A SURVEY ON FREE SUBGROUPS IN THE GROUP OF UNITS OF GROUP RINGS

JAIRO Z. GONÇALVES AND ÁNGEL DEL RÍO

Abstract. In this survey we revise the methods and results on the existence and construction of free groups of units in group rings, with special emphasis in integral group rings over finite groups and group algebras. We also survey results on constructions of free groups generated by elements which are either symmetric or unitary with respect to some involution and other results on which integral group rings have large subgroups which can be constructed with free subgroups and natural group operations.

1. Introduction

All throughout this paper G denotes a group and R is a commutative ring with identity. Most of the time G is assumed to be finite and R is either the ring of integers $\mathbb{Z}$ or a field which we denote by F. In some exceptional cases R is the ring of integers of an algebraic number field. Our main object of interest is the group ring RG. This is a ring which contains R as a subring and G as a subgroup of $\mathcal{U}(RG)$, the group of units of RG. Moreover the elements of R and G commute in RG and G is a basis of RG as left R-module. That is R is formed by the finite formal sums $\sum_{g\in G} \alpha_g g$, with $\alpha_g \in R$, addition defined componentwise and multiplication extended by linearity from the multiplication in R and G, and satisfying $gr = rg$ for every $r \in R$ and $g \in G$.

Historically, the interest to study the properties of RG came mainly from the case $R = \mathbb{C}$, the field of complex numbers and G finite, the background of complex representations of finite groups. Soon, the subject became interesting in itself, with the investigation of RG and its group of units $\mathcal{U}(RG)$ being the meeting point of areas as group theory, algebraic number theory, ring theory, valuation theory, combinatorial group theory, topology, and many others.

In this survey we collect results and techniques about the following problems and other related questions.

Problem 1. *For which rings R and groups G the group $\mathcal{U}(RG)$ contain a free subgroup?*

Problem 2. *Give specific pairs (u, v) of elements of $\mathcal{U}(RG)$ which generate a free group.*

By a free group we always mean a free noncyclic group. By definition, a *free pair* of a group U is a pair (u, v) of elements of a group U which generate a (non-abelian)

2000 *Mathematics Subject Classification.* 20C05, 16S34.

The first author has been partially supported by Grant CNPq 303.756/82-5, and Fapesp Brazil, Proj. Tematico 00/07.291-0 and the second author by the Ministerio de Ciencia y Tecnología of Spain MTM2009-07373 and Fundación Séneca of Murcia 04555/GERM/06.

free group. Clearly Problem 1 has a positive solution for R and G if and only if $\mathcal{U}(RG)$ contains a free group, and Problem 2 consists in exhibiting free pairs in $\mathcal{U}(RG)$ provided they exist.

Historically, the first result on the subject in $\mathbb{Z}G$ was obtained simultaneously by Sehgal [49, Theorem VI 4.2] and Hartley and Pickel [27] as:

Theorem 1.1. [49, 27] *If G is a finite group then $\mathcal{U}(\mathbb{Z}G)$ contains a free subgroup if and only if G is neither abelian nor a Hamiltonian 2-group.*

The existence of a free subgroup in $\mathcal{U}(RG)$ has many consequences on the group structure of $\mathcal{U}(RG)$, mostly in negative form. For example, a group containing a free subgroup cannot be solvable. It should be pointed out that the first proof of Theorem 1.1, in [49], appeared related with solvability questions. There are many other examples of results on units which uses special constructions of free subgroups as part of the arguments in the proofs (see for example Theorems 2.3 and 5.8).

If instead of $\mathbb{Z}G$, we consider a group algebra FG over a field of characteristic 0 then the existence of free subgroups in $\mathcal{U}(FG)$ reduces, via Wedderburn's decomposition theorem, to the existence of free subgroups in $\mathrm{GL}_n(D)$ for D a finite dimensional division algebra over F. Recall that a field is said to be *absolute* if it is an algebraic extension of a finite field. If D is an absolute field then every finite subset of $M_n(D)$ is contained in a finite ring and hence $\mathrm{GL}_n(D)$ does not contain free subgroups. Similarly, if F is an absolute field then $\mathcal{U}(FG)$ does not contain free subgroups. Assume that D is not an absolute field. If $n > 1$ and D is a division algebra over a field of characteristic zero then it is easy to construct free groups in $\mathrm{GL}_n(D)$ using elementary matrices as we will see soon. However, for $n = 1$ this is a extremely hard question, stated as:

> *Lichtman's Conjecture*: [39] The multiplicative group of a non-commutative division ring contains a free subgroup.

If G is a finite group and R is a domain, then $\mathcal{U}(RG)$ is a linear group. Thus it is convenient to put the study of free subgroups of $\mathcal{U}(RG)$ in the context of linear groups. In this direction we have the celebrated Tits' Alternative: If U is a linear group then either U is a solvable-by-locally finite or it contains a (non-abelian) free subgroup. This was proved in the seminal paper of Tits [51] which provides some useful techniques to prove that some pairs are free pairs. Some generalizations of the results of Tits with applications to group rings have been obtained in [19].

The proof of Theorem 1.1 is not constructive and this raised the question of constructing free pairs in $\mathcal{U}(\mathbb{Z}G)$. Marciniak and Sehgal [40] proved that if u is a non-trivial bicyclic unit of $\mathbb{Z}G$ then (u, u^*) is a free pair in $\mathcal{U}(\mathbb{Z}G)$. Here $*$ denotes the natural involution in RG. This settled the question for non-abelian non-Hamiltonian groups. The Hamiltonian case was solved by Ferraz [8] using Bass units. (See Section 4 for the definition of bicyclic and Bass units.) This solves Problem 2 for integral group rings of torsion groups. It also solves the same question for group algebras of zero characteristic. However in positive characteristic bicyclic units have finite order and hence other alternative constructions are needed.

Intimately connected to the existence of free subgroups in $\mathcal{U}(RG)$ is the concept of *involution*. An involution in a group G (respectively, a non-necessarily commutative ring S) is an anti-automorphism of order 2 of G (respectively, of S). If φ is an involution of G then it induces an involution in RG, defining

$(\sum_{g\in G}\alpha_g g)^\varphi = \sum_{g\in G}\alpha_g g^\varphi$. A relevant involution in G is the *natural involution* (also known as classical or standard involution), given by $x^* = x^{-1}$, for each $x \in G$. If u is a non-trivial bicyclic unit in $\mathbb{Z}G$ then (uu^*, u^*u) is a free pair formed by symmetric units, because it is contained in the free group (u, u^*). However, if φ is an arbitrary involution in G then $\langle u, u^\varphi\rangle$ might not be free. For example, this is obviously the case if u is symmetric, anti-symmetric or unitary with respect to φ. This poses the question of constructing free pairs of symmetric, anti-symmetric or unitary units.

Another question that naturally emerges from the construction of free pairs of Marciniak-Sehgal and Ferraz is when $\mathcal{U}(\mathbb{Z}G)$ contains free pairs of Bass units or free pairs formed by a bicyclic unit and a Bass unit, and when a given a Bass unit u is part of a free pair where the "free companion" is also bicyclic or Bass.

Free groups can be considered as free products of infinite cyclic groups. This suggest the question of constructing (non-trivial) free products as subgroups of $\mathcal{U}(RG)$.

A question that naturally emerges from Theorem 1.1 is which portion of $\mathcal{U}(\mathbb{Z}G)$ can a free subgroup cover. More precisely, when does $\mathcal{U}(\mathbb{Z}G)$ contain a free subgroup of finite index or when does $\mathcal{U}(\mathbb{Z}G)$ contain a subgroup of finite index constructed from free subgroups and natural operations as for example direct or free products?

We briefly resume the contents of the paper. In Section 2 we first sketch the proof of Theorem 1.1, and revise the characterization of when a group algebra of a finite group contain a free subgroup and some related results. In Section 3 we present some techniques to prove that a pair of elements of a group is free. This includes some of the methods of Tits and some generalizations from [19], together with some applications on the existence of free products on $\mathcal{U}(\mathbb{Z}G)$. In Section 4 we give concrete constructions of free pairs in $\mathcal{U}(\mathbb{Z}G)$ and $\mathcal{U}(FG)$. Sections 5 is dedicated to the question of existence of free pairs which satisfy some condition relative to an involution of a group algebra. For example, we present some results on the existence of free symmetric or unitary pairs in a FG with respect to an involution of G, extended to the group algebra by linearity. Most results refers to the natural involution. Section 6 consider similar questions on $\mathbb{Z}G$. In Section 7 we consider the question of which pairs of bicyclic or Bass units are free pairs and when $\mathcal{U}(\mathbb{Z}G)$ has a free pair formed by such units. Finally in Section 8 we review some result which characterizes the finite groups G such that $\mathcal{U}(\mathbb{Z}G)$ has a subgroup of finite index with a nice structure with respect to free subgroups, that is, that can be obtained using natural group operations with free groups.

Before we start we establish some basic notation. We use C_n to denote the cyclic group of order n and $C_n = \langle x\rangle_n$ to emphasize that x is a generator of the cyclic group of order n. Other families of groups which will appear in the paper are the dihedral group of order n, denoted D_n, the quaternion group of order n, denoted Q_n and the symmetric group on n letters, denoted Sym_n. We use standard group and ring theoretical notation. For example, $Z(X)$ denotes the center of X (for X a group or a ring), G' the commutator subgroup of the group G, $N_G(X)$ the normalizer of a subset X in the group G, $J(R)$ denotes the Jacobson radical of the ring R, $\langle g_1,\dots,g_n\rangle$ the subgroup generated by $g_1,\dots,g_n$ and $N \rtimes G$ the semidirect product of the N by G with respect to an action of G on N. We use exponential notation for the action of automorphisms and involutions. This includes the notation $g^h = h^{-1}gh$, for g and h elements of a group. The commutator of g and h is $(g,h) = g^{-1}g^h = g^{-1}h^{-1}gh$. The free product of two groups G and H is denoted $G * H$. If X is a set then $|X|$ denotes its cardinality; if g is a group element then $|g|$ denotes its order; and if z is a complex number then $|z|$ denotes its absolute value. More general real absolute values will appear in Section 3 and will be denoted also $|\quad|$.

2. Existence of free subgroups

In this section we consider the question of existence of free groups in group rings, that is, we deal with Problem 1.

We start giving a sketch of how Theorem 1.1 can be proved. We basically follow the proof of Hartley and Pickel [27]. If G is abelian then obviously $\mathcal{U}(\mathbb{Z}G)$ does not contain a free group. Recall that a group is said to be Hamiltonian if it is a non-abelian group with all its subgroups normal. Hamiltonian groups are classified by a result of Baer and Dedekind [44]. Namely, a group G is Hamiltonian if and only if it is isomorphic to $Q_8 \times A$ with Q_8 the quaternion group of order 8 and A a torsion abelian group where the 2-torsion part is elementary abelian. In particular, if G is a finite Hamiltonian 2-group then G is isomorphic to $Q_8 \times A$ with A an elementary abelian 2-group. Then $\mathcal{U}(\mathbb{Z}G) = \pm G$, by a well known result of Higman [28, 29]. This proves the necessary part of Theorem 1.1.

