Multivariable Optimization. Answers to Problems and Multiple choice questions

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Answers to Problems

Problem 1 Answer

Draw the following sets and say if they are convex, closed and bounded.

- a $A = \{(x,y) \in \mathbb{R}^2 / x^2 + y^2 \leq 4\}$
- b $B = \{(x,y) \in \mathbb{R}^2 / y = 2x + 3\}$
- c $C = \{(x,y) \in \mathbb{R}^2 / (x-1)^2 + (y-3)^2 = 9\}$
- d $D = \{(x,y) \in \mathbb{R}^2 / y > x^2, y \leq 1\}$
- e $E = \{(x,y) \in \mathbb{R}^2 / y \geq x\}$
- f $F = \{(x,y) \in \mathbb{R}^2 / x + y \leq 2, x \geq 0, y \geq 0\}$
- g $G = \{(x,y) \in \mathbb{R}^2 / xy \leq 1\}$
- h $H = \{(x,y) \in \mathbb{R}^2 / xy > 1, x \geq 0, y \geq 0\}$

Problem 2

Investigate the convexity of the following sets

a. $A = \{(x, y) \in \mathbb{R}^2 / 0 \leq x \leq 4, 2 \leq y \leq 6\}$
b. $B = \{(x, y, z) \in \mathbb{R}^3 / x + y + 2z \leq 24\}$
c. $C = \{y \in \mathbb{R}^n / y = \alpha x \quad \text{with } \alpha \in \mathbb{R} \text{ and } \; x \in X \subset \mathbb{R}^n \text{ convex}\}$

Problem 3 Answer

Investigate the concavity/convexity for the following functions

a. $f(x,y) = 3x^3 - 2y^2$
b. $f(x,y) = (x-3)^3 + (y+1)^2$
c. $f(x,y) = (x-2)^2 + y^4$
d. $f(x,y,z) = x^2 + y^2 + z^3$
e. $f(x,y,z) = x^2 + y^2 + z^2 + yz$
f. $f(x,y,z) = e^x + y^2 + z^2$
g. $f(x,y,z) = e^{2x} + y^2 z$
h. $f(x,y) = xy$

Problem 4 ▸ Answer

Check the concavity/convexity of the following functions

a. $f(x, y) = \ln y - e^x$
b. $f(x, y) = \ln xy$ for all $x, y > 0$
c. $f(x, y) = \sqrt{x^2 + y^2}$
d. $f(x, y) = x^{\frac{1}{2}} y^{\frac{1}{3}}$ for all $x, y > 0$

Problem 5 Answer

Check the concavity/convexity of the following functions for the different values of parameter a.

a. $f(x, y) = x^2 - 2axy$
b. $g(x, y, z) = ax^4 + 8y - z^2$

Problem 6 ▸ Answer

Investigate the convexity of the following sets

a. $A = \{(x, y) \in \mathbb{R}^2 / (x-1)^2 + (y-1)^2 \leq 2\}$
b. $B = \{(x, y) \in \mathbb{R}^2 / e^{x+y} \leq 12\}$
c. $C = \{(x, y) \in \mathbb{R}^2 / 3x^2 + 4y^2 \geq 10\}$
d. $D = \{(x, y) \in \mathbb{R}^2 / x + y \leq 2, x \geq 0, y \geq 1\}$
e. $E = \{(x, y) \in \mathbb{R}^2 / x^2 + y^2 - 4x - 2y \leq 3, x \leq 2y\}$
f. $F = \{(x, y) \in \mathbb{R}^2 / x + y \leq 3, 2x + 5y = 10, x \geq 0, y \geq 0\}$

Problem 1

Provide a graphical resolution of the following optimization problems:

a
$$\begin{cases} \max. : 6x + y \\ s.t. : 2x + y \leq 6 \\ \quad x + y \geq 1 \\ \quad y \leq 3 \\ \quad x, y \geq 0 \end{cases}$$

b
$$\begin{cases} opt. : x + y \\ s.t. : x^2 + y^2 = 1 \\ \quad x, y \geq 0 \end{cases}$$

Problem 1

Provide a graphical resolution of the following optimization problems:

c $$\begin{cases} opt.: (x-2)^2+(y-1)^2 \\ s.t.: x^2-y \leq 0 \\ \qquad x+y \leq 2 \\ \qquad x,y \geq 0 \end{cases}$$

d $$\begin{cases} opt.: x-y^2 \\ s.t.: (x-1)(y-2) \geq 0 \\ \qquad 2 \leq x \leq 4 \end{cases}$$

Problem 1

Provide a graphical resolution of the following optimization problems:

e $$\begin{cases} opt.: 3x + 2y \\ s.t.: -x + y \leq 2 \\ \qquad x - y \leq 2 \end{cases}$$

f $$\begin{cases} opt.: (x-2)^2 + (y-2)^2 \\ s.t.: x + y \geq 1 \\ \qquad -x + y \leq 1 \end{cases}$$

Problem 1

Provide a graphical resolution of the following optimization problems:

g $$\begin{cases} \min . : x + y \\ s.t. : x^2 + y^2 \geq 4 \\ \qquad x^2 + y^2 \leq 1 \end{cases}$$

h $$\begin{cases} \max . : x + y \\ s.t. : x - y^2 \geq 0 \\ \qquad x + y \leq 2 \end{cases}$$

Problem 2 ▸ Answer

A firm produces two goods. The profit obtained after the purchase of each are 10 and 15 monetary units respectively. To produce one unit of good 1 requires 4 hours of man-labor and 3 hours of machine work. Each unit of good 2 needs 7 hours of man-labor and 6 hours of machine work. The maximum man-labor time available is 300 hours and for the machines 500 hours. Find the quantities produced of each good which maximize the profit.

