GROUP ALGEBRAS OF KLEINIAN TYPE AND GROUPS OF UNITS

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Abstract. The algebras of Kleinian type are finite dimensional semisimple rational algebras A such that the group of units of an order in A is commensurable with a direct product of Kleinian groups. We classify the Schur algebras of Kleinian type and the group algebras of Kleinian type. As an application, we characterize the group rings RG, with R an order in a number field and G a finite group, such that the group of units of RG is virtually a direct product of free-by-free groups.

The study of Kleinian groups goes back to the works of Poincaré [Poi] and Bianchi [Bia] and it has been an active field of research ever since. During the last decades, Kleinian groups have been strongly related to the Geometrization Program of Thurston for the classification of 3-manifolds [EGM, MR, Mas, Thu]. The use of the methods of Kleinian groups to the study of the groups of units of group rings was started in [PRR] and led to the notions of algebras of Kleinian type and finite groups of Kleinian type.

Let K be a number field, A a central simple K-algebra and R an order in A. (By order we mean always a $\mathbb{Z}$-order.) Let R^1 denote the group of units of R of reduced norm 1. Every embedding $\sigma : K \to \mathbb{C}$ induces an embedding $\overline{\sigma} : A \to M_d(\mathbb{C})$, where d is the degree of A. Furthermore, $\overline{\sigma}(R^1) \subseteq \mathrm{SL}_d(\mathbb{C})$. We say that A is of *Kleinian type* if either $A = K$ or A is a quaternion algebra over K and $\overline{\sigma}(R^1)$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$ for some embedding σ of K in $\mathbb{C}$. More generally, an *algebra of Kleinian type* [PRR] is, by definition, a direct sum of simple algebras of Kleinian type. A *finite group* G is of *Kleinian type* if and only if the rational group algebra $\mathbb{Q}G$ is of Kleinian type.

The finite groups of Kleinian type have been classified in [JPRRZ] where it has been also proved that a finite group G is of Kleinian type if and only if the group of units $\mathbb{Z}G^*$ of its integral group ring $\mathbb{Z}G$ is commensurable with a direct product of free-by-free groups. Recently, Alan Reid asked us in a private communication about the consequences of replacing the ring of rational integers by another ring of integers. This leads to the following two problems:

Problem 1. Classify the group algebras of Kleinian type of finite groups over number fields.

Problem 2. Given a group algebra of Kleinian type KG, describe the structure of the group of units of the group ring RG for R an order in K.

The simple factors of KG are Schur algebras over their centers. So, in order to solve Problem 1, it is natural to start classifying the Schur algebras of Kleinian type. This is obtained in Section 1. Using this classification and that of finite groups of Kleinian type given in [JPRRZ] we obtain the classification of the group algebras of Kleinian type in Section 2. In Section 3 we obtain a partial solution for Problem 2.

1. Schur algebras

Throughout K is a number field. We refer to the field homomorphisms $K \to \mathbb{R}$ as real embeddings of K. By abuse of notation, a pair of complex embeddings of K is, by definition, a pair of conjugate field homomorphisms $K \to \mathbb{C}$ whose images are not embedded in $\mathbb{R}$. Recall that the infinite places of K correspond to the real embeddings of K and the pairs of complex embeddings of K. Furthermore, the finite places of K correspond to the prime ideals of the ring of integers of K.

A *Schur algebra* over K is a central simple K-algebra A which is generated over K by a finite subgroup of the group of units A^* of A. Equivalently, a Schur algebra over K is a simple factor, with center K, of a group algebra of a finite group.

If L/K is a finite cyclic extension of degree n, with $\mathrm{Gal}(L/K) = \langle \sigma \rangle$ and $a \in K^*$, then $(L/K, \sigma, a)$ denotes the cyclic algebra $L[u|uxu^{-1} = \sigma(x), u^n = a]$. Sometimes we abbreviate $(L/K, \sigma, a)$ by writing $(L/K, a)$, if

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the generator σ is either clear from the context or not relevant. Recall that $(L/K,a)$ is split if and only if a is a norm of the extension L/K [Rei, Theorem 30.4].

A *cyclic cyclotomic algebra* is a cyclic algebra $(L/K,a)$ with L/K a cyclotomic extension and a a root of unity. A cyclic cyclotomic algebra $(L/K,a)$ is a Schur algebra because it is generated over K by the finite metacyclic group $\langle u,\zeta\rangle$, where ζ is a root of unity of L such that $L=K(\zeta)$. Conversely, every algebra generated by a finite metacyclic group is cyclic cyclotomic.

Quaternion algebras are cyclic algebras of degree 2 and take the form $\left(\frac{a,b}{K}\right)=K[i,j|i^2=a,j^2=b,ji=-ij]$, for $a,b\in K^*$. We abbreviate $\mathbb{H}(K)=\left(\frac{-1,-1}{K}\right)$. If $A=\left(\frac{a,b}{K}\right)$ and σ is a real embedding of K then A is said to *ramify* at σ if $\mathbb{R}\otimes_{\sigma(K)}A\simeq\mathbb{H}(\mathbb{R})$, or equivalently, if $\sigma(a),\sigma(b)<0$. Recall that a *totally definite quaternion algebra* is a quaternion algebra A over a totally real field which is ramified at every real embedding of its center.

Let G be a finite group. A *strong Shoda pair* of G is a pair (M,L) of subgroups of G such that $L\trianglelefteq M\trianglelefteq G$, M/L is cyclic and M/L is maximal abelian in $N_G(L)/L$. (The definition in [ORS] is more general but for our purposes we do not need such a generality.) If $M=L$ (and hence $M=G$), then let $\varepsilon(M,M)=\widehat{M}=\frac{1}{|M|}\sum_{m\in M}m\in\mathbb{Q}M$; otherwise, let $\varepsilon(M,L)=\prod(\widehat{L}-\widehat{S})$, where S runs on the minimal subgroups of M containing L properly. Finally, let $e(G,M,L)$ denote the sum of the different G-conjugates of $\varepsilon(M,L)$.

For every positive integer n, ζ_n denotes a complex primitive n-th root of unity. We quote the following result from [ORS].

Proposition 1.1. *Let G be a finite group and (M,L) a strong Shoda pair of G. Let $N=N_G(L)$, $k=[M:L]$ and $n=[G:N]$. Then $e=e(G,M,L)$ is a primitive central idempotent of $\mathbb{Q}G$ and $\mathbb{Q}Ge$ is isomorphic with $M_n(\mathbb{Q}(\zeta_k)*^{\sigma}_{\tau}N/M)$, an $n\times n$-matrix algebra over a crossed product of N/M over the cyclotomic field $\mathbb{Q}(\zeta_k)$, with defining action and twisting given as follows: Let x be a generator of M/L and let $\gamma:N/M\to N/L$ be a left inverse of the natural epimorphism $N/L\to N/M$. Then, for every $a,b\in N/M$, one has*

$$\zeta_k^{\sigma(a)}=\zeta_k^i,\ \text{if } x^{\gamma(a)}=x^i \quad \text{and} \quad \tau(a,b)=\zeta_k^j,\ \text{if } \gamma(ab)^{-1}\gamma(a)\gamma(b)=x^j.$$

In [ORS] it was proved that if G is a finite metabelian group, then every primitive central idempotent of $\mathbb{Q}G$ is of the form $e(G,M,L)$ for some strong Shoda pair (M,L) of G. This can be used to compute the Wedderburn decomposition of $\mathbb{Q}G$ for G metabelian. A method to compute the Wedderburn decomposition of $\mathbb{Q}G$ for G an arbitrary finite group is given in [Olt]. It has been implemented in the GAP package `wedderga` [BKOOR]. We will make use several times of this method at different places of the paper.

We quote the following proposition from [JPRRZ].

Proposition 1.2. *The following statements are equivalent for a central simple algebra A over a number field K.*

1. *A is of Kleinian type.*
2. *A is either a number field or a quaternion algebra which is not ramified at at most one infinite place.*
3. *One of the following conditions holds:*
   - (a) *$A=K$.*
   - (b) *A is a totally definite quaternion algebra.*
   - (c) *$A\simeq M_2(\mathbb{Q})$.*
   - (d) *$A\simeq M_2(\mathbb{Q}(\sqrt{d}))$, for d a square-free negative integer.*
   - (e) *A is a quaternion division algebra, K is totally real and A ramifies at all but one real embeddings of K.*
   - (f) *A is a quaternion division algebra, K has exactly one pair of complex (non-real) embeddings and A ramifies at all real embeddings of K.*

We need the following lemmas.