The proof of the sufficient part is more involved. The first observation is that $\mathbb{Z}G$ is an order in $\mathbb{Q}G$, i.e. it is finitely generated as $\mathbb{Z}$-module and contains a basis of $\mathbb{Q}G$ over $\mathbb{Q}$. If $\mathcal{O}$ is any other order in $\mathbb{Q}G$ then $\mathcal{U}(\mathbb{Z}G) \cap \mathcal{U}(\mathcal{O})$ has finite index in both $\mathcal{U}(\mathcal{O})$ and $\mathcal{U}(\mathbb{Z}G)$ (see e.g. [50, Lemma 4.2]). Thus if (u,v) is a free pair in $\mathcal{U}(\mathcal{O})$ then (u^n, v^n) is a free pair in $\mathcal{U}(\mathbb{Z}G)$ for some integer n. By Maschke Theorem $\mathbb{Q}G$ is semisimple and hence, by Wedderburn's Theorem, $\mathbb{Q}G$ decomposes as a direct sum of matrix rings $M_{n_i}(D_i)$ over division rings D_i. If $\mathcal{O}_i$ is an order in D_i for each i, then $\mathcal{O} = \bigoplus_i M_{n_i}(\mathcal{O}_i)$ is an order in $\mathbb{Q}G$. Thus, if $\mathrm{GL}_{n_i}(\mathcal{O}_i)$ has a free pair then so does $\mathcal{U}(\mathbb{Z}G)$. If $n_i > 1$ then it is easy to prove that $\mathrm{GL}_{n_i}(\mathcal{O}_i)$ contains a free subgroup using the following classical result of Sanov.

Theorem 2.1. [48] *If z and w are complex numbers with $|z|, |w| \geq 2$ then $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}$ form a free pair.*

Complex numbers z for which $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$ form a free pair are called free points. For example, the unique integers which are not free points are $0, 1$ and -1 by Sanov Theorem and the equality

$$\left(\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right)^4 = I.$$

See [1] for a list of results on free points.

The above considerations proves Theorem 1.1 for the case when $\mathbb{Q}G$ is not a product of division rings. Otherwise all the subgroups of G are normal. Indeed, if $\mathbb{Q}G$ is a sum of division rings then all the idempotents of $\mathbb{Q}G$ are central. In particular, if H is a subgroup of G then the idempotent $\frac{1}{|H|}\sum_{h\in H} h$ is central in $\mathbb{Q}G$. An easy calculation shows that this implies that H is normal in G as desired.

Thus it remains to show that if G is a Hamiltonian group which is not a 2-group then $\mathcal{U}(\mathbb{Z}G)$ contains a free subgroup. As G is a Hamiltonian but it is not a 2-group then it contains a subgroup isomorphic to $Q_8 \times C_p$ with p and odd prime integer, and we may assume without loss of generality that $G = Q_8 \times C_p$. Then one of the components of $\mathbb{Q}G$ is isomorphic to the Hamiltonian quaternion algebra $\mathbb{H}(\mathbb{Q}(\zeta))$, where ζ is a complex primitive p-th root of unity. Recall that if F is field of characteristic $\neq 2$ then the Hamiltonian quaternion algebra $\mathbb{H}(F)$ has a basis $\{1, i, j, k\}$ over F subject to the following relations: $i^2 = j^2 = -1$, $k = ij = -ji$. As $\mathcal{O} = \mathbb{H}(\mathbb{Z}[\zeta])$ is an order in $\mathbb{H}(\mathbb{Q}(\zeta))$, to prove Theorem 1.1 it only remains to prove that $((1+\zeta i)^n, (1+\zeta j)^n)$ is a free pair of $\mathcal{U}(\mathcal{O})$ for some positive integer n. First of all observe that $1+\zeta i$ and $1+\zeta j$ are units in $\mathcal{O}$ because $(1+\zeta i)(1-\zeta i) = (1+\zeta j)(1-\zeta j) = 1+\zeta^2$ and $1+\zeta^2$ is a unit in $\mathbb{Z}[\zeta]$, as can be seen by putting $X = -1$ in the identity

$$\frac{X^p - 1}{X - 1} = \prod_{i=1}^{p-1} (X - \zeta^i)$$

Consider $\mathbb{H}(\mathbb{Q}(\zeta))$ included in $M_2(\mathbb{C})$ via the map

$$a + bi + cj + dk \mapsto \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$$

Then the image of $u = 1+\zeta i$ and $v = 1+\zeta j$ are

$$S = \begin{pmatrix} 1+\zeta & 0 \\ 0 & 1-\zeta \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & \zeta \\ -\zeta & 1 \end{pmatrix},$$

respectively. The eigenvalues of S are $\lambda_1 = 1+\zeta$ and $\lambda_2 = 1-\zeta$ and the eigenvalues of T are $\mu_1 = 1+\zeta i$ and $\mu_2 = 1-\zeta i$. They satisfy $|\lambda_1| \neq |\lambda_2|$ and $|\mu_1| \neq |\mu_2|$ because the order of ζ is an odd prime p. Moreover the eigenspaces of S are T are all different. Using this it is easy to prove that S and T satisfy the hypothesis of Tits' Criterion (Theorem 3.2) with respect to the standard absolute value on $\mathbb{C}$. Thus (S^n, T^n) is a free pair for n sufficiently large and hence (u^n, v^n) is a free pair for n sufficiently large.

Theorem 1.1 solves Problem 1 for $R = \mathbb{Z}$ and G a finite group. We now deal with the same question for $\mathcal{U}(FG)$ with F a field and G a finite group. As in Theorem 1.1, there are two situations where obviously $\mathcal{U}(FG)$ cannot contain free groups, namely if G is abelian or F is absolute. For a prime integer p let $\mathbb{O}_p(G)$ denote the maximal normal p-subgroup of G. We also denote $\mathbb{O}_0(G) = 1$. The solution of Problem 1 in this case is given by the following

Theorem 2.2. [9] *Let FG be the group ring of the finite group G over the field F of characteristic $p \geq 0$. Then $\mathcal{U}(FG)$ contains a free subgroup if and only if F is not absolute and $G/\mathbb{O}_p(G)$ is not abelian.*

We prove the sufficient part of Theorem 2.2 for the case where $p > 0$. The proof uses the fact that if D is a non-commutative division algebra finite dimensional over its center then $\mathcal{U}(D)$ contains a free subgroup [9]. Assume that F is a non-absolute field of characteristic $p > 0$ and $\mathcal{U}(FG)$ does not contain a free subgroup. We have to prove that $G/\mathbb{O}_p(G)$ is abelian. Let K be the prime subfield of F. As F is not absolute it contains an element t transcendental over K and we may assume without lost of generality that $F = K(t)$. As G is finite, $FG/J(FG)$ is semisimple and units of $FG/J(FG)$ lift to units in FG. Then $FG/J(FG) \cong \oplus_{i=1}^k M_{n_i}(D_i)$

with D_i a division ring for each i, each D_i is finite dimensional over its center and $\mathcal{U}(FG/J(FG))$ does not contain free subgroups. Hence $\mathcal{U}(D_i)$ does not contain a free subgroup and thus D_i is a field containing F. Let

$$S = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad P = \begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix} \quad T = PSP^{-1}.$$

If $n_i > 1$ for some i, then $M_{n_i}(D_i)$ contains a subgroup isomorphic to $\langle S, T\rangle$. Calculating the eigenvalues of S and T we observe that they satisfy the assumptions of Tits's Criterion (Theorem 3.2) with respect to the absolute value $\left|t^n \frac{g}{h}\right| = 2^{-n}$ for $g, h \in K[t]$ with $g(t)h(t) \neq 0$. Then (S^n, T^n) is a free pair in $\mathrm{GL}_{n_i}(D_i)$ contradicting the assumptions. Thus $n_i = 1$ for each $i = 1, \dots, k$. We claim that $\mathbb{O}_p(G)$ is a Sylow subgroup of G. Otherwise G contains an element x such that $x\mathbb{O}_p(G)$ has order p^m for some $m \geq 1$. Since $\mathbb{O}_p(G) = (1 + J(FG)) \cap G$ (see e.g. [9]), we deduce that the natural projection of $x - 1$ on $FG/J(FG)$ is non-zero and nilpotent, contradicting the fact that $FG/J(FG)$ is a product of fields. This proves the claim and we conclude that $F(G/\mathbb{O}_p(G))$ is a semisimple ring and units of $F(G/\mathbb{O}_p(G))$ lift to units in FG. Arguing as above we deduce that $F(G/\mathbb{O}_p(G))$ is a product of fields, so that $G/\mathbb{O}_p(G)$ is abelian, as desired.

The solution of Problem 1 for the case where R is the ring of integers of an algebraic number field and G is finite is given by

Theorem 2.3. [11] *Let R be the ring of integers of an algebraic number field K, and let G be a finite group. Then*

1. *If K is totally real then the following conditions are equivalent:*
    - (a) *$\mathcal{U}(RG)$ does not contain a free subgroup*
    - (b) *$\mathcal{U}(RG)$ is nilpotent*
    - (c) *$\mathcal{U}(RG)$ is FC*
    - (d) *$\mathcal{U}(RG)$ is solvable*
    - (e) *The torsion $T(\mathcal{U}(RG)) = \pm G$*
    - (f) *$T(\mathcal{U}(RG))$ is a subgroup of $\mathcal{U}(RG)$*
    - (g) *G is either abelian or a Hamiltonian 2-group.*
2. *If K is not totally real, then the following conditions are equivalent:*
    - (a) *$\mathcal{U}(RG)$ does not contain a free subgroup*
    - (b) *G is abelian.*

Having obtained the above results on the existence of free subgroups in full group rings, we can ask which conditions on a subgroup U of $\mathcal{U}(\mathbb{Z}G)$ implies the existence of a free subgroup in U. Of course U should not be abelian-by-finite. A similar situation in encountered in [10, Theorem 2.1], where it is proved that if D is a non-commutative division ring finite-dimensional over its center then every non-central subnormal subgroup of $\mathcal{U}(D)$ contains a free subgroup. We can ask whether this holds for integral group rings. Formally

Problem 3. *For which finite groups G, every non-central subnormal subgroup of $\mathcal{U}(\mathbb{Z}G)$ contains a free subgroup?*

To partially answer this question we need to introduce the following concept. We say that a subgroup M of the group U is *almost subnormal* in U if M is subnormal in a subgroup of finite index of U.

Theorem 2.4. [12] *Let G be a finite group. Every almost subnormal subgroup of $\mathcal{U}(\mathbb{Z}G)$ containing G contains a free subgroup, unless G is an abelian or a Hamiltonian 2-group.*

3. Free products in linear groups

A linear group is a subgroup of $\mathrm{GL}_n(F)$ for F a field. If G is a finite group and F is a field then the group algebra FG can be considered as a subalgebra of a matrix algebra $M_n(F)$, with $n = |G|$ via the regular representation and hence $\mathcal{U}(FG)$ is a subgroup of $\mathrm{GL}_n(F)$. In other words $\mathcal{U}(FG)$ is a linear group. More generally if R is a commutative domain then $\mathcal{U}(RG)$ is a linear group.

In this section we review some techniques on linear groups. Along the way we will prove Sanov Theorem (Theorem 2.1) and will give a sketch of the proof of Tits Criterion which have been used in the proof of Theorem 1.1 given in Section 2.

The first important breakthrough on free groups of linear groups appears in the seminal paper [51] of Tits which contains the celebrated *Tits' Alternative*: Every linear group is either solvable-by-locally finite or contains a free subgroup. The proof of Tits' Alternative uses much beautiful and difficult mathematics which are beyond the scope of this paper. However some of the basic ideas will appear here in the form presented in [19]. One of the main tools is a simple but powerful lemma with an obvious and suggestive name.

