?

(3) Does there exist a right Goldie ring R, and an idempotent $e \in E_1(R)$, for which eRe is not a right Goldie ring?

(4) Classify the rings R for which there is a positive solution to the Isomorphism Problem for Incidence Rings over partially ordered (resp. preordered) sets with coefficients in R.

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University of Colorado

Colorado Springs CO 80933

email addresses: abrams@math.uccs.edu haefner@math.uccs.edu

Universidad de Murcia

30071 Murcia Spain

email address: adelrio@fcu.um.es