Lemma 1.3. *If $K=\mathbb{Q}(\sqrt{d})$ with d a square-free negative integer then*

1. *$\mathbb{H}(K)$ is a division algebra if and only if $d\equiv 1 \mod 8$.*
2. *$\left(\frac{-1,-3}{K}\right)$ is a division algebra if and only if $d\equiv 1 \mod 3$.*

Proof. (1) Writing $\mathbb{H}(K)$ as $(K(\zeta_4)/K,-1)$, one has that $\mathbb{H}(K)$ is a division algebra if and only if -1 is a sum of two squares in K. It is well known that this is equivalent to $d\equiv 1 \mod 8$ [FGS].

(2) Assume first that $A = \left(\frac{-1,-3}{K}\right)$ is not split. Then A is ramified at at least two finite places p_1 and p_2. Writing A as $(K(\zeta_3)/K, -1)$ and using [Rei, Theorem 14.1], one deduces that p_1 and p_2 are divisors of 3. Thus 3 is totally ramified in K and this implies that $\left(\frac{D}{3}\right) = 1$, where D is the discriminant of K [BS, Theorem 3.8.1]. Since $D = d$ or $D = 4d$ and $\left(\frac{p}{3}\right) \equiv p \mod 3$, for each rational prime p, one has $d = \left(\frac{d}{3}\right) = \left(\frac{D}{3}\right) \equiv 1 \mod 3$.

Conversely, assume that $d \equiv 1 \mod 3$. Then 3 is totally ramified in K. Let p be a prime divisor of 3 in K. Then the residue field of K_p has order 3 and $K_p(\zeta_4)/K_p$ is the unique unramified extension of degree 2 of K_p [Rei, Theorem 5.8]. Since $v_p(-3) = 1$, we deduce from [Rei, Theorem 14.1] that -3 is not a norm of the extension $K_p(\zeta_4)/K_p$. Thus $K_p \otimes_K A = (K_p(\zeta_4)/K_p, -3)$ is a division algebra, hence so is A. □

Lemma 1.4. *Let D be a division quaternion Schur algebra over a number field K. Then D is generated over K by a metabelian subgroup of D^*.*

Proof. By means of contradiction we assume that D is not generated over K by a metabelian group. Using Amitsur's classification of the finite subgroups of division rings (see [Ami] or [SW]) we deduce that D is generated by a group G which is isomorphic to one of the following three groups: $\mathcal{O}^*$, the binary octahedral group of order 48; $\mathrm{SL}(2,5)$, the binary icosahedral group of order 120; or $\mathrm{SL}(2,3) \times M$, where M is a metacyclic group. Recall that $\mathcal{O}^* = \langle x, y, a, b | x^4 = x^2y^2 = x^2b^2 = a^3 = 1, a^b = a^{-1}, x^y = x^{-1}, x^b = y, x^a = x^{-1}y, y^a = x^{-1}\rangle$. We may assume without loss of generality that G is one of these three groups. Let D_1 be the rational subalgebra of D generated by G. It is enough to show that D_1 is generated over $\mathbb{Q}$ by a metabelian group. So we may assume that D is generated over $\mathbb{Q}$ by G, and so D is one of the factors of the Wedderburn decomposition of $\mathbb{Q}G$.

Computing the Wedderburn decomposition of $\mathbb{Q}(\mathcal{O}^*)$ and $\mathbb{Q}\,\mathrm{SL}(2,5)$ and having in mind that D has degree 2 we obtain that $D \simeq (\mathbb{Q}(\zeta_8)/\mathbb{Q}(\sqrt{2}), -1)$, if $G = \mathcal{O}^*$, and $D \simeq (\mathbb{Q}(\zeta_5)/\mathbb{Q}(\sqrt{5}), -1)$, if $G = \mathrm{SL}(2,5)$. In both cases D is generated over its center by a finite metacyclic group.

Finally, assume that $G = \mathrm{SL}(2,3) \times M$, with M metacyclic. Then D is a simple factor of $A_1 \otimes_{\mathbb{Q}} A_2$, where A_1 is a simple epimorphic image of $\mathbb{Q}\,\mathrm{SL}(2,3)$ and A_2 is a simple epimorphic image of $\mathbb{Q}M$. Since A_2 is generated by a metacyclic group, it is enough to show that so is A_1. This is clear if A_1 is commutative. Assume otherwise that A_1 is not commutative. Having in mind that D is a division quaternion algebra, one deduces that so is A_1 and, computing the Wedderburn decomposition of $\mathbb{Q}\,\mathrm{SL}(2,3)$, one obtains that A_1 is isomorphic to $\mathbb{H}(\mathbb{Q})$. This finishes the proof because $\mathbb{H}(\mathbb{Q})$ is generated over $\mathbb{Q}$ by a quaternion group of order 8. □

For a positive integer n we set

$$\eta_n = \zeta_n + \zeta_n^{-1} \quad \text{and} \quad \lambda_n = \zeta_n - \zeta_n^{-1}.$$

Observe that $\eta_n^2 - \lambda_n^2 = 4$ and hence $\mathbb{Q}(\eta_n^2) = \mathbb{Q}(\lambda_n^2)$. Furthermore, if $n \geq 3$, then λ_n^2 is totally negative because $\zeta_n^{2i} + \zeta_n^{-2i} \leq 1$, for every $i \in \mathbb{Z}$ prime with n. Therefore, if $\lambda_n^2 \in K$ then $\left(\frac{\lambda_n^2,-1}{K}\right)$ ramifies at every real embedding of K.

We are ready to classify the Schur algebras of Kleinian type.

Theorem 1.5. *Let K be a number field and let A be a non-commutative central simple K-algebra. Then A is a Schur algebra of Kleinian type if and only if A is isomorphic to one of the following algebras:*

(1) $M_2(K)$, *with $K = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{d})$ for d a square-free negative integer.*
(2) $\mathbb{H}(\mathbb{Q}(\sqrt{d}))$, *for d a square-free negative integer, such that $d \equiv 1 \mod 8$.*
(3) $\left(\frac{-1,-3}{\mathbb{Q}(\sqrt{d})}\right)$, *for d a square-free negative integer, such that $d \equiv 1 \mod 3$.*
(4) $\left(\frac{\lambda_n^2,-1}{K}\right)$, *where $n \geq 3$, $\eta_n \in K$ and K has at least one real embedding and at most one pair of complex (non-real) embeddings.*

Proof. That the algebras listed are of Kleinian type follows at once from Proposition 1.2. Let K be a field. Then $M_2(K)$ is an epimorphic image of KD_8 and if $\lambda_n^2 \in K$ then the algebra $\left(\frac{\lambda_n^2,-1}{K}\right)$ is an epimorphic image of KQ_{2n}. This shows that the algebras listed are Schur algebras because $\mathbb{H}(K) = \left(\frac{\zeta_4^2,-1}{K}\right)$ and $\left(\frac{-3,-1}{K}\right) = \left(\frac{\zeta_6^2,-1}{K}\right)$.

Now we prove that if A is a Schur algebra of Kleinian type then one of the cases (1)-(4) holds. If A is not a division algebra, Proposition 1.2 implies that $A = M_2(K)$ for $K = \mathbb{Q}$ or an imaginary quadratic extension of $\mathbb{Q}$, so (1) holds.

In the remainder of the proof we assume that A is a division Schur algebra of Kleinian type. By Lemma 1.4, A is generated over K by a finite metabelian group G. Then $A = K \otimes_L B$, where B is a simple epimorphic image of $\mathbb{Q}G$ with center L and, by Proposition 1.1, B is a cyclic cyclotomic algebra $(\mathbb{Q}(\zeta_n)/L, \zeta_n^a)$ of degree 2. Since A is of Kleinian type, so is B.