Lemma 3.1 (Ping-Pong Lemma)**.** *Let G_1 and G_2 be subgroups of a group G with $|G_1| > 2$. Assume that G acts on a set P which contains two different non-empty subsets P_1 and P_2 such that $g(P_i) \subseteq P_j$ if $1 \neq g \in G_i$ and $\{i, j\} = \{1, 2\}$. Then $\langle G_1, G_2\rangle = G_1 * G_2$.*

Here $G_1 * G_2$ stands for the free product of G_1 and G_2, hence $\langle G_1, G_2\rangle = G_1 * G_2$ means that if $g_1, \dots, g_k$ is a list of non-trivial elements belonging to G_1 and G_2 alternatively then $g_1 \cdots g_k \neq 1$. The proof of Lemma 3.1 is almost obvious: Let $g_1, \dots, g_k$ be as above and assume that $g_1 \cdots g_k = 1$. After some conjugation, and using that $|G_1| > 2$ we may assume without loss of generality that the extreme elements g_1 and g_k belong to G_1. Then $P_1 = (g_1 \cdots g_k)(P_1) \subseteq P_2$ and if $1 \neq g \in G_2$ then $P_2 = (gg_1 \cdots g_k g^{-1})(P_2) \subseteq P_1$. Thus $P_1 = P_2$, contradicting the hypothesis.

To illustrate the use of the Ping-Pong Lemma we now prove Theorem 2.1: Consider $\mathrm{GL}_2(\mathbb{C})$ acting on $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$, the compactification of $\mathbb{C}$ with one point, via Möebius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d}.$$

Let $z \in \mathbb{C}$ with $|z| \geq 2$ and let $a(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$. Consider $P_1 = \{x \in \mathbb{C} : |x| \leq 1\}$ and $P_2 = \{x \in \mathbb{C} : |x| \geq 1\}$. Let $g = a(z)^n$ with $n \neq 0$. If $x \in P_1$ then $g \cdot x = x + nz \in P_2$, because $|z| \geq 2$. If $x \in P_2$ then $|nzx| \leq |nzx+1|+1$ and hence $|g^T \cdot x| = \frac{1}{|nz+1|} \leq \frac{1}{|n|zx|-1|} \leq 1$, because $|zx| \geq 2$. Hence, if z and w are complex numbers with $|z|, |w| \geq 2$ then $G_1 = \langle a(z)\rangle$, $G_2 = \langle a(w)^T\rangle$, P_1 and P_2 satisfy the hypothesis of Lemma 3.1 and hence Theorem 2.1 follows.

In the remainder of the section F is a locally compact field, i.e. a field with a nontrivial real absolute value $|\ \ |$ such that F is locally compact relatively to the topology defined by $|\ \ |$. One of the ingredients in the proof of Tits' Alternative is the following theorem, known as *Tits's Criterion.*

Theorem 3.2. [51] *Let F be a locally compact field with respect to a real absolute value $|\ \ |$. Let S and T be diagonalizable endomorphisms of a vector space of dimension 2 over F. Let λ_1 and λ_2 be the eigenvalues of S and μ_1 and μ_2 be the eigenvalues of T. Assume that $|\lambda_1| \neq |\lambda_2|$, $|\mu_1| \neq |\mu_2|$ and the eigenspaces of S and T are all different. Then (S^n, T^n) is a free pair for n sufficiently large.*

We now present some results from [19] which extends Tits' Criterion. Let V be a F-vector space and let $T : V \to V$ be a nonsingular diagonalizable operator. We say that $V = X_+ \oplus X_0 \oplus X_-$ is a T-decomposition of V if there exist real numbers $r > s > 0$ with $X_+ \neq 0$ spanned by the eigenspaces of T corresponding to the eigenvalues of absolute value $\geq r$, $X_- \neq 0$ spanned by the eigenspaces of T corresponding to the eigenvalues of absolute values $\leq s$, and with X_0 the span of the remaining eigenspaces. Observe that $X_+ \neq X_-$ and, in particular $\dim_F(V) \geq 2$. Moreover, T must have infinite multiplicative order for either r or s is different from 1 and hence V has an element v with $g^n(v) \neq v$ for every $n \geq 1$.

Next result extends Tits Criterion (Theorem 3.2). In fact, as explained in [19], it is equivalent to Tits Criterion.

Theorem 3.3. [19] *Let F be a locally compact field. Let V a finite dimensional F-vector space and let $S, T : V \to V$ be two nonsingular operators. Suppose that S and T are both diagonalizable with $V = X_+ \oplus X_0 \oplus X_-$ and $V = Y_+ \oplus Y_0 \oplus Y_-$ being S- and T-decompositions of V, respectively. If the eight intersections $X_\pm \cap (Y_0 \oplus Y_\pm)$ and $Y_\pm \cap (X_0 \oplus X\pm)$ are trivial, then there is a positive integer n_0 such that (S^m, T^n) is a free pair for every $m, n \geq n_0$.*

We briefly sketch the main idea of the proof of Theorem 3.3 (see [19] for details). The action of $\mathrm{GL}_n(V)$ on V induces an action on the projective space $\mathbb{P}(V)$. The absolute value $|\ \ |$ induces a distance between disjoint subsets of $\mathbb{P}(V)$. The image in $\mathbb{P}(V)$ of X_+ is an attractor of S. This means that the sequences $(S^n(v))$, with v an element of $\mathbb{P}(V)$ representing an element of $V \setminus (X_0 + X_-)$, approximates the projective subset represented by X_+. Similarly X_-, Y_+ and Y_- are attractors of S^{-1}, T and T^{-1}, respectively. Using this one can prove that some neighborhoods P_1 of $X_+ \cup X_-$ and P_2 of $Y_+ \cup Y_-$ satisfy the hypothesis of Lemma 3.1 with respect to $G_1 = \langle S \rangle$ and $G_2 = \langle T \rangle$.

Suppose that $T : V \to V$ is a non-trivial transvection, i.e. $T = 1 + \tau$, with $\tau : V \to V$ a linear operator with $\tau^2 = 0 \neq \tau$. We are going to consider free products in which elements T of this form are involved. Tits Criterion does not apply for these kind of elements because they are not diagonalizable. Hence it is convenient to have an alternative to Tits Criterion for these kind of elements. Since $T^n = 1 + n\tau$, we see that T has infinite order if $char(F) = 0$, and it has prime order p if $char(F) = p > 0$. So in the positive characteristic case the resulting free products are not going to be free groups. We have to exclude the case $p = 2$. Otherwise D_∞, the infinite dihedral group can appear, and this group does not contain free subgroups. Another restriction concerning this operators is the following. Let us use $1\mathbb{Z}$ to denote the set of integer multiples of 1 in F, so that $1\mathbb{Z} = \mathbb{Z}$ if $char(F) = 0$, and $1\mathbb{Z} = GF(p)$, the field with p elements, if $char(F) = p > 0$. In the latter case it is clear that $|v| = 1$ for every element $v \in 1\mathbb{Z} \setminus \{0\}$. We abbreviate this writing $|1\mathbb{Z} \setminus 0| = 1$. However, in the former situation a number of possibilities exist. Thus the hypothesis $|1\mathbb{Z} \setminus 0| \geq 1$ comes into play only in the characteristic 0 situation. This hypothesis is really needed, as we can see in [19]. We can now state the analogous of Theorem 3.3 for transvections, in the form of

Theorem 3.4. [19] *Let F be a locally compact field with respect to an absolute value $|\ \ |$. Assume that $|1\mathbb{Z} \setminus 0| \geq 1$, and char$(F) \neq 2$. Let V be a finite-dimensional F-vector space, and let $\sigma, \tau : V \to V$ nonzero linear operators of square zero and $a, b \in F \setminus \{0\}$. Write $I = \sigma(V)$, $K = \ker \sigma$, $J = \tau(V)$ and $L = \ker \tau$. If the intersections $I \cap L$ and $J \cap K$ are both trivial, then $\langle 1+a\sigma, 1+b\tau\rangle = \langle 1+a\sigma\rangle * \langle 1+b\tau\rangle$ for $|a|$ and $|b|$ sufficiently large.*

We consider also the mixed case of a diagonalizable operator and a transvection.

Theorem 3.5. [19] *Let F be a locally compact field with respect to a absolute value $|\ \ |$. Assume that $|1\mathbb{Z} \setminus 0| \geq 1$. Let V be a finite-dimensional F-vector space. Let S be diagonalizable operator of V with an S-decomposition given by $V = X_+ \oplus X_0 \oplus X_-$. Furthermore, let $\tau : V \to V$ be a nonzero operator of square zero and set $I = \tau(V)$, and $K = \ker \tau$. If the four intersections $X_\pm \cap K$ and $I \cap (X_0 \oplus X_\pm)$ are trivial, then $\langle S^n, 1 + a\tau\rangle = \langle S^n\rangle * \langle 1 + a\tau\rangle$ for sufficiently large integers n and all $a \in F$ with $|a|$ sufficiently large.*

Observe that the hypotheses of Theorem 3.3 implies $\dim_F X_+ = \dim_F X_- = \dim_F Y_+ = \dim_F Y_-$ and the hypotheses of Theorem 3.5 implies $\dim_F X_+ = \dim_F X_- = \dim_F I$.

Finally, another situation for which we found an interesting application is

Theorem 3.6. [19] *Let F be a locally compact field with respect to a real absolute value $|\ \ |$. Assume that $|1\mathbb{Z} \setminus 0| \geq 1$. Let V be a finite-dimensional F-vector space, and let G be a nonidentity finite subgroup of the general linear group $\mathrm{GL}(V)$. Assume, in fact, that $|G| \geq 3$ when char$(F) = 2$. Furthermore, let $\tau : V \to V$ be a nonzero linear transformation of square zero, and write $K = \ker \tau$ and $I = \tau(V)$. If $gI \cap K = 0$ for all $1 \neq g \in G$. Then for all $a \in F$ of sufficiently large absolute value, we have $\langle G, T\rangle \cong G * \langle T\rangle$ where $T = 1 + a\tau$.*

We can consider a free group of rank 2 as a free product of two infinite cyclic groups. So, it is natural to ask when $\mathcal{U}(\mathbb{Z}G)$ contain a free product of the form $\mathbb{Z}_p * \mathbb{Z}$, with p prime. In this direction, making use of Theorem 3.6, one can prove the following

Theorem 3.7. [18] *Let G be a finite group. Then $\mathcal{U}(\mathbb{Z}G)$ contains the free product $\mathbb{Z}_p * \mathbb{Z}$ for some prime p, if and only if G has a noncentral element of order p. Moreover, when this occurs, there exists $u \in \mathcal{U}(\mathbb{Z}G)$ and a noncentral element y of G of order p such that $\langle y, u\rangle = \langle y\rangle * \langle u\rangle \cong \mathbb{Z}_p * \mathbb{Z}$.*

It is also possible to extend the above for $p = 2$ and G torsion.

Theorem 3.8. [18] *Let G be a torsion group. Then $\mathcal{U}(\mathbb{Z}G)$ contains the free product $\mathbb{Z}_2 * \mathbb{Z}$ if and only if G has a noncentral element of order 2. Moreover, when this occurs, there exists $u \in \mathcal{U}(\mathbb{Z}G)$ and a noncentral element y of G of order 2 such that $\langle y, u\rangle = \langle y\rangle * \langle u\rangle \cong \mathbb{Z}_2 * \mathbb{Z}$.*

The unit u which appears in Theorem 3.7 and Corollary 3.8 is a kind of “bicyclic” unit, that is, it has the form $u = 1 + \mu$, with $\mu \in \mathbb{Z}G$, $\mu^2 = 0$.

4. Constructing free pairs

The proof of Theorem 1.1 is not constructive. This raises the question of giving specific generic constructions of free pairs in $\mathcal{U}(\mathbb{Z}G)$. This was solved by Marciniak and Sehgal [40] for non-commutative non-Hamiltonian groups and by Ferraz [8] for the remaining cases. To present these free pairs we need to introduce two of the most relevant construction of units.

Let g be an element of G of order n. Then let $\widehat{g} = \sum_{k=0}^{n-1} g^k$, an element of $\mathbb{Z}G$. If k and m are integers such that $k^m \equiv 1 \mod n$, then

$$u_{k,m}(g) = (1+g+g^2+\cdots+g^{k-1})^m + \frac{1-k^m}{n}\widehat{g}$$

is a unit of $\mathbb{Z}G$. Units of this form were introduced by Bass in [2] and are called *Bass cyclic units*, or simply *Bass units*. We also say that $u_{k,m}(g)$ is based on g.

If h is another element of G then form $\mu = (1-g)h\widehat{g}$. It is clear that $\mu^2 = 0$ and hence $1+\mu = 1+(1-g)h\widehat{g}$ is a unit of $\mathbb{Z}G$ with inverse $1-\mu$. Similarly, $1+\widehat{g}h(1-g)$ is a unit of $\mathbb{Z}G$. These units where introduced by Ritter and Sehgal in [46] and are called *bicyclic units*. Moreover $\mu \neq 0$ if, and only if $x \not\in N_G\langle g\rangle$, the normalizer of $\langle g\rangle$ in G.

In a groundbreaking work, Marciniak and Sehgal proved:

Theorem 4.1. [40] *Let $*$ be the natural involution in G. If $\mathbb{Z}G$ contains a nontrivial bicyclic unit u, then (u,u^*) is a free pair of $\mathcal{U}(\mathbb{Z}G)$.*

An easy consequence of this theorem is the following extension of Theorem 1.1.

Corollary 4.2. *If G is a group such that $\mathcal{U}(\mathbb{Z}G)$ does not contain a free subgroup then every finite subgroup of G is normal and the torsion elements of G form a subgroup of G which is either abelian or Hamiltonian 2-group. In particular, if G is a torsion group then $\mathcal{U}(\mathbb{Z}G)$ contains a free subgroup, unless G is either an abelian, or a Hamiltonian 2-group.*

Proof. Assume that $\mathcal{U}(\mathbb{Z}G)$ does not contain a free subgroup. If G has a non-normal finite subgroup then it has a non-normal cyclic subgroup $\langle g\rangle$. If $h \in G \setminus N_G(\langle g\rangle$ then $u = 1+(1-g)h\widehat{g}$ is a non-trivial bicyclic unit. Thus (u,u^*) is a free pair of $\mathcal{U}(\mathbb{Z}G)$, by Theorem 4.1, contradicting the hypothesis. Let g and h be two torsion elements of orders n and m respectively. Then $g^h = g^r$ for some integer r. Therefore $(gh)^m = g^{1+r+\cdots+r^{m-1}}$ and hence gh is torsion. This proves that the torsion elements form a subgroup T of G. Moreover, every subgroup of T is normal and hence T is either abelian or Hamiltonian. In the second case $T \cong Q_8 \times A$ with A abelian and A is a 2-group by Theorem 1.1. □

A similar argument, using now Theorem 2.2, proves the following

Corollary 4.3. *Let FG be the group algebra of the torsion group G over the field F of characteristic 0. Then $U(FG)$ contains a free subgroup, unless G is abelian.*

Problem 1, for $R = \mathbb{Z}$, is still open for non-abelian non-torsion groups. Observe that for torsion-free groups the question is related with the still unsolved Unit Conjecture (sometimes known as Kaplanski Conjecture): If G is a torsion-free group and F is a field then every unit of FG is of the form ug with $u \in \mathcal{U}(F)$ and $g \in G$. If the Unit Conjecture has a positive solution for a torsion-free group G then the $\mathcal{U}(\mathbb{Z}G)$ has a free subgroup if and only if so does G.

Theorem 4.1 gives an answer to the question of producing free pairs for non-Hamiltonian 2-groups. If G is a Hamiltonian non 2-group then it contains $Q_8 \times C_p$ for p an odd prime. Thus the following theorem of Ferraz provides a construction of free pairs for the case not covered by Theorem 4.1. In the following theorem, ϕ stands for Euler totient function.

Theorem 4.4. [8] *Let $G = Q_8 \times \langle g \rangle$ with $Q_8 = \langle x, y \rangle$ the quaternion group of order 8 and g and element of odd prime order p.*

1. *If $p = 3$ then $(u_{5,4}(xg), u_{5,4}(yg))$ is a free pair in $\mathcal{U}(\mathbb{Z}G)$.*
2. *If $p > 3$ then $(u_{3,\phi(4p)}(xg), u_{3,\phi(4p)}(yg))$ is a free pair in $\mathcal{U}(\mathbb{Z}G)$.*

Somehow the proofs of Theorem 4.1 and Theorem 4.4 are parallel to the proof of Theorem 1.1 for the Hamiltonian and non-Hamiltonian cases, respectively. For example, the proof of Theorem 4.4 consists in proving that the natural image in $\mathbb{H}(\mathbb{Z}[\zeta])$ of the given units form a free pair. For that one uses similar arguments than those of Section 2 for the pair $(1+\zeta i, 1+\zeta j)$. For the proof of Theorem 4.1 one construct a group homomorphism $\langle u, u^* \rangle \to \mathrm{GL}_2(\mathbb{C})$ which maps u and u^* to $a(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ and $a(z)^T$ with $|z| \geq 2$. Then Sanov Theorem (Theorem 2.1) applies.

The treatment of [40] raised the following question.

Problem 4. *Let R be a torsion-free ring and let $a, b \in R$ with $a^2 = b^2 = 0$. When does $1+a$ and $1+b$ generate a free subgroup?*

In this direction Salwa has obtained the following result.

Theorem 4.5. [47] *Let R be a torsion free ring. Let $a, b \in R$ such that $a^2 = b^2 = 0$ and ab is not nilpotent. Then $(1+ma, 1+mb)$ is a free pair of $\mathcal{U}(R)$ for every sufficiently large m.*

The non-nilpotence assumption in Theorem 4.5 is needed. For example, if R is a finite dimensional F-algebra with F a subfield of $\mathbb{C}$ and a and b are elements of R with $a^2 = b^2 = 0$ and ab is nilpotent then $\langle 1+a, 1+b \rangle$ is nilpotent [36].

Now we deal with Problem 2 for a group algebra FG of a finite group G. So assume that $\mathcal{U}(FG)$ contains a free subgroup and the goal is to construct a concrete free pair in $\mathcal{U}(FG)$. This is easy in zero characteristic. Indeed, if G is not Hamiltonian then the free pair of Theorem 4.1 belongs to $\mathcal{U}(FG)$. If G is Hamiltonian then it contains Q_8 and hence it is enough to construct a free pair in $\mathcal{U}(\mathbb{Q}Q_8)$. We claim that if $Q_8 = \langle a, b \rangle$ then $(u = 1+2a, v = 1+2b)$ is a free pair in $\mathcal{U}(\mathbb{Q}Q_8)$. Indeed, we have an isomorphism $f = (f_1, f_2) : \mathbb{Q}Q_8 \cong \mathbb{Q}(G/\langle a^2 \rangle) \times \mathbb{H}(\mathbb{Q})$ where the restriction of f_1 to G is the natural projection $Q_8 \to Q_8/\langle a^2 \rangle$ and $f_2(a) = i$ and $f_2(b) = j$. Then $f_1(u)f_1(1-2a) = f_1(v)f_1(1-2b) = -3$ and $f_2(u) = 1+2i$ and $f_2(v) = 1+2j$. Then u and v are units of $\mathbb{Q}Q_8$ and $(f_2(u), f_2(v))$ is a free pair in $\mathbb{H}(\mathbb{Q})$ because it satisfies the hypothesis of the following theorem for $R = \mathbb{Z}$, $a = b = -1$, $\alpha = \beta = 2$ and ν the p-adic valuation on $\mathbb{Q}(i)$ with $p = 1+2i$.

Theorem 4.6. [13] *Let R be an integral domain of characteristic $\neq 2$ and let F be the field of fraction of R. Let $0 \neq a, b, \alpha, \beta \in R$ and suppose that there is a valuation ν on the field $F(\sqrt{a}, \sqrt{b})$, such that $\nu(a) = \nu(b) = \nu(\alpha) = \nu(\beta) = 0$, $\nu(1+\alpha\sqrt{a}) \neq \nu(1-\alpha\sqrt{a})$ and $\nu(1+\beta\sqrt{b}) \neq \nu(1-\beta\sqrt{b})$. Then $(1+\alpha i, 1+\beta j)$ is a free pair in the group of units of the quaternion F-algebra $\left(\frac{a,b}{F}\right) = F[i, j : i^2 = a, j^2 = b, ji = -ij]$.*

If F has positive characteristic then bicyclic units cannot be used to construct free pairs in FG because, in this case, they have order p (or 1). This is also true for Bass units because they have coefficients in the prime field. With this limitations in mind one can ask if some version of Corollary 4.3 holds in positive characteristic. The answer was obtained in

Theorem 4.7. [15] *Let F be a field of characteristic $p > 0$ containing an element t transcendental over its prime field. Let G be a group which has two elements x and y such that x has finite order, y does not normalize $\langle x\rangle$, and the subgroup $\langle x, y^{-1}xy\rangle$ has no p-torsion. If we let*

$$a = (1-x)y\widehat{x}, \quad b = \widehat{x}y^{-1}(1-x^{\delta})$$

where $\delta = (-1)^p$, then

$$\langle 1+ta, 1+tbab, 1+t(1-b)aba(1+b)\rangle \cong \mathbb{Z}_p * \mathbb{Z}_p * \mathbb{Z}_p$$

Theorem 4.7 solves Problem 2 in some cases. Indeed, let x, y and t be as in Theorem 4.7 and let $u_1 = 1+ta$, $u_2 = 1+tbab$ and $u_3 = 1+t(1-b)aba(1+b)$. Then $u = u_1u_2$ and $v = u_2u_3$ is a free pair in $\mathcal{U}(FG)$. Problem 2 is still open for non-absolute fields F of characteristic $p > 0$ and torsion groups G such that $G/\mathbb{O}_p$ is not abelian but G does not contain elements x and y satisfying the hypotheses of Theorem 4.7.

Another way of constructing free pairs in $\mathcal{U}(FG)$ was obtained in [17], as we describe below.

Assume that G is a finite group and F is a nonabsolute field. Let p be a prime integer and $x \in G$ of p-power order. We say that $u_x \in \mathcal{U}(FG)$ is a *special unit* with respect to x, if one of the following three conditions is satisfied:

(1) $char(F) = 0$ and $u_x = (x-r)(x-s)^{-1}$, for suitable integers r and s, with $r, s \geq 2$.
(2) $p \neq char(F) > 0$ and $u_x = (x-t^r)(x-t^s)^{-1}$, for suitable integers r and s and $t \in F$ is transcendental over the prime subfield of F.
(3) $char(F) = p$ and $u_x = 1+t\widehat{x}$, where $t \in F$ is transcendental over the prime subfield of F.

Theorem 4.8. [17] *Let G be a finite group and F a field such that $\mathcal{U}(FG)$ contains a free subgroup. Then G has two elements of prime power order x and y and a free pair (u_x, u_y) form by special units with respect to x and y respectively. Moreover, if* $\mathrm{Char}(F) \neq 0$ *then u_x and u_y are constructed with the same preselected transcendental element.*

5. Free groups and involutions in group algebras

In this section we consider free pairs in $\mathcal{U}(RG)$ formed by elements which are either symmetric or unitary with respect to some involution φ. These pairs are called *free symmetric pairs* and *free unitary pairs* respectively. In the first part of the section $\varphi = *$, the natural involution. In the last part of the section we consider other types of involutions on G extended linearly to an involution in RG.