Now we prove that L is totally real. Otherwise, since L is a Galois extension of $\mathbb{Q}$, L is totally complex and therefore K is also totally complex. By Proposition 1.2, both L and K are imaginary quadratic extensions of $\mathbb{Q}$ and so $L = K$ and $\varphi(n) = 4$, where φ is the Euler function. Then either (a) $n = 8$ and $K = \mathbb{Q}(\zeta_4)$ or $K = \mathbb{Q}(\sqrt{-2})$; or (b) $n = 12$ and $K = \mathbb{Q}(\zeta_4)$ or $K = \mathbb{Q}(\zeta_3)$. If $n = 8$, then B is generated over $\mathbb{Q}$ by a group of order 16 containing an element of order 8. Since B is a division algebra, $G = Q_{16}$ and so $B = \mathbb{H}(\mathbb{Q}(\sqrt{2}))$, a contradiction. Thus $n = 12$ and hence $B = (\mathbb{Q}(\zeta_{12})/\mathbb{Q}(\zeta_d), \zeta_d^a)$, where $d = 6$ or 4. Since ζ_6 is a norm of the extension $\mathbb{Q}(\zeta_{12})/\mathbb{Q}(\zeta_6)$, necessarily $d = 4$. So $A = B = (\mathbb{Q}(\zeta_{12})/\mathbb{Q}(\zeta_4), \zeta_4^a) = \left(\frac{\zeta_4^a, -3}{\mathbb{Q}(\zeta_4)}\right)$. Since $X = 1 + \zeta_4$, $Y = \zeta_4$ is a solution of the equation $\zeta_4 X^2 - 3Y^2 = 1$, ζ_4 is a norm of the extension $\mathbb{Q}(\zeta_{12})/\mathbb{Q}(\zeta_4)$, and hence so is ζ_4^a, yielding a contradiction.

So L is a totally real field of index 2 in $\mathbb{Q}(\zeta_n)$. Then $L = \mathbb{Q}(\eta_n)$ and necessarily B is isomorphic to $(\mathbb{Q}(\zeta_n)/\mathbb{Q}(\eta_n), -1) \simeq \left(\frac{\lambda_n^2, -1}{L}\right)$. This implies that $A \simeq \left(\frac{\lambda_n^2, -1}{K}\right)$. If K has some embedding in $\mathbb{R}$, then (4) holds. Otherwise $K = \mathbb{Q}(\sqrt{d})$, for some square-free negative integer d. This implies that $L = \mathbb{Q}$. Then $n = 3$, 4 or 6 and so A is isomorphic to either $\mathbb{H}(K)$ or $\left(\frac{-1,-3}{K}\right)$. Since A is a division algebra, Lemma 1.3 implies that, in the first case, $d \equiv 1 \mod 8$ and condition (2) holds, and, in the second case, $d \equiv 1 \mod 3$ and condition (3) holds. $\square$

2. Group algebras

In this section we classify the group algebras of Kleinian type, that is the number fields K and finite groups G such that KG is of Kleinian type. The classification for $K = \mathbb{Q}$ was given in [JPRRZ].

We start with some notation. The cyclic group of order n is usually denoted by C_n. To emphasize that $a \in C_n$ is a generator of the group, we write C_n either as $\langle a \rangle$ or $\langle a \rangle_n$. Recall that a group G is *metabelian* if G has an abelian normal subgroup N such that $A = G/N$ is abelian. We simply denote this information as $G = N : A$. To give a concrete presentation of G we will write N and A as direct products of cyclic groups and give the necessary extra information on the relations between the generators. By $\overline{x}$ we denote the coset xN. For example, the dihedral group of order $2n$ and the quaternion group of order $4n$ can be given by

$$\begin{array}{rcl} D_{2n} & = & \langle a \rangle_n : \langle \overline{b} \rangle_2, \quad b^2 = 1, a^b = a^{-1}. \\ Q_{4n} & = & \langle a \rangle_{2n} : \langle \overline{b} \rangle_2, \quad a^b = a^{-1},\ b^2 = a^n. \end{array}$$

If N has a complement in G then A can be identify with this complement and we write $G = N \rtimes A$. For example, the dihedral group can be also given by $D_{2n} = \langle a \rangle_n \rtimes \langle b \rangle_2$ with $a^b = a^{-1}$ and the semidihedral groups of order 2^{n+2} can be described as

$$\begin{array}{rcl} D^+_{2^{n+2}} & = & \langle a \rangle_{2^{n+1}} \rtimes \langle b \rangle_2, \quad a^b = a^{2^n+1}. \\ D^-_{2^{n+2}} & = & \langle a \rangle_{2^{n+1}} \rtimes \langle b \rangle_2, \quad a^b = a^{2^n-1}. \end{array}$$

Following the notation in [JPRRZ], for a finite group G, we denote by $\mathcal{C}(G)$ the set of isomorphism classes of noncommutative simple quotients of $\mathbb{Q}G$. We generalize this notation and, for a semisimple group algebra KG, we denote by $\mathcal{C}(KG)$ the set of isomorphism classes of noncommutative simple quotients of KG. For simplicity, we represent $\mathcal{C}(G)$ by listing a set of representatives of its elements. For example, using the isomorphisms

$$\mathbb{Q}D^-_{16} \cong 4\mathbb{Q} \oplus M_2(\mathbb{Q}) \oplus M_2(\mathbb{Q}(\sqrt{-2})) \quad \text{and} \quad \mathbb{Q}D^+_{16} \cong 4\mathbb{Q} \oplus 2\mathbb{Q}(i) \oplus M_2(\mathbb{Q}(i))$$

one deduces that $\mathcal{C}(D^+_{16}) = \{M_2(\mathbb{Q}(i))\}$ and $\mathcal{C}(D^-_{16}) = \{M_2(\mathbb{Q}), M_2(\mathbb{Q}(\sqrt{-2}))\}$.

The following groups play an important role in the classification of groups of Kleinian type.

$\mathcal{W} = \left(\langle t\rangle_2 \times \langle x^2\rangle_2 \times \langle y^2\rangle_2\right) : (\langle \overline{x}\rangle_2 \times \langle \overline{y}\rangle_2)$, with $t=(y,x)$ and $Z(\mathcal{W}) = \langle x^2, y^2, t\rangle$.

$\mathcal{W}_{1n} = \left(\prod_{i=1}^n \langle t_i\rangle_2 \times \prod_{i=1}^n \langle y_i\rangle_2\right) \rtimes \langle x\rangle_4$, with $t_i=(y_i,x)$ and $Z(\mathcal{W}_{1n}) = \langle t_1,\dots,t_n,x^2\rangle$.

$\mathcal{W}_{2n} = \left(\prod_{i=1}^n \langle y_i\rangle_4\right) \rtimes \langle x\rangle_4$, with $t_i=(y_i,x)=y_i^2$ and $Z(\mathcal{W}_{2n}) = \langle t_1,\dots,t_n,x^2\rangle$.

$\mathcal{V} = \left(\langle t\rangle_2 \times \langle x^2\rangle_4 \times \langle y^2\rangle_4\right) : (\langle \overline{x}\rangle_2 \times \langle \overline{y}\rangle_2)$, with $t=(y,x)$ and $Z(\mathcal{W}) = \langle x^2, y^2, t\rangle$.

$\mathcal{V}_{1n} = \left(\prod_{i=1}^n \langle t_i\rangle_2 \times \prod_{i=1}^n \langle y_i\rangle_4\right) \rtimes \langle x\rangle_8$, with $t_i=(y_i,x)$ and $Z(\mathcal{V}_{1n}) = \langle t_1,\dots,t_n,y_1^2,\dots,y_n^2,x^2\rangle$.

$\mathcal{V}_{2n} = \left(\prod_{i=1}^n \langle y_i\rangle_8\right) \rtimes \langle x\rangle_8$, with $t_i=(y_i,x)=y_i^4$ and $Z(\mathcal{V}_{2n}) = \langle t_i,x^2\rangle$.

$\mathcal{U}_1 = \left(\prod_{1\le i<j\le 3} \langle t_{ij}\rangle_2 \times \prod_{k=1}^3 \langle y_k^2\rangle_2\right) : \left(\prod_{k=1}^3 \langle \overline{y_k}\rangle_2\right)$, with $t_{ij}=(y_j,y_i)$ and $Z(\mathcal{U}_1) = \langle t_{12},t_{13},t_{23},y_1^2,y_2^2,y_3^2\rangle$.

$\mathcal{U}_2 = \left(\langle t_{23}\rangle_2 \times \langle y_1^2\rangle_2 \times \langle y_2^2\rangle_4 \times \langle y_3^2\rangle_4\right) : \left(\prod_{k=1}^3 \langle \overline{y_k}\rangle_2\right)$, with $t_{ij}=(y_j,y_i)$, $y_2^4=t_{12}$, $y_3^4=t_{13}$ and $Z(\mathcal{U}_2) = \langle t_{12},t_{13},t_{23},y_1^2,y_2^2,y_3^2\rangle$.