The usual setting is to consider $char(K) = p \neq 2$, G finite with p not dividing $|G|$, and apply Wedderburn Theorem. But this will lead us to study involutions in full matrix rings over division rings, a difficult task. An alternative is to use a group theoretical approach, guessing the result and proving the theorem by induction. In this case we look for the minimal counterexample and show that, in fact it does not exists, constructing concrete free pairs, either of symmetric or unitary units.

We start recalling some definitions for an arbitrary involution φ in an arbitrary ring R. An element $x \in R$ is said to be *symmetric* (respectively, *unitary*) with respect to φ if $x^\varphi = x$ (respectively, $xx^\varphi = x^\varphi x = 1$). A pair (u,v) of elements of R is *stable under* φ if $\{u,v\}$ is stable under φ (i.e. either u and v are symmetric or $v = u^\varphi$). If R has a free symmetric pair then it is a stable free pair. Conversely, if (u,v) is a stable free pair which is not symmetric then $v = u^\varphi$ and hence then $(uu^\varphi, u^\varphi u)$ is a free symmetric pair.

Theorem 5.1. [14] *Let G be a finite group and F a nonabsolute field of characteristic $p \neq 2$. Assume that $G/\mathbb{O}_p(G)$ is neither abelian nor a Hamiltonian 2-group. Then $\mathcal{U}(FG)$ contains a stable free pair with respect to the natural involution in FG.*

If $\mathcal{U}(FG)$ has a free symmetric pair then the symmetric units of FG do not commute. This only happens in special cases as the following Theorem shows.

Theorem 5.2. [3] *Let G be a torsion group and R a commutative ring of characteristic different from 2. Then the symmetric units of RG commute if and only if G is either abelian or Hamiltonian 2-group.*

Combining Theorems 5.1 and 5.2 we can characterize the the group algebras over finite groups which contain free symmetric pairs.

Corollary 5.3. *Let G be a finite group and F a field of characteristic $p \neq 2$. Then $\mathcal{U}(FG)$ contains a symmetric free pair with respect to the natural involution if and only if F is nonabsolute and $G/\mathbb{O}_p(G)$ is neither abelian nor a Hamiltonian 2-group.*

Proof. Indeed, if F and G satisfy the given conditions then $\mathcal{U}(FG)$ contains a stable free pair (u,v) by Theorem 5.1. Then either (u,v) or (uu^*, u^*u) is a free symmetric pair. Assume otherwise that F and G do not satisfy the hypothesis of Theorem 5.1. We claim that then $\mathcal{U}(FG)$ does not have free symmetric pairs. This is clear if F is an absolute field. Otherwise $\overline{G} = G/\mathbb{O}_p(G)$ is either abelian or a Hamiltonian 2-group and hence the symmetric units of $F\overline{G}$ commute by Theorem 5.2. Thus, if u and v are symmetric units of FG then $uv - vu$ belong to the kernel K of the natural map $FG \to F\overline{G}$. Moreover, K is nilpotent, and hence the elements of $1+K$ form a nilpotent subgroup of $\mathcal{U}(FG)$. Thus, the group generated by the symmetric units is nilpotent-by-abelian and hence it does not contain a free subgroup. □

The existence of free unitary pairs with respect to the natural involution was accomplished in

Theorem 5.4. [16] *Let G be a finite group, F a nonabsolute field of characteristic $p \neq 2$, $P = \mathbb{O}_p(G)$ and $\overline{G} = G/P$. Then the group of unitary units with respect to the natural involution of $\mathcal{U}(FG)$ does not contain a free subgroup if, and only if:*

(1) *$p = 0$ or, $p > 0$ and P is a Sylow p-subgroup of G; and*
(2) *either $\overline{G}$ is abelian, or it has an abelian subgroup $\overline{A}$ of index 2 and, if the latter occurs, then either $\overline{G} = \overline{A} \rtimes \langle \overline{y} \rangle$ is dihedral, or $\overline{A}$ is an elementary abelian 2 group.*

We now sketch the proof of Theorem 5.4. The initial difficulty is to have enough unitary units. If R is a ring with an involution * and if $\alpha \in R$ commutes with α^*, then $\alpha(\alpha^*)^{-1}$ is a unitary unit provided, of course, that α and α^* are invertible. If R is a F-algebra, then it is convenient to introduce a second parameter. More precisely, if α commutes with α^* and $k \in F$, then we have that

$$u_k(\alpha) = (k - \alpha)(k - \alpha^*)^{-1}$$

is a unitary element of $\mathcal{U}(FG)$, provided that $k-\alpha$ (equivalently, $k-\alpha^*$) is invertible in R. Using this, unitary units in FG, with respect to the natural involution, are constructed and then Tits's Criterion (Theorem 3.2) is applied to show that a convenient power of these elements generate a free group. But setting the stage to apply Tits' Criterion is not so easy. We need to begin with a locally compact field F with respect to a real absolute value. This appears in the following lemma, with the real absolute value determined by the non-archimedean valuation ν by $|x| = 2^{-\nu(x)}$.

Lemma 5.5. [51, Lemma 2.2] *Let G be a finite group of order n and let F be either $\mathbb{Q}$ or $K(t)$, the rational function field in one variable over a finite prime field K. In the latter case we also assume that the characteristic of K does not divide n. Then there exists a field extension E of F, containing all n-th roots of unity, such that E is locally compact with respect to the topology induced by a non-archimedean valuation ν. Let $\Sigma = \{x \in E : x^n = 1\}$ and $F' = F(\Sigma)$.*

(1) *If $F = \mathbb{Q}$ and $\epsilon \in \Sigma$ then there exist infinitely many integers k such that $\nu(k - \epsilon) > 0$ and $\nu(k - \delta) = 0$ for every $\delta \in \Sigma \setminus \{\epsilon\}$.*
(2) *If $F = K(t)$ and $0 \neq \epsilon \in F'$ then there exist infinitely many integers k such that $\nu(t^k - \epsilon) > 0$ and $\nu(t^k - \delta) = 0$ for every $\delta \in F' \setminus \{\epsilon\}$.*

The proof of Theorem 5.4 (the necessity part) is then reduced to a finite number of types of groups. In each one we consider a convenient representation of G, apply Tits' Criterion and Lemma 5.5 to construct a free group of unitary units. For example, one important step in it is

Theorem 5.6. [16, Proposition 2.4] *Let F be a nonabsolute field of characteristic $\neq 2$. If G is any of the groups listed below, then the $\mathcal{U}(FG)$ contains a free unitary pair with respect to the natural involution.*

(1) *$G = C_q \rtimes C_r$, with q and r odd primes different,* $\mathrm{Char}(F) \neq q$ *and C_r acts faithfully on C_q.*
(2) *$G = Q_{4r}$, the quaternion group of order $4r$ with r is relatively prime to the characteristic of F.*
(3) *$G = (\langle x \rangle_4 \times \langle w \rangle_4) \rtimes \langle y \rangle_2$ with $x^y = xw$ and $w^y = x^2 w$.*
(4) *$G = A \rtimes C_4$, where A is abelian of odd order relatively prime to* $\mathrm{Char}(F)$ *and C_4 acts in a fixed point free manner on A.*
(5) *G is nonabelian of order relatively prime to the characteristic of F, the center of G is not elementary abelian 2-group, and all irreducible representations of G over the algebraic closure of F have degree at most* 2.

In the remainder of the section φ is an arbitrary involution of the group G extended by linearity to an involution on RG, for R a commutative ring. Again a necessary condition for $\mathcal{U}(RG)$ to admit a free symmetric pair with respect to φ is that symmetric units should not commute and, in particular, RG should have non-commuting symmetric elements. Group rings with non-commuting symmetric elements with respect to an involution in G have been characterized in [35]. We include here the case of characteristic different of 2.

Theorem 5.7. [35] *Let G be a non-abelian group with an involution φ and R a commutative ring of characteristic $\neq 2$. Then the following are equivalent:*

(1) *The symmetric elements of RG with respect to φ commute.*
(2) *$G/Z(G) \cong C_2 \times C_2$ and the involution is given by*

$$g^{\varphi} = \begin{cases} g, & \text{if } g \in Z(G); \\ sg, & \text{if } g \notin Z(G); \end{cases}$$

where $G' = \{1, s\}$.

Recall that a ring R is Von Neumman regular if for every $x \in R$ there is $y \in R$ with $x = xyx$. It is well known that a group algebra FG is Von Neumann regular if and only if G is locally finite and the characteristic of F is either 0 or it is p and G has no elements of order p [4, 52] (see also [41, Theorem 1.5]). A far reaching result on the existence of symmetric units is the following.

Theorem 5.8. [7] *Let FG be a Von Neumann regular group algebra over a non-absolute field F and let φ be an involution of G. Let φ be an involution of G and suppose that either F is uncountable or for every finite subgroup H of G, every division ring in the Wedderburn components of FH is either a field or a quaternion algebra. Then the following conditions are equivalent.*

(1) *The symmetric elements of FG commute (see Theorem 5.7).*
(2) *The symmetric units of FG satisfy a group identity.*
(3) *$\mathcal{U}(FG)$ does not admit a free symmetric pair with respect to φ.*

A more precise result was obtained in [25]. At the beginning of this section we mentioned that we avoid a direct attack to the existence of free symmetric or unitary pairs via Wedderburn Theorem, due to our lack of knowledge on the structure of a division ring with an involution. This gap was filled in [24]. This was used in the proof of the following result.

Theorem 5.9. [25] *Let FG be a non-commutative Von Neumann regular group algebra over a non-absolute field F and let φ be an involution of G. Then FG contains free symmetric pairs and free unitary pairs with respect to φ unless G has an abelian subgroup A of index 2 such that $A^{\varphi} = A$, and one of the following holds:*

(1) *$a^{\varphi} = a$ for all $a \in A$. Then $\mathcal{U}(FG)$ contains free symmetric pairs, but no free unitary pairs.*
(2) *For all $a \in A$ and all $g \in G \setminus A$ we have $a^{\varphi} = a^g$ and $g^{\varphi} = g$. Then $\mathcal{U}(FG)$ contains free symmetric pairs, but no free unitary pairs.*
(3) *There exist central elements $u \neq v$ of order 2 such that $G' = \langle u \rangle$, $\{a^{-1}a^{\varphi} | a \in A\} = \langle v \rangle$, $C = \{(a^{-1})^g a^{\varphi} | a \in A\} = \langle uv \rangle$, and $g^{-1}g^{\varphi} \in C$ for all $g \in G \setminus A$. Then $\mathcal{U}(FG)$ contains free symmetric pairs, but no free unitary pairs.*
(4) *$G' = \langle u \rangle$ is central of order 2, and for all $a \in A$ and all $g \in G \setminus A$ we have $a^{\varphi} = a^g$ and $g^{\varphi} = gu$. Then $\mathcal{U}(FG)$ contains free unitary pairs, but no free symmetric pairs.*

It is also possible to say something in the modular case, as

Theorem 5.10. [25] *Let G be a locally finite group, let φ be an involution of G, and let F be a nonabsolute field of positive characteristic $p \neq 2$. Then the group ring FG contains free symmetric pairs and free unitary pairs with respect to φ, except when for every finite, φ-stable subgroup X of G, the group $G = X/\mathbb{O}_p(X)$ is either abelian or one of the exceptions listed in Theorem 5.9*

6. Free groups and involutions in integral group rings

In this section φ is an involution on the group G and we consider free pairs in $\mathcal{U}(\mathbb{Z}G)$ satisfying same condition with respect to φ. The first natural question is whether Theorem 4.1 still holds true for an arbitrary involution. The answer is negative and this can be easily deduced from Theorem 5.7. Indeed, if $G/Z(G) \cong C_2 \times C_2$ and φ satisfies the conditions of Theorem 5.7 then $\mathcal{U}(\mathbb{Z}G)$ does not admit free symmetric pairs and hence it does not admits stable free pairs either. So (u, u^φ) is not a free pair for any non-trivial unit. In particular, if G is not Hamiltonian (for example, $G = Q_8 \times C_4$) then $\mathbb{Z}G$ has non-trivial bicyclic units u but (u, u^φ) is not a free group for any of them.