$\mathcal{T} = (\langle t\rangle_4 \times \langle y\rangle_8) : \langle \overline{x}\rangle_2$, with $t=(y,x)$ and $x^2=t^2=(x,t)$.

$\mathcal{T}_{1n} = \left(\prod_{i=1}^n \langle t_i\rangle_4 \times \prod_{i=1}^n \langle y_i\rangle_4\right) \rtimes \langle x\rangle_8$, with $t_i=(y_i,x)$, $(t_i,x)=t_i^2$ and $Z(\mathcal{T}_{1n}) = \langle t_1^2,\dots,t_n^2,x^2\rangle$.

$\mathcal{T}_{2n} = \left(\prod_{i=1}^n \langle y_i\rangle_8\right) \rtimes \langle x\rangle_4$, with $t_i=(y_i,x)=y_i^{-2}$ and $Z(\mathcal{T}_{2n}) = \langle t_1^2,\dots,t_n^2,x^2\rangle$.

$\mathcal{T}_{3n} = \left(\langle y_1^2 t_1\rangle_2 \times \langle y_1\rangle_8 \times \prod_{i=2}^n \langle y_i\rangle_4\right) : \langle \overline{x}\rangle_2$, with $t_i=(y_i,x)$, $(t_i,x)=t_i^2$, $x^2=t_1^2$, $Z(\mathcal{T}_{3n}) = \langle t_1^2, y_2^2,\dots,y_n^2,x^2\rangle$ and, if $i\ge 2$ then $t_i=y_i^2$,

$\mathcal{S}_{n,P,Q} = C_3^n \rtimes P = (C_3^n \times Q) : \langle \overline{x}\rangle_2$, with Q a subgroup of index 2 in P and $z^x=z^{-1}$ for each $z\in C_3^n$.

We collect the following lemmas from [JPRRZ].

Lemma 2.1. *Let G be a finite group and A an abelian subgroup of G such that every subgroup of A is normal in G. Let $\mathcal{H} = \{H \mid H$ is a subgroup of A with A/H cyclic and $G' \not\subseteq H\}$. Then $\mathcal{C}(G) = \cup_{H\in\mathcal{H}} \mathcal{C}(G/H)$.*

Lemma 2.2. *Let A be a finite abelian group of exponent d and G an arbitrary group.*

1. *If $d|2$ then $\mathcal{C}(A\times G) = \mathcal{C}(G)$.*
2. *If $d|4$ and $\mathcal{C}(G) \subseteq \left\{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}), \left(\frac{-1,-3}{\mathbb{Q}}\right), M_2(\mathbb{Q}(\zeta_4))\right\}$ then $\mathcal{C}(A\times G) \subseteq \mathcal{C}(G) \cup \{M_2(\mathbb{Q}(\zeta_4))\}$.*
3. *If $d|6$ and $\mathcal{C}(G) \subseteq \left\{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}), \left(\frac{-1,-3}{\mathbb{Q}}\right), M_2(\mathbb{Q}(\zeta_3))\right\}$ then $\mathcal{C}(A\times G) \subseteq \mathcal{C}(G) \cup \{M_2(\mathbb{Q}(\zeta_3))\}$.*

Lemma 2.3. (1) $\mathcal{C}(\mathcal{W}_{1n}) = \{M_2(\mathbb{Q})\}$.

(2) $\mathcal{C}(\mathcal{W}) = \mathcal{C}(\mathcal{W}_{2n}) = \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q})\}$.

(3) $\mathcal{C}(\mathcal{V}), \mathcal{C}(\mathcal{V}_{1n}), \mathcal{C}(\mathcal{V}_{2n}), \mathcal{C}(\mathcal{U}_1), \mathcal{C}(\mathcal{U}_2), \mathcal{C}(\mathcal{T}_{1n}) \subseteq \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}), M_2(\mathbb{Q}(\zeta_4))\}$.

(4) $\mathcal{C}(\mathcal{T}), \mathcal{C}(\mathcal{T}_{2n}), \mathcal{C}(\mathcal{T}_{3n}) \subseteq \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}), M_2(\mathbb{Q}(\zeta_4)), \mathbb{H}(\mathbb{Q}(\sqrt{2})), M_2(\mathbb{Q}(\sqrt{-2}))\}$.

(5) *Let $G = \mathcal{S}_{n,P,Q}$.*

   (a) *If $P = \langle x\rangle$ is cyclic of order 2^n then $\mathcal{C}(G) = \mathcal{C}(G/\langle x^2\rangle) \cup \left\{\left(\frac{\zeta_{2^{n-1}},-3}{\mathbb{Q}(\zeta_{2^{n-1}})}\right)\right\}$. In particular, if $P=C_2$ then $\mathcal{C}(G) = \{M_2(\mathbb{Q})\}$, if $P=C_4$ then $\mathcal{C}(G) = \left\{M_2(\mathbb{Q}), \left(\frac{-1,-3}{\mathbb{Q}}\right)\right\}$ and if $P=C_8$ then $\mathcal{C}(G) = \left\{M_2(\mathbb{Q}), \left(\frac{-1,-3}{\mathbb{Q}}\right), M_2(\mathbb{Q}(\zeta_4))\right\}$.*

   (b) *If $P = \mathcal{W}_{1n}$ and $Q = \langle y_1,\dots,y_n,t_1,\dots,t_n,x^2\rangle$ then $\mathcal{C}(G) = \left\{M_2(\mathbb{Q}), \left(\frac{-1,-3}{\mathbb{Q}}\right), M_2(\mathbb{Q}(\zeta_3))\right\}$.*

   (c) *If $P = \mathcal{W}_{21}$ and $Q = \langle y_1^2, x\rangle$ then $\mathcal{C}(G) = \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}(\sqrt{3})), M_2(\mathbb{Q}(\zeta_4)), M_2(\mathbb{Q}(\zeta_3))\}$.*

We are ready to present our classification of the group algebras of Kleinian type.

Theorem 2.4. *Let K be a number field and G a finite group. Then KG is of Kleinian type if and only if G is either abelian or an epimorphic image of $A\times H$, for A an abelian group, and one of the following conditions holds:*

(1) *$K=\mathbb{Q}$ and one of the following conditions holds.*

(a) *A has exponent* 6 *and* H *is either* $\mathcal{W}$, $\mathcal{W}_{1n}$ *or* $\mathcal{W}_{2n}$, *for some* n, *or* $H = \mathcal{S}_{m,\mathcal{W}_{1n},Q}$ *with* $Q = \langle y_1, \dots, y_m, t_1, \dots, t_m, x^2\rangle$, *for some* n *and* m.

(b) *A has exponent* 4 *and* H *is either* $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{V}$, $\mathcal{V}_{1n}$, $\mathcal{V}_{2n}$ *or* $\mathcal{S}_{n,C_8,C_4}$, *for some* n.

(c) *A has exponent* 2 *and* H *is either* $\mathcal{T}$, $\mathcal{T}_{1n}$, $\mathcal{T}_{2n}$, $\mathcal{T}_{3n}$ *or* $\mathcal{S}_{n,\mathcal{W}_{21},Q}$ *with* $Q = \langle y_1^2, x\rangle$, *for some* n.

(2) $K \neq \mathbb{Q}$ *and has at most one pair of complex (non-real) embeddings, A has exponent* 2 *and* $H = Q_8$.

(3) K *is an imaginary quadratic extension of* $\mathbb{Q}$, *A has exponent* 2 *and* H *is either* $\mathcal{W}$, $\mathcal{W}_{1n}$, $\mathcal{W}_{2n}$ *or* $\mathcal{S}_{n,C_4,C_2}$, *for some* n.

(4) $K = \mathbb{Q}(\zeta_3)$, *A has exponent* 6 *and* H *is either* $\mathcal{W}$, $\mathcal{W}_{1n}$ *or* $\mathcal{W}_{2n}$, *for some* n, *or* $H = \mathcal{S}_{m,\mathcal{W}_{1n},Q}$ *with* $Q = \langle y_1, \dots, y_m, t_1, \dots, t_m, x^2\rangle$, *for some* n *and* m.