Unfortunately, there is no general result as Theorem 4.1 for an arbitrary involution of $\mathbb{Z}G$, even in the case when symmetric elements do not commute as the following example shows.

Example 6.1. [20] *Let p be an odd prime and let P be an extra-special p-group, i.e. $Z(P) = P'$ and it has order p. Let φ be an involution of P such that $x^\varphi x^{-1} \in Z(G)$ for every $x \in G$. Then $\mathbb{Z}P$ has no free bicyclic pairs of the form (u, u^φ).*

The bulk of the proof of the statement of Example 6.1 consist in proving that for all integers i, j either $u^i(u^\varphi)^i$ and $u^j(u^\varphi)^j$, or $(u^\varphi)^i u^i$ and $(u^\varphi)^j u^j$ commute, and so (u, u^φ) is not a free pair.

Thus the question of possible generalizations of Theorem 4.1 should be reformulated in a different form. We propose the following.

Problem 5. *For a non-Hamiltonian group G, classify the involutions φ of G and the bicyclic units u of $\mathbb{Z}G$ such that (u, u^φ) is a free pair.*

Problem 6. *For a non-Hamiltonian group G, classify the involutions φ of G such that $\mathbb{Z}G$ admits a free pair of the form (u, u^φ) for some bicyclic unit u.*

The following result gives an answer to Problem 5 for the finite groups G with $G/Z(G) \cong C_2 \times C_2$ (i.e. the groups appearing in Theorem 5.2.)

Theorem 6.2. [6] *Let φ be an involution on a finite group G such that $G/Z(G) \cong C_2 \times C_2$ and let s be the unique non-trivial commutator of G. Let $x, y \in G$ be such that $u = 1 + (1 - x)y\widehat{x}$ is a non-trivial bicyclic unit of $\mathbb{Z}G$. Put*

$$T = \begin{cases} \langle x^2, x^\varphi x^{-1} \rangle, & \text{if } x^\varphi x^{-1} \in Z(G); \\ \langle x^2, (x^\varphi)^2 \rangle, & \text{otherwise.} \end{cases}$$

Then (u, u^φ) is a free pair in $\mathcal{U}(\mathbb{Z}G)$ if and only if $s \in T$. Otherwise, $\langle u, u^\varphi \rangle$ is a torsion-free abelian group.

A more general situation is considered in [20]. For example we have

Theorem 6.3. [20] *Let G a group with an involution φ and let $x, y \in G$ such that $u = 1 + (1 - x)y\widehat{x}$ is a non-trivial bicyclic unit of $\mathbb{Z}G$. Assume that $\langle x^\varphi \rangle = \langle x \rangle$ and $x^y \notin \langle x \rangle$.*

(1) *If* $y^\varphi xy \in \langle x \rangle$, *then* (u, u^φ) *is a free pair.*
(2) *Assume that* $y^\varphi y \in \langle x \rangle$. *Then* (u, u^φ) *is a free pair, unless* $x^\varphi = x$ *and* $|x| = 3$. *In the exceptional case* $(u^2, (u^\varphi)^2)$ *is a free pair.*

Statement (2) of Theorem 6.3 generalizes Theorem 4.1 because for the natural involution both $x^* = x^{-1}$ and $y^* y = 1$ belong to $\langle x \rangle$ and if $|x| = 3$, then $x^* \neq x$.

In contrast with Example 6.1 we have the following affirmative answer to Problem 6 in a less general setting.

Theorem 6.4. [20] *Let* G *be a finite nonabelian group and* φ *an involution on* G. *If all Sylow subgroups of* G *are abelian, then* $\mathcal{U}(\mathbb{Z}G)$ *contains a free pair* (u, u^φ) *for some bicyclic unit* u.

We briefly sketch the proof of Theorem 6.4. Let σ be the composition of φ and the natural involution. Then σ is automorphism of G such that $\sigma^2 = 1$ and a subgroup G is σ invariant if and only if it is φ-invariant. Observe that the hypothesis about the Sylow subgroups of G is inherited by subgroups and quotients of G. Hence, arguing by induction on the order of G one may assume that every σ-invariant proper subgroup of G is abelian. It can be also proved that one may assume that G/N is abelian for every non-trivial σ-stable normal subgroup of G because if u is a bicyclic unit of G/N then u^n is the image of a bicyclic unit of G for some integer n. Thus we may assume that every σ-invariant proper subgroup of G is abelian and every proper quotient G/N, with N a normal σ-invariant subgroup of G, is abelian. This is a strong restriction on G as Lemma 6.5 below shows. As G cannot be a p-group we conclude that $G = A \rtimes X$ as in case (2) of Lemma 6.5. The proof concludes by proving that some concrete bicyclic unit u satisfy the desired condition. The form of u varies depending on whether $|G|$ is even and A contains a non-trivial element a such that $a^\varphi = a$ and $\langle a \rangle \trianglelefteq G$ or not.

Lemma 6.5. [20] *Let* G *be a finite nonabelian group and* σ *and automorphism of* G *with* $\sigma^2 = 1$. *Assume that every proper* σ*-stable subgroup of* G *is abelian and every quotient* G/N *with* N *a non-trivial* σ*-stable normal subgroup of* G *is abelian. Then one of the following conditions hold.*

(1) G *is a* p*-group for some prime* p, $|G'| = p$ *and* $|G : Z(G)| = p^2$, *or*
(2) G *is the semidirect product* $G = A \rtimes X$, *where* A *is an elementary abelian* q*-group for some prime* q, X *is cyclic of prime order* p, $p \neq q$, *and* $Z(G) = 1$.

Another interesting question that can be asked it the following: *Is there an analogous of Theorem 4.1 for Bass units?* We can see immediately that the answer is no for the natural involution, since in this case, $u = u_{k,m}(x)$ is a polynomial in x and hence u and u^* commute. In this situation we also have the additional complication that not all element x of G produces a noncentral Bass unit of infinite order. This yields the following question.

Problem 7. *Let* φ *be an involution of* G *and* $u = u_{k,m}(x)$ *a Bass unit of infinite order such that* x *and* x^φ *do not commute (in particular* $x^\varphi \notin \langle x \rangle$ *and hence* φ *is not the natural involution). Under which conditions* (u, u^φ) *is a free pair?*

We have the following partial solution which provides stable free pairs of Bass units.

Theorem 6.6. [21] *Let* G *be a finite nonabelian group of order prime to* 6 *and* φ *an involution of* G. *Then, for some prime* p *and appropriate parameters* k *and* m, *we have either:*

(1) *There exist two p-elements x and y such that $(u_{k,m}(x), u_{k,m}(y))$ is a free φ-symmetric pair of Bass units, or*
(2) *There exists a p-element x such that $(u_{k,m}(x), u_{k,m}(x^\varphi))$ is a free pair.*

For symmetric groups we have.

Theorem 6.7. [21] *Let n be an odd integer with $n \geq 5$. Let φ be an involution on the symmetric group Sym_n other than the natural involution. Then Sym_n has a d-cycle x and suitable parameters k and m such that $(u_{k,m}(x), u_{k,m}(x^\varphi))$ is a free pair.*

7. Free pairs formed by bicyclic or Bass units

Bass units and bicyclic units have had a relevant role in the history of units in integral group rings. For example, Bass proved that if G is a cyclic group then the group generated by the Bass units of $\mathbb{Z}G$ has finite index in $\mathcal{U}(\mathbb{Z}G)$ [2]. This result was extended by Bass and Milnor to arbitrary abelian groups (see [50]). For non-abelian groups, Sehgal and Ritter and Jespers and Leal have proved that the group generated by Bass and bicyclic units have finite index in $\mathcal{U}(\mathbb{Z}G)$ for many groups (see [45, 46, 32, 50]). Furthermore Bass units and bicyclic units have been important ingredients of many results in the previous sections. The following questions arise naturally

Problem 8. *Find conditions for two arbitrary Bass units (respectively, two bicyclic units, one bicyclic unit and a Bass unit) to form a free pair.*

Problem 9. *Construct free pairs formed by two bicyclic units, two Bass units or one bicyclic and one Bass unit.*

Of course some minimum requirements for the group should be imposed. For instance, the group should not be abelian and in the versions of Problem 9 where bicyclic unit are involved the group should not be Hamiltonian. With this assumption Theorem 4.1 solves Problem 9 for the case of two bicyclic units of different type (i.e one bicyclic of the form $1 + (1-g)h\widehat{g}$ and the other of the form $1 + \widehat{g}h(1-g)$). For bicyclic units of the same type we have the following example

Example 7.1. [36, 37] *Let $G = D_{2n}$, be the dihedral group of order $2n$. Assume that n is not multiple of 12. Let u and v bicyclic units of $\mathbb{Z}G$ of the same type. Then either u and v commute or form a free pair.*

It is not clear what happens if 12 divides n. In fact the statement of Example 7.1 holds true if 12 divides n if and only if $\sqrt{3}$ is a free point. Apparently this is a hard question. The requirement of only using bicyclic units of the same type is necessary. For example, if $D_6 = \langle a \rangle_3 \rtimes \langle b \rangle_2$ and we take the bicyclic units of different type $u = 1 + (1-b)a(1+b)$ and $v = 1 + (1+ab)a(1-ab)$ then $\langle u, v \rangle$ contains a [5].

Non-trivial Bass units may have finite order. More precisely, consider the Bass unit $u_{k,m}(g)$ and let n be the order of g. If $k \equiv 1 \mod n$ then $u_{k,m}(g) = 1$; if $k \equiv -1 \mod n$ then $u_{k,m}(g) = g^{-m}$; otherwise (i.e. if $k \not\equiv \pm 1 \mod n$) then $u_{k,m}(g)$ has infinite order. Thus, if there is a Bass unit of infinite order based on g then $|g| \neq 1, 2, 3, 4$ and 6. Therefore the version of Problem 9 which involves a Bass unit requires some condition on the order of the group. This partially justify the assumption on the order of G in the following theorem.

Theorem 7.2. *Let G be a finite nonabelian group of order prime to 6. Then*

(1) [19] *$\mathbb{Z}G$ has a free pair formed by Bass units based on elements of prime power order.*
(2) [22] *$\mathbb{Z}G$ has a bicyclic unit u and a Bass unit v such that (u^n, v) is a free pair, for sufficiently large positive integer n.*

The proof of Theorem 7.2 is a real "tour de force", and it is done by induction on the order of the group to reduce to a few cases and for these cases one uses representation theory to be able to apply Theorem 3.3 for statement (1) and Theorem 3.5 for statement (2). . In [42], the interested reader can find a gentle introduction to the proof of (1). We now present an overview of the proof of Theorem 7.2.

Using the induction argument as in the sketch of the proof of Theorem 6.4 one may assume without loss of generality that every proper subgroup and every proper quotient of G is abelian. By [43], it follows that $G = A \rtimes X$, where X is cyclic of prime order p and one of the following condition hold for A:

(1) A is cyclic of prime power order.
(2) A is elementary abelian of order p^2
(3) A is elementary abelian q-group for some prime $q \neq p$ and X acts faithfully and irreducible on A.