(5) $K = \mathbb{Q}(\zeta_4)$, *A has exponent* 4 *and* H *is either* $\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}, \mathcal{V}_{1n}, \mathcal{V}_{2n}, \mathcal{T}_{1n}$ *or* $\mathcal{S}_{n,C_8,C_4}$, *for some* n.

(6) $K = \mathbb{Q}(\sqrt{-2})$, *A has exponent* 2 *and* H *is either* D_{16}^- *or* $\mathcal{T}_{2n}$, *for some* n.

Proof. To avoid trivialities, we assume that G is non-abelian. The main theorem of [JPRRZ] states that $\mathbb{Q}G$ is of Kleinian type if and only if G is an epimorphic image of $A \times H$ for A abelian and A and H satisfy one of the conditions (a)-(c) from (1). So, in the remainder of the proof, we assume that $K \neq \mathbb{Q}$.

First we prove that if one of the conditions (2)-(6) holds, then KG is of Kleinian type. For that we compute $\mathcal{C}(KG)$ and check that it is formed by algebras satisfying one of the conditions of Theorem 1.5. Since $\mathcal{C}(K\overline{G}) \subseteq \mathcal{C}(KG)$, for $\overline{G}$ an epimorphic image of G, it is enough to compute $\mathcal{C}(KG)$ for $G = A \times H$ and K, A and H satisfying one of the conditions (2)-(6). We use repeatedly Lemmas 2.2 and 2.3 which provide an approximation of $\mathcal{C}(G)$ and $\mathcal{C}(KG) = \{KZ(A) \otimes_{Z(A)} A : A \in \mathcal{C}(G)\}$.

If (2) holds, then $\mathcal{C}(KG) = \{\mathbb{H}(K)\}$.

If (3) holds, then $\mathcal{C}(G) \subseteq \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}), \left(\frac{-1,-3}{\mathbb{Q}}\right)\}$, and so $\mathcal{C}(KG) \subseteq \{M_2(K), \mathbb{H}(K), \left(\frac{-1,-3}{K}\right)\}$.

Similarly, if (4) holds then $\mathcal{C}(G) \subseteq \mathcal{C}(H) \cup \{M_2(\mathbb{Q}(\zeta_3))\} \subseteq \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}), \left(\frac{-1,-3}{\mathbb{Q}}\right), M_2(\mathbb{Q}(\zeta_3))\}$. Hence $\mathcal{C}(KG) = \{M_2(\mathbb{Q}(\zeta_3))\}$, by Lemma 1.3.

Arguing similarly one deduces that if (5) holds then $\mathcal{C}(KG) = \{M_2(\mathbb{Q}(\zeta_4))\}$.

Finally, assume that (6) holds. If $H = D_{16}^-$ then $\mathcal{C}(G) = \{M_2(\mathbb{Q}), M_2(\mathbb{Q}(\sqrt{-2}))\}$ and so $\mathcal{C}(KG) = \{M_2(\mathbb{Q}(\sqrt{-2}))\}$. Otherwise $H = \mathcal{T}_{2n}$ for some n. In this case we show that $\mathcal{C}(KG) = \{M_2(\mathbb{Q}(\sqrt{-2}))\}$. For that, we need a better approximation of $\mathcal{C}(G)$ that the one given in Lemma 2.3. Namely, we show that $\mathcal{C}(G) = \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}), M_2(\mathbb{Q}(\sqrt{-2}))\}$. Let L be a proper subgroup of H' (the derived subgroup of H) such that H'/L is cyclic. Using that $(y,x)y^2 = 1$, for each $y \in \langle y_1, \dots, y_n\rangle$, one has that $\mathcal{T}_{2n}/L$ is an epimorphic image of $\mathcal{T}_{21} \times C_2^{n-1}$. Then Lemmas 2.1 and 2.2 imply that $\mathcal{C}(G) = \mathcal{C}(\mathcal{T}_{21})$. So we may assume that $G = \mathcal{T}_{21}$. Now take $B = Z(\mathcal{T}_{21}) = \langle t^2, x^2\rangle \simeq C_2^2$ and L a subgroup of B such that B/L is cyclic. If $t^2 \in L$, then $\mathcal{T}_{21}/L$ is an epimorphic image of $\mathcal{W}$ and therefore $\mathcal{C}(\mathcal{T}_{21}/L) \subseteq \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q})\}$. Otherwise $L = \langle x^2\rangle$ or $L = \langle x^2t^2\rangle$; hence $\mathcal{T}_{21}/L \simeq D_{16}^-$ and so $\mathcal{C}(\mathcal{T}_{21}/L) \subseteq \{M_2(\mathbb{Q}), M_2(\mathbb{Q}(\sqrt{-2}))\}$. Using Lemma 2.1, one deduces that $\mathcal{C}(\mathcal{T}_{2n}) = \{\mathbb{H}(\mathbb{Q}), M_2(\mathbb{Q}), M_2(\mathbb{Q}(\sqrt{-2}))\}$ as claimed.

Conversely, assume that KG is of Kleinian type (and G is non-abelian and $K \neq \mathbb{Q}$). Then $\mathbb{Q}G$ is of Kleinian type, that is G is an epimorphic image of $A \times H$ for A and H satisfying one of the conditions (a)-(c) from (1). Furthermore, K has at most one pair of complex embeddings, by Theorem 1.5. We have to show that K, A and H satisfy one of the conditions (2)-(6). We consider several cases.

Case 1. Every element of $\mathcal{C}(G)$ *is a division algebra.*

This implies that G is Hamiltonian and so $G \simeq Q_8 \times E \times F$ with E an elementary abelian 2-group and F abelian of odd order [Rob, 5.3.7]. If $F = 1$, then (2) holds. Otherwise, $\mathcal{C}(KG)$ contains $\mathbb{H}(K(\zeta_n))$, where n is the exponent of F. Therefore $n = 3$ and $K = \mathbb{Q}(\zeta_3)$, by Theorem 1.5. Since Q_8 is an epimorphic image of $\mathcal{W}$, condition (4) holds.

In the remainder of the proof we assume that $\mathcal{C}(G)$ contains a non-division algebra B. Then $B = M_2(L)$ for some field L and therefore $M_2(KL) \in \mathcal{C}(KG)$. Since $K \neq \mathbb{Q}$, KL is an imaginary quadratic extension of $\mathbb{Q}$ and $L \subseteq K$. Let E be the center of an element of $\mathcal{C}(G)$. Then KE is the center of an element of $\mathcal{C}(KG)$. If $KE \neq K$, then the two complex embeddings of K extends to more than two complex embeddings of KE, yielding a contradiction. This shows that K contains the center of each element of $\mathcal{C}(G)$.

Case 2. The center of each element of $\mathcal{C}(G)$ *is* $\mathbb{Q}$.

Then Lemmas 2.2 and 2.3 imply that $\mathcal{C}(G) \subseteq \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}), \left(\frac{-1,-3}{\mathbb{Q}}\right)\}$. Using this and the main theorem of [LR] one has that $G = A \times H$, where A is an elementary abelian 2-group and H is an epimorphic image of $\mathcal{W}$, $\mathcal{W}_{1n}$, $\mathcal{W}_{2n}$ or $\mathcal{S}_{n,C_4,C_2}$, for some n. So G satisfies (3).

Case 3. At least one element of $\mathcal{C}(G)$ has center different from $\mathbb{Q}$.

Then the center of each element of $\mathcal{C}(G)$ is either $\mathbb{Q}$ or K. Using Lemmas 2.2 and 2.3 one has: If A and H satisfy condition (1.a) then $K = \mathbb{Q}(\zeta_3)$, hence (4) holds. If either A and H satisfy (1.b) or they satisfy (1.c) with $H = \mathcal{T}_{1n}$, for some n, then $K = \mathbb{Q}(\zeta_4)$ and condition (5) holds.

Otherwise, A has exponent 2 and H is either $\mathcal{T}, \mathcal{T}_{2n}, \mathcal{T}_{3n}$, or $\mathcal{S}_{n,\mathcal{W}_{21},\langle y_1^2, x\rangle}$, for some n. Since A has exponent 2, one may assume that $G = A \times H_1$, for H_1 an epimorphic image of H and H_1 is not an epimorphic image of any of the groups considered above. We use the standard bar notation for the images of $\mathbb{Q}H$ in $\mathbb{Q}H_1$.