Now enters the non-linear complex irreducible representations of the group $G = A \rtimes X$ as above. They are all induced from the $\mathbb{C}$-linear representations of A and, in particular, they have degree p. Our objective is to apply Theorem 3.3 on $S = \rho(u)$ and $T = \rho(v)$ and Theorem 3.5 on $S = \rho(u)$ and $\tau = \rho(w-1)$ for suitable Bass units u and v and a bicyclic unit w and ρ a non-linear irreducible representation of G. If $u = u_{k,m}(g)$, with $|g| = d$, then S is diagonalizable and the eigenvalues of $\rho(u)$ are of the form $u_{k,m}(\varepsilon^i)$ with ε a primitive d-root of unity. Thus we need to have some control of the maximum and minimum values of $u_{k,m}(\epsilon)$. This is accomplished with the following

Lemma 7.3. [42] *Let $p \geq 5$ be a prime, set $d = p^n$, and let ϵ be a primitive complex d-th root of unity. Assume that $k \not\equiv 0, \pm 1 \pmod p$.*

(1) *$|u_{k,m}(\epsilon^a)| = |u_{k,m}(\epsilon^b)|$ if and only if $a \equiv \pm b \pmod d$.*
(2) *$u_{k,m}(\epsilon^a) = u_{k,m}(\epsilon^b)$ if and only if either $a \equiv b \pmod d$, or $a \equiv -b \pmod d$, and d divides ma.*

This lemma tells that if we are restricted to Bass units of $\mathbb{Z}G$, with G as above, based on elements of order p^n, in our case $p \geq 5$, then the maximum r (respectively, the minimum s) of $|u_{k,m}(\epsilon^a)|$ is reached by two values of a modulo p^n. So, with the notation of Theorem 3.3, it is necessary to verify that the eight intersections $X_\pm \cap (Y_0 \oplus Y_\pm)$ and $Y_\pm \cap (X_0 \oplus X_\pm)$ are all trivial. The proof of statement (1) is now complete by checking this for suitable Bass units, which are chosen differently in the three cases (1), (2) and (3). For the proof of statement (2) we should check that the four intersection $K \cap X_\pm = I \cap (X_0 \oplus X_\pm)$ are trivial (now we are using the notation of Theorem 3.5). Unfortunately this does not hold in some cases. This difficulty is surpassed by observing that $W = (X_+ \cap K) + (X_- \cap K)$ is invariant by the endomorphisms S and τ (provided that u is selected in an appropriated way). Thus they induce endomorphisms $\overline{S}$ and $\overline{\mu}$ in $\overline{V} = V/W$. The proof finishes checking the hypothesis of Theorem 3.5 for $\overline{S}$ and $\overline{\mu}$.

Stretching Problem 9 we have the following

Problem 10. *Given a bicyclic (respectively, Bass) unit u in $\mathbb{Z}G$. Construct a bicyclic or Bass cyclic unit v such that (u, v) is a free pair, or at least (u^n, v^n) is a free pair for some n.*

Again we should impose some minimal condition on the unit u. At least u should be non-trivial. This is enough for bicyclic units, by Theorem 4.1, provided that one allows using bicyclic units of different type. However we do not know the answer for the following question.

Problem 11. *Does for every non-trivial bicyclic unit u in $\mathbb{Z}G$ exist a bicyclic unit v of the same type such that (u, v) is a free pair?*

If $u = u_{k,m}(g)$ and $n = |g|$, one should require at least that $k \not\equiv \pm 1 \mod n$, for otherwise u have finite order. However, this is not enough. For example, if G is a dihedral group then some power of u is central in $\mathbb{Z}G$. Thus, we at least should require that u has infinite order modulo the center. In the positive side we have the following result.

Theorem 7.4. [23] *Let G be a solvable group. If u is a Bass unit of $\mathbb{Z}G$ based on an element of prime order then $\mathbb{Z}G$ contains a Bass unit, or a bicyclic unit v, such that (u^n, v^n) is free pair for some integer n.*

We observe that there are finite solvable groups G having a Bass unit u satisfying the hypothesis of Theorem 7.4 which does not admit free pairs (u^n, v^n) with v a Bass unit [23, Example 2.2] and other examples which does not admit free pairs (u^n, v^n) with v a bicyclic unit [23, Example 2.3].

Another type of unit to which we can draw our attention are the *alternating units*. They are defined as follows. For an odd integer consider the polynomial

$$f_c = \sum_{i=0}^{c-1} (-1)^i X^i.$$

For g a group element of finite odd order n and c a positive integer with $(c, n) = 1$, the alternating unit based on g with parameter c is

$$u_c(g) = \begin{cases} f_c(g), & \text{if } c \text{ is odd;} \\ f_{c+n}(g), & \text{if } c \text{ is even.} \end{cases}$$

See [50, Lemma 10.6] for a proof that $u_c(g)$ is a unit in $\mathbb{Z}\langle g \rangle$. Notice $u_{c+2n}(g) = u_c(g)$, so that it is enough to use $1 < c < n$. Moreover, if $n < 5$ then the only existing alternating units are trivial.

We can ask if there is a result analogous to *Theorem* 7.4 for alternating units. In this direction we have

Theorem 7.5. [26] *Let G be a group of odd order with a non-central element $x \in G$. Assume that the order of x is either prime ≥ 5 or of the form 3^l, with $l \geq 2$. Then there exist an alternating unit $u = u_c(x)$ and a unit v, being either a bicyclic or an alternating unit, such that (u^m, v^m) is a free pair for all sufficiently large integers m.*

8. Subgroups of finite index constructed from free subgroups

There is very little information available on the structure of $\mathcal{U}(ZG)$ for G a finite non-abelian group. Basically the only general results known on $\mathcal{U}(\mathbb{Z}G)$ is that it is finitely presented and has a torsion-free subgroup of finite index. Given the ubiquity of free subgroups in $\mathcal{U}(\mathbb{Z}G)$ it is natural to ask whether one can construct large subgroups of $\mathcal{U}(\mathbb{Z}G)$ using free subgroups as building blocks and some natural operations with this groups. A somehow naive question is whether $\mathcal{U}(\mathbb{Z}G)$ might have a free subgroup of finite index. The answer is kind of disappointing:

Theorem 8.1. [30] *Let G be a finite group. Then $\mathcal{U}(\mathbb{Z}G)$ has a (non-abelian) free subgroup of finite index if and only if G is isomorphic to either D_6, D_8, Q_{12} or $C_4 \rtimes C_4$.*

A more satisfactory result is the following theorem. In the remainder of the section free groups include cyclic or even trivial groups.

Theorem 8.2. [31, 38, 35] *The following conditions are equivalent for a finite group G:*

(1) *$\mathcal{U}(\mathbb{Z}G)$ has a subgroup of finite index which is a (finite) direct product of free products of (finitely many finitely generated) abelian groups.*
(2) *$\mathcal{U}(\mathbb{Z}G)$ has a subgroup of finite index which is a (finite) direct product of (finitely generated) free groups.*
(3) *Every non-abelian simple quotient of $\mathbb{Q}G$ is isomorphic to either $M_2(\mathbb{Q})$, $\left(\frac{-1,-3}{\mathbb{Q}}\right)$ or $\mathbb{H}(K)$ with $K = \mathbb{Q}, \mathbb{Q}(\sqrt{2})$ or $\mathbb{Q}(\sqrt{3})$.*
(4) *G is either abelian or isomorphic to $H \times C_2^k$, where H is one of the following groups:*
    - $\langle x, y \mid x^4 = y^4 = (x^2, y) = (x, y^2) = (x, (x, y)) = (y, (x, y)) = 1\rangle$,
    - $\langle x, y_1, \dots, y_n \mid x^4 = y_i^2 = (y_i, y_j) = (x^2, y_i) = ((x, y_i), y_j) = ((x, y_i), x) = 1\rangle$,
    - $\langle x, y_1, \dots, y_n \mid x^4 = y_i^4 = y_i^2(x, y_i) = (y_i, y_j) = (x^2, y_i) = (y_i^2, x) = 1\rangle$,
    - $\langle x, y_1, \dots, y_n \mid x^2 = y_i^2 = (y_i, y_j) = ((x, y_i), y_j) = (x, y_i)^2 = 1\rangle$,
    - $\langle x, y_1, \dots, y_n \mid x^2 = y_i^4 = y_i^2(x, y_i) = (y_i, y_j) = ((x, y_i), x) = 1\rangle$,
    - $\langle x, y_1, \dots, y_n \mid x^4 = y_i^4 = x^2 y_1^2 = y_i^2(x, y_i) = (y_i, y_j) = (y_i^2, x) = 1\rangle$,
    - $\langle x, y_1, \dots, y_n \mid x^4 = x^2 y_i^4 = y_i^2(x, y_i) = (y_i, y_j) = 1\rangle$,
    - *$Z \rtimes \langle x\rangle$ where Z is an elementary abelian 3-group, x has order 2 or 4 and $z^x = z^{-1}$ for every $z \in Z$,*
    - *$Z \rtimes \langle x, y\rangle$ where Z is an elementary abelian 3-group, $Q_8 \cong \langle x, y\rangle$ and $z^x = z^y = z^{-1}$ for every $z \in Z$.*

Finally the following theorem classify the finite groups G such that $\mathcal{U}(\mathbb{Z}G)$ contains a subgroup of finite index which is a direct product of free-by-free groups. A group U if free-by-free if it has a normal free subgroup F such that U/F is free, equivalently U is a semidirect product of a free normal subgroup by a free subgroup. The notation $N : A$ used in the following theorem represents a group G with a normal subgroup N such that $G/N \cong A$. In all the cases N and A is abelian and are represented as direct product of cyclic groups and some additional information is provided to completely describe G. The natural image of an element of $x \in G$ in $G/N \cong A$ is denoted $\overline{x}$.

Theorem 8.3. [33] *For a finite group G the following statements are equivalent.*

(1) *$\mathcal{U}(\mathbb{Z}G)$ has a subgroup of finite index which is a direct product of free-by-free groups.*
(2) *Every simple quotient of $\mathbb{Q}G$ is either a field, a totally definite quaternion algebra or $M_2(K)$, where K is either $\mathbb{Q}$, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$ or $\mathbb{Q}(\sqrt{-3})$.*

(3) *G is either abelian or an epimorphic image of $A \times H$, where A is abelian and one of the following conditions holds:*
  (a) *A has exponent 6 and H is one of the following groups:*
  - $W = \left(\langle t\rangle_2 \times \langle x^2\rangle_2 \times \langle y^2\rangle_2\right) : (\langle \overline{x}\rangle_2 \times \langle \overline{y}\rangle_2)$, *with* $t = (y,x)$ *and* $Z(W) = \langle x^2, y^2, t\rangle$.
  - $W_{1n} = \left(\prod_{i=1}^n \langle t_i\rangle_2 \times \prod_{i=1}^n \langle y_i\rangle_2\right) \rtimes \langle x\rangle_4$, *with* $t_i = (y_i, x)$ *and* $Z(W_{1n}) = \langle t_1, \dots, t_n, x^2\rangle$.
  - $W_{2n} = \left(\prod_{i=1}^n \langle y_i\rangle_4\right) \rtimes \langle x\rangle_4$, *with* $t_i = (y_i, x) = y_i^2$ *and* $Z(W_{2n}) = \langle t_1, \dots, t_n, x^2\rangle$.

  (b) *A has exponent 4 and H is one of the following groups:*
  - $V = \left(\langle t\rangle_2 \times \langle x^2\rangle_4 \times \langle y^2\rangle_4\right) : (\langle \overline{x}\rangle_2 \times \langle \overline{y}\rangle_2)$, *with* $t = (y,x)$ *and* $Z(W) = \langle x^2, y^2, t\rangle$.
  - $V_{1n} = \left(\prod_{i=1}^n \langle t_i\rangle_2 \times \prod_{i=1}^n \langle y_i\rangle_4\right) \rtimes \langle x\rangle_8$, *with* $t_i = (y_i, x)$ *and* $Z(V_{1n}) = \langle t_1, \dots, t_n, y_1^2, \dots, y_n, x^2\rangle$.
  - $V_{2n} = \left(\prod_{i=1}^n \langle y_i\rangle_8\right) \rtimes \langle x\rangle_8$, *with* $t_i = (y_i, x) = y_i^4$ *and* $Z(V_{2n}) = \langle t_i, x^2\rangle$.
  - $\mathcal{U}_1 = \left(\prod_{1\le i<j\le 3} \langle t_{ij}\rangle_2 \times \prod_{k=1}^3 \langle y_k^2\rangle_2\right) : \left(\prod_{k=1}^3 \langle \overline{y_k}\rangle_2\right)$, *with* $t_{ij} = (y_j, y_i)$ *and* $Z(\mathcal{U}_1) = \langle t_{12}, t_{13}, t_{23}, y_1^2, y_2^2, y_3^2\rangle$
  - $\mathcal{U}_2 = \left(\langle t_{23}\rangle_2 \times \langle y_1^2\rangle_2 \times \langle y_2^2\rangle_4 \times \langle y_3^2\rangle_4\right) : \left(\prod_{k=1}^3 \langle \overline{y_k}\rangle_2\right)$, *with* $t_{ij} = (y_j, y_i)$, $y_2^4 = t_{12}$, $y_3^4 = t_{13}$ *and* $Z(\mathcal{U}_2) = \langle t_{12}, t_{13}, t_{23}, y_1^2, y_2^2, y_3^2\rangle$.

  (c) *A has exponent 2 and H is one of the following groups:*
  - $T = (\langle t\rangle_4 \times \langle y\rangle_8) : \langle \overline{x}\rangle_2$, *with* $t = (y,x)$ *and* $x^2 = t^2 = (x,t)$.
  - $T_{1n} = \left(\prod_{i=1}^n \langle t_i\rangle_4 \times \prod_{i=1}^n \langle y_i\rangle_4\right) \rtimes \langle x\rangle_8$, *with* $t_i = (y_i, x)$, $(t_i, x) = t_i^2$ *and* $Z(T_{1n}) = \langle t_1^2, \dots, t_n^2, x^2\rangle$.
  - $T_{2n} = \left(\prod_{i=1}^n \langle y_i\rangle_8\right) \rtimes \langle x\rangle_4$, *with* $t_i = (y_i, x) = y_i^{-2}$ *and* $Z(T_{2n}) = \langle t_1^2, \dots, t_n^2, x^2\rangle$.
  - $T_{3n} = \left(\prod_{i=2}^n \langle y_1^2 t_1\rangle_2 \times \langle y_1\rangle_8 \times \prod_{i=2}^n \langle y_i\rangle_4\right) : \langle \overline{x}\rangle_2$, *with* $t_i = (y_i, x)$, $(t_i, x) = t_i^2$, $x^2 = t_1^2$, $Z(T_{3n}) = \langle t_1^2, y_2^2, \dots, y_n^2, x^2\rangle$ *and, if* $i \ge 2$ *then* $t_i = y_i^2$*).*

  (d) *$H = M \rtimes P = (M \times Q) : \langle \overline{u}\rangle_2$, where M is an elementary abelian 3-group, $P = Q : \langle \overline{u}\rangle_2$, $m^u = m^{-1}$ for every $m \in M$, and one of the following conditions holds:*
  - *A has exponent 4 and $P = C_8$.*
  - *A has exponent 6, $P = W_{1n}$ and $Q = \langle y_1, \dots, y_n, t_1, \dots, t_n, x^2\rangle$.*
  - *A has exponent 2, $P = W_{21}$ and $Q = \langle y_1^2, x\rangle$.*

References

[1] J. Bamberg, *Non-free points for groups generated by a pair of 2×2 matrices*, J. London Math. Soc. (2) 62 (2000) 795–801.

[2] H. Bass, *The Dirichlet unit theorem, induced characters and Whitehead groups of finite groups*, Topology 4 (1966) 391–410.

[3] O. Broche Cristo, *Commutativity of symmetric elements in group rings*, J. Group Theory 9 (2006) 673–683.

[4] I.G. Connell, *On the group rings*, Can. J. Math. 15 (1963) 650–685.

[5] A. Dooms and E. Jespers, *Normal complements of the trivial units in the unit group of some integral group rings*, Comm. Algebra 31 (2003) 475–482.

[6] A. Dooms, E. Jespers and M. Ruiz, *Free groups and subgroups of finite index in the unit group of an integral group ring*, Comm. Algebra, 35 (2007), 2879–2888.

[7] A. Dooms and M. Ruiz, *Symmetric units satisfying a group identity*, J. Algebra 308 (2007) 742–750.

[8] R. Ferraz, *Free subgroups in the units of* $\mathbb{Z}[K_8 \times C_p]$, Comm. Algebra 31 no. 9 (2003) 4291–4299.

[9] J. Z. Gonçalves, *Free subgroup of units in group rings*, Canad. Math. Bulletin, 27, 3 (1984), 309–312.

[10] ———, *Free groups in subnormal subgroups and the residual nilpotence of the group of units of group rings*, Canad. Math. Bulletin, 27, 3 (1984), 365–370.

[11] ———, *Free subgroups in the group of units of group rings II*, J. Number Theory 21, 2 (1985), 121–127.

[12] J.Z. Gonçalves, J. Ritter, S. K. Sehgal, *Subnormal subgroups in* $\mathcal{U}(\mathbb{Z}G)$, Proceedings AMS 103, 2 (1988), 375–382.

[13] J. Z. Gonçalves, A. Mandel and M. Shirvani, *Free products of units in algebras I. Quaternion algebras*, J. Algebra, 214 (1999), 301–316.

[14] ———, *Free products of units in algebras II. Crossed products*, J. Algebra 233 (2000), 567–593.

[15] J. Z. Gonçalves and D. S. Passman, *Construction of free subgroups in the group of units of modular group algebras*, Comm. Algebra 24 (1996), 4211–4215.

[16] ———, *Unitary units in group algebras*, Israel J. Math. 125 (2001), 131–155.

[17] ———, *Free unit groups in group algebras*, J. Algebra, 246 (2001), 226–252.

[18] ———, *Embedding free products in the unit group of an integral group ring.* Archiv der Mathematik 82 (2004) 97—102.

[19] ——— *Linear groups and group rings.* J. Algebra 295 (2006), 94–118.

[20] ———, *Involutions and free pairs of bicyclic units in integral group rings*, J. Group Theory 13 (2010), 721–742.

[21] ———, *Involutions and free pairs of Bass cyclic units in integral group rings*, J. Algebra and Applications, 10 (2011), 711–725.

[22] J. Z. Gonçalves and Á. del Río, *Bicyclic units, Bass cyclic units and free groups*, J. Group Theory 11 (2008), 247–265.

[23] ———, *Bass cyclic units as factors in a free group.* International J. Algebra Computation, 21 (2011) 531–545.

[24] J. Z. Gonçalves and M. Shirvani, *Free symmetric and unitary pairs in central simple algebras with involution*, Group, Rings and Algebras, W. Chin, J. Osterburg and D. Quinn, Contemporary Math. 420 (2006), 121–140.

[25] ———, *Free symmetric and unitary pairs in group algebras with involution*, preprint.

[26] J. Z. Gonçalves and P. M. Veloso, *Alternating units as free factors in the group of units of integral group rings.* Proceedings Edinburgh Math. Soc. (2011) 54, 695–709.

[27] B. Hartley and P.F. Pickel, *Free subgroups in the unit groups of integral group rings*, Canad. J. Math. 32 (1980), no. 6, 1342–1352.

[28] G. Higman, Units in group rings, D. Phil. Thesis, Univ. Oxford, Oxford, 1940.

[29] G. Higman, The units of group-rings, Proc. London. Math. Soc. 46 (1940) 231–248.

[30] E. Jespers, *Free normal complements and the unit group in integral group rings*, Proc. Amer. Math. Soc., 122 (1994) 59–66.

[31] E. Jespers, G. Leal and Á. del Río, *Products of free groups in the unit group of integral group rings*, J. Algebra 180, (1996) 22–40.

[32] E. Jespers and G. Leal, *Generators of large subgroups of the unit group of integral group rings*, Manuscripta Math. 78 no. 3 (1993) 303–315.

[33] E. Jespers, A. Pita, Á. del Río, M. Ruiz and P. Zalesski, *Groups of units of integral group rings commensurable with direct products of free-by-free groups* Adv. Math. 212 nÂ° 2 (2007) 692–722.

[34] E. Jespers and Á. del Río, *A structure theorem for the unit group of the integral group ring of some finite groups*, Journal für die Reine und Angewandte Mathematik 521 (2000) 99–117.

[35] E. Jespers and M. Ruiz, *On symmetric elements and symmetric units in group rings*, Comm. Algebra 34 (2006) 727–736.

[36] E. Jespers, Á. del Río and M. Ruiz, *Groups generated by two bicyclic units in integral group rings*, J. Group Theory 5 (2002), no. 4, 493–511.

[37] V. Jiménez, A trigonometric inequality and its application to the theory of integral group rings, Comm. Algebra, 36 (2008) 63–76.

[38] G. Leal and Á. del Río, *Products of free groups in the unit group of integral group rings II*, J. Algebra 191 (1997) 240–251.

[39] A. I. Lichtman *On the subgroups of the multiplicative group of skewfields.* Proceedings AMS 63 (1977), 15–16.

[40] Z.S. Marciniak and S.K. Sehgal, *Constructing free subgroups of integral group rings*, Proc. AMS 125 (1997) 1005–1009.

[41] D.S. Passman, *The algebraic structure of group rings*, John Wiley & Sons 1977.

[42] ——— *Free subgroups in linear groups and group rings*, Contemporary Math. 456 (2008), 151–164.

[43] G. Miller and H. Moreno, *Non-abelian groups in which every subgroup is abelian*, Trans. Amer. Math. Soc. 4 (1903), 398–404.

[44] D.J.S. Robinson, *A course in the theory of groups*, Graduate Texts in Mathematics, vol. 80, Springer-Verlag, New York, 1982.

[45] J. Ritter and S.K. Sehgal, *Construction of units in integral group rings of finite nilpotent groups*, Trans. Amer. Math. Soc. 324 (1991), no. 2, 603–621.

[46] ——— *Construction of units in integral group rings of finite nilpotent groups*, Bull Amer. Math. Soc. 20 (1989), 165–168.

[47] A. Salwa, *On free subgroups of units of rings*, Proc. AMS 127 (1999) 2569–2572.

[48] I. N. Sanov, *A property of a representation of a free group.* Doklady Akad.Nauk SSSR (NS), 57 (1947), 657–659.

[49] S. K.Sehgal *Topics in group rings*, Marcel Dekker, New York, 1978.

[50] ——— *Units in integral group rings*, Longman Scientific & Technical, Pitman Monographs, Surveys in Pure and Applied Mathematics 69, 1993.

[51] J. Tits, *Free groups in linear groups.* J. Algebra 20 (1972), 250–270.

[52] O. Villamayor, *On weak dimension of algebras*, Pac. J. Math. 9 (1959) 941–951.

Department of Mathematics, University of São Paulo, 05508-090, Brazil
E-mail address: jz.goncalves@usp.br

Departamento de Matemáticas, Universidad de Murcia, Murcia 30100, Spain
E-mail address: adelrio@um.es