Assume first that $H = \mathcal{T}$. Then $(M = \langle y, t\rangle, L = \langle ty^{-2}\rangle)$ is a strong Shoda pair of $\mathcal{T}$ and, by using Proposition 1.1, one deduces that if $e = e(\mathcal{T}, M, L) = \widehat{L}\frac{1-y^4}{2}$, then $\mathbb{Q}Ge \simeq \mathbb{H}(\mathbb{Q}(\sqrt{2}))$. Since $\mathbb{H}(\mathbb{Q}(\sqrt{2}))$ is not of Kleinian type, we have that $\overline{e} = 0$, or equivalently $\overline{y}^4 \in \overline{L}$. Hence either $\overline{y}^4 = 1$, $\overline{t}^2 = 1$ or $\overline{t} = \overline{y}^{-2}$. So H_1 is an epimorphic image of either $\mathcal{T}/\langle y^4\rangle$, $\mathcal{T}/\langle t^2\rangle$ or $\mathcal{T}/\langle ty^2\rangle$. In the first case H_1 is an epimorphic image of $\mathcal{T}_{11}$ and in the second case H_1 is an epimorphic image of $\mathcal{V}$. In both cases we obtain a contradiction with the hypothesis that H_1 is not an epimorphic image of the groups considered above. Thus H_1 is an epimorphic image of $\mathcal{T}/\langle ty^2\rangle \simeq D_{16}^-$. In fact $H_1 = D_{16}^-$, because every proper non-abelian quotient of D_{16}^- is an epimorphic image of $\mathcal{W}$. Then $\mathcal{C}(G) = \mathcal{C}(D_{16}^-) = \{M_2(\mathbb{Q}), M_2(\mathbb{Q}(\sqrt{-2}))\}$ and so $K = \mathbb{Q}(\sqrt{-2})$. Hence condition (6) holds.

Assume now that $H = \mathcal{T}_{2n}$. By the first part of the proof we have $\mathcal{C}(H) = \{\mathbb{H}(\mathbb{Q}), M_2(\mathbb{Q}), M_2(\mathbb{Q}(\sqrt{-2}))\}$. Since we are assuming that one element of $\mathcal{C}(G)$ has center different from $\mathbb{Q}$, then $M_2(\mathbb{Q}(\sqrt{-2})) \in \mathcal{C}(H_1)$ and so $K = \mathbb{Q}(\sqrt{-2})$. Hence condition (6) holds.

In the two remaining cases we are going to obtain some contradiction.

Suppose that $H = \mathcal{T}_{3n}$. We may assume that n is the minimal positive integer such that G is an epimorphic image of $\mathcal{T}_{3n} \times A$, for A an elementary abelian 2-group. This implies that $\langle \overline{t_1^2}, \overline{t_2}, \dots, \overline{t_n}\rangle$ is elementary abelian of order 2^n and hence $\langle \overline{y_1}^{-2}, \overline{y_2}, \dots, \overline{y_n}\rangle \simeq C_4^n$. Let $M = \langle y_1, y_2, \dots, y_n\rangle$ and $L_1 = \langle t_1 y_1^{-2}, y_2, y_3, \dots, y_n\rangle$. Then (M, L_1) is a strong Shoda pair of H. By using Proposition 1.1, we obtain that $\mathbb{Q}He(H, M, L_1) \simeq \mathbb{H}(\mathbb{Q}(\sqrt{2}))$. This implies that $\overline{e(H, M, L_1)} = 0$, or equivalently $\overline{t_1^2} = \overline{y_1^4} \in \overline{L_1}$. Since $\overline{t_1^2}$ has order 2 and $\overline{t_1^2} \not\in \langle t_2, \dots, t_n\rangle$ one has $\overline{t_1^2} = \overline{t_1 y_1^{-2} t_2^{\alpha_2} \cdots t_n^{\alpha_n}}$ for some $\alpha_1, \dots, \alpha_n \in \{0, 1\}$. Since, by assumption, H_1 is not an epimorphic image of $\mathcal{T}_{2n}$, we have $\alpha_i \neq 0$ for some $i \geq 2$. After changing generators one may assume that $\alpha_2 = 1$ and $\alpha_i = 0$ for $i \geq 3$. Thus $\overline{t_1} = \overline{y_1^{-2} t_2}$. Let now $L_2 = \langle t_1 y_1^{-2}, y_2 y_1^{-2}, y_3, \dots, y_n\rangle$. Then (M, L_2) is also a strong Shoda pair of H and $\mathbb{Q}Ge(H, M, L_2) \simeq \mathbb{H}(\mathbb{Q}(\sqrt{2}))$. The same argument shows that $\overline{y_1}^4 = \overline{t_1^2} \in \overline{L_2} = \langle \overline{t_1 y_1^{-2}}, \overline{y_2 y_1^{-2}}, \overline{y_3}, \dots, \overline{y_n}\rangle$. This yields a contradiction because $\overline{t_1 y_1^{-2}} = (\overline{y_2 y_1^{-2}})^2 \in \langle \overline{y_2 y_1^{-2}}, \overline{y_3}, \dots, \overline{y_n}\rangle$ and $\overline{y_1}^4 \not\in \langle \overline{y_2 y_1^{-2}}, \overline{y_3}, \dots, \overline{y_n}\rangle$.

Finally assume that $H = \mathcal{S}_{n,\mathcal{W}_{21},Q}$, with $Q = \langle y_1^2, x\rangle$ and set $y = y_1$. Since, by assumption, G does not satisfy (1.a), one has $\overline{t} = \overline{t_1} \neq 1$. Moreover, as in the previous case, one may assume that n is minimal (for G to be a quotient of $H \times A$, with A elementary abelian 2-group). Let $M = \langle C_3^n, x, t\rangle$ and $L = \langle Z_1, tx^2\rangle$, where Z_1 is a maximal subgroup of $Z = C_3^n$. Then (M, L) is a strong Shoda pair of H and $\mathbb{Q}He(H, M, L) \simeq \mathbb{H}(\mathbb{Q}(\sqrt{3}))$. Thus $0 = \overline{e(H, M, L)} = \overline{\widehat{L}(1 - \widehat{z})(1 - \widehat{t})}$, where $z \in Z \setminus Z_1$. Comparing coefficients and using the fact that $\overline{t} \neq \overline{z}$, for each $z \in Z$, we have $\overline{\widehat{L}(1 - \widehat{z})} = 0$, that is $\overline{L} = \overline{Z}$ and this contradicts the minimality of n. □

3. Groups of units

In this section we study the virtual structure of RG^*, for G a finite group and R an order in a number field K. More precisely, we characterize the finite groups G and number fields K for which RG^* is finite, virtually abelian, virtually a direct product of free groups or virtually a direct product of free-by-free groups.

We say that a group *virtually* satisfies a group theoretical condition if it has a subgroup of finite index satisfying the given condition. Notice that the virtual structure of RG^* does not depend on the order R and, in fact, if S is any order in KG, then S^* and RG^* are commensurable (see e.g. [Seh, Lemma 4.6]). Recall that two subgroups of a given group are said to be *commensurable* if their intersection has finite index in both. It is easy to show that a group commensurable with a free group (respectively a free-by-free group, a direct product of free groups, a direct product of free-by-free groups) is also virtually free (respectively a free-by-free group, a direct product of free groups, a direct product of free-by-free groups).

One important tool is the following lemma.

Lemma 3.1. *Let $A = \prod_{i=1}^n A_i$ be a finite dimensional rational algebra with A_i simple for every i. Let S be an order in A and S_i an order in A_i.*

1. *S^* is finite if and only if for each i, A_i is either $\mathbb{Q}$, an imaginary quadratic extension of $\mathbb{Q}$ or a totally definite quaternion algebra over $\mathbb{Q}$.*
2. *S^* is virtually abelian if and only if for each i, A_i is either a number field or a totally definite quaternion algebra.*
3. *S^* is virtually a direct product of free groups if and only if for each i, A_i is either a number field, a totally definite quaternion algebra or $M_2(\mathbb{Q})$.*
4. *S^* is virtually a direct product of free-by-free groups if and only if for each i, S_i^1 is virtually free-by-free.*

Proof. We are going to use the following facts:

(a) S^* is commensurable with $\prod_{i=1}^n S_i^*$ and S_i^* is commensurable with $Z(S_i)^* \times S_i^1$. (This is because S and $\prod_{i=1}^n S_i$ are both orders in A.)
(b) S_i^1 is finite if and only if A_i is either a field or a totally definite quaternion algebra (see [Kle1] or [Seh, Lemma 21.3]).
(c) If A_i is neither a field nor a totally definite quaternion algebra and S_i^1 is commensurable with a direct product of groups G_1 and G_2 then either G_1 or G_2 is finite [KR].
(d) S_i^1 is infinite and virtually free if and only if $A_i \simeq M_2(\mathbb{Q})$ (see e.g. [Kle2, page 233]).

(1) By (a), S^* is finite if and only if S_i^* is finite for each i if and only if $Z(S_i)^*$ and S_i^1 are finite for each i. By the Dirichlet Unit Theorem, if A_i is a number field, then S_i^* is finite if and only if A_i is either $\mathbb{Q}$ or an imaginary quadratic extension of $\mathbb{Q}$. Using this and (b), one deduces that, if A_i is not a number field then $Z(S_i)^*$ and S_i^1 are finite if and only if A_i is a totally definite quaternion algebra over $\mathbb{Q}$.

(2) By (a), S^* is virtually abelian if and only if S_i^1 is virtually abelian for each i. If A_i is either a number field or a totally definitive quaternion then S_i^1 is finite, by (b). Conversely, assume that S_i^1 is virtually abelian. We argue by contradiction to show that A_i is either a number field or a totally definite quaternion algebra. Otherwise S_i^1 is virtually infinite cyclic by (b) and (c) and the fact that S_i^1 is finitely generated. Then (d) implies that $A_i = M_2(\mathbb{Q})$ and so S_i^1 is commensurable with $\mathrm{SL}_2(\mathbb{Z})$. This yields a contradiction because $\mathrm{SL}_2(\mathbb{Z})$ is not virtually cyclic.

(3) By (a) and [JR, Lemma 3.1], S^* is virtually a direct product of free groups if and only if so is S_i^1 for each i. As in the previous proof, if A_i is neither commutative nor a totally definite quaternion algebra, then S_i^1 is virtually a direct product of free groups if and only if it is virtually free if and only if $A_i \simeq M_2(\mathbb{Q})$.

(4) Is proved in [JPRRZ, Theorem 2.1]. □

The characterization of when RG^* is finite (respectively virtually abelian, virtually a direct product of free groups) is an easy generalization of the corresponding result for integral group rings.

Theorem 3.2. *Let R be an order in a number field K and G a finite group. Then RG^* is finite if and only if one of the following conditions holds:*

1. *$K = \mathbb{Q}$ and G is either abelian of exponent dividing 4 or 6, or isomorphic to $Q_8 \times A$, for A an elementary abelian 2-group.*
2. *K is an imaginary quadratic extension of $\mathbb{Q}$ and G is an elementary abelian 2-group.*
3. *$K = \mathbb{Q}(\zeta_3)$ and G is abelian of exponent 3 or 6.*
4. *$K = \mathbb{Q}(\zeta_4)$ and G is abelian of exponent 4.*

Proof. If $K = \mathbb{Q}$, then $R = \mathbb{Z}$ and it is well known that $\mathbb{Z}G^*$ is finite if and only if G is abelian of exponent dividing 4 or 6 or it is isomorphic to $Q_8 \times A$, for A an elementary abelian 2-group [Seh].

If one of the conditions (1)-(4) holds, then KG is a direct product of copies of $\mathbb{Q}$, imaginary quadratic extensions of $\mathbb{Q}$ and $\mathbb{H}(\mathbb{Q})$. Then RG^* is finite by Lemma 3.1.

Conversely, assume that RG^* is finite and $K \neq \mathbb{Q}$. Then $\mathbb{Z}G^*$ is finite and therefore G is either abelian of exponent dividing 4 or 6 or isomorphic to $Q_8 \times A$, for A an elementary abelian 2-group. Moreover, R^* is finite and so $K = \mathbb{Q}(\sqrt{d})$ for d a square-free negative integer. If the exponent of G is 2 then (2) holds. If the exponent of G is 3 or 6, then one of the simple components of KG is $\mathbb{Q}(\sqrt{d}, \zeta_3)$, hence $d = -3$, and therefore (3) holds. If the exponent of G is 4 then one of the simple components of KG is $\mathbb{Q}(\sqrt{d}, \zeta_4)$ and therefore $d = -1$, that is (4) holds. □

Theorem 3.3. *Let R be an order in a number field K and G a finite group. Then RG^* is virtually abelian if and only if either G is abelian or K is totally real and $G \simeq Q_8 \times A$, for A an elementary abelian 2-group.*

Proof. As in the proof of Theorem 3.2, the sufficient condition is a direct consequence of Lemma 3.1.

Conversely, assume that RG^* is virtually abelian. Then $\mathbb{Z}G^*$ is virtually abelian and therefore it does not contain a non-abelian free group. Then G is either abelian or isomorphic to $G \simeq Q_8 \times A$, for A an elementary abelian 2-group [HP]. In the latter case, one of the simple components of KG is $\mathbb{H}(K)$ and hence K is totally real, by Lemma 3.1. □

Theorem 3.4. *Let R be an order in a number field K and G a finite group. Then RG^* is virtually a direct product of free groups if and only if either G is abelian or one of the following conditions holds:*

(1) *$K = \mathbb{Q}$ and $G \simeq H \times A$, for A an elementary abelian 2-group and H is either $\mathcal{W}$, $\mathcal{W}_{1n}$, $\mathcal{W}_{1n}/\langle x^2\rangle$, $\mathcal{W}_{2n}$, $\mathcal{W}_{2n}/\langle x^2\rangle$, $\mathcal{W}_{2n}/\langle x^2 t_1\rangle$, $\mathcal{T}_{3n}$ or $H = \mathcal{S}_{n,C_{2t},C_t}$, for some n and $t = 1, 2$ or 4.*

(2) *K is totally real and $G \simeq Q_8 \times A$, for A an elementary abelian 2-group.*

Proof. The finite groups G such that $\mathbb{Z}G^*$ is virtually a direct product of free groups were classified in [JR] and are the abelian groups and those satisfying condition (1). So, in the remainder of the proof, one may assume that $R \neq \mathbb{Z}$, or equivalently $K \neq \mathbb{Q}$, and we have to show that RG^* is virtually a direct product of free groups if and only if either G is abelian or (2) holds.

If either G is abelian or (2) holds then RG^* is virtually abelian, hence RG^* is virtually a direct product of free groups, because it is finitely generated.

Conversely, assume that RG^* is virtually a direct product of free groups and G is non-abelian. Since $K \neq \mathbb{Q}$, $M_2(\mathbb{Q})$ is not a simple quotient of KG, hence Lemma 3.1 implies that every simple quotient of KG is either a number field or a totally definite quaternion algebra. In particular, G is Hamiltonian, that is $G = Q_8 \times A \times F$, where A is an elementary abelian 2-group and F is abelian of odd order. If n is the exponent of F then $\mathbb{H}(K(\zeta_n))$ is a simple quotient of KG and this implies that $n = 1$ and K is totally real. □

Now we prove the main result of this section which provides a characterization of when RG^* is virtually a direct product of free-by-free groups.

Theorem 3.5. *Let R be an order in a number field K and G a finite group. Then RG^* is virtually a direct product of free-by-free groups if and only if either G is abelian or one of the following conditions holds:*

(1) *G is an epimorphic image of $A \times H$ with A abelian and K, A and H satisfy one of the conditions (1), (4), (5) or (6) of Theorem 2.4.*

(2) *K is totally real and $G \simeq A \times Q_8$, for A an elementary abelian 2-group.*

(3) *$K = \mathbb{Q}(\sqrt{d})$, for d a square-free negative integer, $SL_2(\mathbb{Z}[\sqrt{d}])$ is virtually free-by-free, and $G \simeq A \times H$ where A is an elementary abelian 2-group and one of the following conditions holds:*
   (a) *H is either $\mathcal{W}_{1n}$, $\mathcal{W}_{1n}/\langle x^2\rangle$, $\mathcal{W}_{2n}/\langle x^2\rangle$ or $\mathcal{S}_{n,C_2,1}$, for some n.*
   (b) *H is either $\mathcal{W}$, $\mathcal{W}_{2n}$ or $\mathcal{W}_{2n}/\langle x^2 t_1\rangle$, for some n and $d \not\equiv 1 \mod 8$.*
   (c) *$H = \mathcal{S}_{n,C_4,C_2}$ for some n and $d \not\equiv 1 \mod 3$.*

Proof. To avoid trivialities we assume that G is not abelian.

We first show that if K and G satisfy one of the listed conditions then RG^* is virtually a direct product of free-by-free groups. By Lemma 3.1, this is equivalent to show that for every $B \in \mathcal{C}(KG)$ and S an (any) order in B, we have that S^1 is virtually free-by-free. By using Lemmas 2.2 and 2.3 and the computation of $\mathcal{C}(KG)$ in the proof of Theorem 2.4, it is easy to show that if K and G satisfy one of the conditions (1) or (2), then every element of $\mathcal{C}(KG)$ is either a totally definite quaternion algebra or isomorphic to $M_2(K)$ for $K = \mathbb{Q}(\sqrt{d})$, with $d = 0, -1, -2$ or -3. In the first case S^1 is finite and in the second case S^1 is virtually free-by-free (see Lemma 3.1 and [JPRRZ, Lemma 3.1] or alternatively [Kle1], [MR, page 137] and [WZ]). If K and G satisfy condition (3) then Lemmas 1.3, 2.2 and 2.3 imply that $\mathcal{C}(KG) = \{M_2(\mathbb{Q}(\sqrt{d}))\}$. Since S^1 and $\mathrm{SL}_2(\mathbb{Z}[\sqrt{d}])$ are commensurable and, by assumption, the latter is virtually free-by-free, we have that S^1 is virtually free-by-free.

Conversely, assume that RG^* is virtually a direct product of free-by-free groups. Let B be a simple factor of KG and S an order in B. By Lemma 3.1, S^1 is virtually free-by-free and hence the virtual cohomological dimension of S^1 is at most 2. Then B is of Kleinian type by [JPRRZ, Corollary 3.4]. This proves that KG is of Kleinian type. Furthermore, B is of one of the types (a)-(f) from Proposition 1.2. However, the virtual cohomological dimension of S^1 is 0, if B is of type (a) or (b); 1 if B is of type (c); 2 if it is of type (d) or (e); and 3 if B is of type (f) [JPRRZ, Remark 3.5]. Thus B is not of type (f). Since every simple factor of KG contains K, either K is totally real or K is an imaginary quadratic extension of $\mathbb{Q}$ and KG is split.

By Theorem 2.4, G is an epimorphic image of $A \times H$ with A abelian and K, A and H satisfying one of the conditions (1)-(6) of Theorem 2.4. If they satisfy one of the conditions (1), (4), (5) or (6) of Theorem 2.4, then condition (1) (of Theorem 3.5) holds. So, we assume that K, A and H satisfy either condition (2) or (3) of Theorem 2.4. Since A is an elementary abelian 2-group, one may assume that $G = A \times H_1$ with H_1 an epimorphic image of H.

Assume first that K, A and H satisfy condition (2) of Theorem 2.4. Then one of the simple quotient of KG is isomorphic to $\mathbb{H}(K)$. If K is totally real then condition (2) holds. Otherwise $K = \mathbb{Q}(\sqrt{d})$ for d a square-free negative integer and $\mathbb{H}(K)$ is split. By Lemma 1.3, $d \not\equiv 1 \mod 8$. Since $Q_8 \simeq \mathcal{W}_{21}/\langle x^2 t_1\rangle$, condition (3b) holds.

Secondly assume that K, A and H satisfy condition (3) of Theorem 2.4 and set $K = \mathbb{Q}(\sqrt{d})$ for d a square-free negative integer. Then $\mathcal{C}(G) \subseteq \left\{\mathbb{H}(\mathbb{Q}), M_2(\mathbb{Q}), \left(\frac{-1,-3}{\mathbb{Q}}\right)\right\}$, by Lemma 2.3. By the main theorem of [JR], H_1 is isomorphic to either $\mathcal{W}$, $\mathcal{W}_{1n}$, $\mathcal{W}_{2n}$, $\mathcal{W}_{1n}/\langle x_1^2\rangle$, $\mathcal{W}_{2n}/\langle x_1^2\rangle$, $\mathcal{W}_{2n}/\langle x_1^2 t_1\rangle$, $\mathcal{S}_{n,C_2,1}$ or $\mathcal{S}_{n,C_4,C_2}$, for some n. If H is either $\mathcal{W}_{1n}$, $\mathcal{W}_{1n}/\langle x_1^2\rangle$, $\mathcal{W}_{2n}/\langle x_1^2\rangle$ or $\mathcal{S}_{n,C_2,1}$, then (3a) holds. If H is either $\mathcal{W}$, $\mathcal{W}_{2n}$ or $\mathcal{W}_{2n}/\langle x^2 t_1\rangle$ then $\mathcal{C}(G) = \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q})\}$ and, arguing as in the previous paragraph, one deduces that $d \not\equiv 1 \mod 8$. In this case condition (3b) holds. Finally, if $G = \mathcal{S}_{n,C_4,C_2}$ then $\mathcal{C}(G) = \{M_2(\mathbb{Q}), \left(\frac{-1,-3}{\mathbb{Q}}\right)\}$ and using the second part of Lemma 1.3 one deduces that $d \not\equiv 1 \mod 3$, and condition (3c) holds. □

The main theorem of [JPRRZ] states that a finite group G is of Kleinian type if and only if $\mathbb{Z}G^*$ is commensurable with a direct product of free-by-free groups. One implication is still true when $\mathbb{Z}$ is replaced by an arbitrary order in a number field. This is a consequence of Theorems 2.4 and 3.5.

Corollary 3.6. *Let R be an order in a number field K and G a finite group. If RG^* is commensurable with a direct product of free-by-free groups then KG is of Kleinian type.*

It also follows from Theorems 2.4 and 3.5 that the converse of Corollary 3.6 fails. The group algebras KG of Kleinian type for which the group of units of an order in KG is not virtually a direct product of free-by-free groups occur under the following circumstances, where $G = A \times H$ for A an elementary abelian 2-group:

(1) K is a number field of index ≥ 3 over $\mathbb{Q}$ with exactly one pair of complex embeddings and at least one real embedding and $H = Q_8$.
(2) $K = \mathbb{Q}(\sqrt{d})$, for d a square-free negative integer with $d \equiv 1 \mod 8$ and $H = \mathcal{W}$, $\mathcal{W}_{2n}$ or $\mathcal{W}_{2n}/\langle x^2 t_1\rangle$, for some n.
(3) $K = \mathbb{Q}(\sqrt{d})$ for d a square-free negative integer with $d \equiv 1 \mod 3$ and $H = \mathcal{S}_{n,C_4,C_2}$.
(4) $K = \mathbb{Q}(\sqrt{d})$ and d and H satisfy one of the conditions (3a)-(3c) from Theorem 3.5, but $\mathrm{SL}_2(\mathbb{Z}[\sqrt{d}])$ is not virtually free-by-free.

Resuming, if KG is of Kleinian type, then we have a good description of the virtual structure of RG^* for R an order in K, except in the four cases above. It has been conjectured that $\mathrm{SL}_2(\mathbb{Z}[\sqrt{d}])$ is virtually free-by-cyclic for every negative integer. This conjecture has been verified for $d = -1, -2, -3, -7$ and -11 (see [MR] and [WZ]). Thus, maybe the last case does not occur and the hypothesis of $\mathrm{SL}_2(\mathbb{Z}[\sqrt{d}])$ being virtually free-by-free in Theorem 3.5 is superfluous.

In order to obtain information on the virtual structure of RG^* in the four cases (1)-(4) above, one should investigate the groups S, for S an order in the following algebras: $\mathbb{H}(K)$, for K a number field with exactly one pair of complex embeddings and $\mathbb{H}(K)$ is not split, $\mathbb{H}(\mathbb{Q}(\sqrt{d}))$ with $d \equiv 1 \mod 8$, $\left(\frac{-1,-3}{\mathbb{Q}(\sqrt{d})}\right)$ with $d \equiv 1 \mod 3$ and, of course, $M(\mathbb{Q}(\sqrt{d}))$ for d a square-free negative integer. Notice that $K = \mathbb{Q}(\sqrt{-7})$ and $G = Q_8 = \mathcal{W}_{11}/\langle x^2 t_1\rangle$ is an instance of cases (2) above and, if R is an order in K, then RQ_8^* is commensurable with $\mathbb{H}(R)^*$. A presentation of $\mathbb{H}(R)^*$, for R the ring of integers of $\mathbb{Q}(\sqrt{-7})$ has been computed in [CJLR].

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