Problem 3 Answer

Maximize the utility function $U(x, y) = xy$, where x and y are the quantities consumed of two goods. The price of each unit of these goods is 2 and 1 monetary units respectively and the available budget is 100 monetary units. Formulate the optimization problem the consumer must solve in order to achieve the maximum utility. Calculate the optimal consumed quantities of goods x and y.

Problem 4 ▸ Answer

Which of the following optimization problems satisfy the Weierstrass' theorem conditions?

a) $\begin{cases} \min. : x^2 + y^2 \\ s.t. : x + y = 3 \end{cases}$

b) $\begin{cases} \text{max.} : x + y^2 \\ s.t. : 3x^2 + 5y \leq 4 \\ \qquad x, y \geq 0 \end{cases}$

c) $\begin{cases} opt. : 2x + y \\ s.t. : x + y = 1 \\ \qquad x^2 + y^2 \leq 9 \end{cases}$

d) $\begin{cases} opt. : x + \ln y \\ s.t. : x - 5y^2 \geq -1 \\ \qquad x + y^2 \leq 1 \end{cases}$

e) $\begin{cases} \max. : x^2 + y^2 \\ s.t. : x + y \geq 4 \\ \qquad 2x + y \geq 5 \\ \qquad x, y \geq 0 \end{cases}$

f) $\begin{cases} \min. : e^{x+y} \\ s.t. : 0 \leq x \leq 1 \\ \qquad 0 \leq y \leq 1 \end{cases}$

Problem 1 ▸ Answer

Classify the stationary points of

a. $f(x,y) = 2x^2 + xy + 2y^2 - 4x - y$
b. $f(x,y) = (x^2 + y^2)\mathrm{e}^{x^2-y^2}$
c. $f(x,y) = 2x - 2e^x + 3y - y^3 + 4$
d. $f(x,y) = x\ln(y+1)$
e. $f(x,y,z) = e^{-x^2-y^2-x+z^2}$
f. $f(x,y,z) = x^2 - xy^2 + y^4 - 3yz + z^3$
g. $f(x,y) = x^3 + 3x^2 + y^3 + 6y^2$

Problem 2 Answer

A firm produces an output good using two inputs, denoted by x and y, according to the following production function

$$Q = x^{1/2}y^{1/3}.$$

If $p_1 = 2$, $p_2 = 1$ and $p_3 = 1$ are the prices of output and inputs respectively, maximize the firm's profit.

Problem 3 Answer

The output production function of a firm is

$$Q = x^{1/2}y^{1/3}.$$

where x and y are the units for two different inputs. If p_1, p_2 and p_3 are the prices of output and inputs respectively, and the firm seeks to maximize profits

a. Find the demand of inputs functions.
b. Suppose that p_3 rises while the rest of parameters remain constant; what is the effect upon the demand for input y?
c. If p_1 rises while p_2 and p_3 remain constant; what is the effect upon the demand for x and y?

Problem 4 ▸ Answer

Find the maxima and minima point of the function $f(x, y) = 2x^3 + ay^3 + 6xy$ for different values of parameter $a \in \mathbb{R}$.

Problem 5

Answer

A firm produces three output goods in units x, y and z respectively. If profit is given by

$$B(x,y,z) = -x^2 + 6x - y^2 + 2yz + 4y - 4z^2 + 8z - 14,$$

find the units of each good that maximize profit and find the maximum profit.

Problem 6

Answer

A monopolistic firm produces two goods whose demand functions are

$$p_1 = 12 - x_1, \quad p_2 = 36 - 5x_2$$

where x_1 and x_2 are the quantities of the two goods produced and p_1 and p_2 the prices of a unit of each good. Knowing that the cost function is $C(x_1, x_2) = 2x_1x_2 + 15$, solve the corresponding profit maximizing problem.

Problem 7 ▸ Answer

Solve the output production maximizing problem

$$\max Q(x, y) = -x^3 - 3y^2 + 3x^2 + 24y$$

where x and y are the necessary inputs. Find the maximum production.

Problem 8 Answer

In a competitive market, a firm produces good Q according to the function

$$Q(K, L) = 8K^{1/2}L^{1/4}$$

where K and L are capital and labor respectively. Given the unitary prices of 5 m.u for output and 2 m.u. and 10 m.u. for inputs, find the maximum profit.

Problem 9 Answer

The output production function of a firm and its cost function are given, respectively, by

$$\begin{aligned} Q(x,y) &= 7x^2+7y^2+6xy \\ C(x,y) &= 4x^3+4y^3 \end{aligned}$$

where x and y are the productive inputs. Knowing that the selling price of a unit of good is 3 m.u., find the maximum point for both productive inputs, x and y, and find the maximum profit.

Problem 1 ▸ Answer

Solve the following problems using the substitution method and also the Lagrange multipliers method (the understanding of the problems can be improved using a graphical resolution approach).

a. Min. $f(x, y) = (x-1)^2 + y^2$ subject to $y - 2x = 0$.
b. Max. $f(x, y) = xy$ subject to $2x + 3y = 6$.
c. Opt. $f(x, y) = 2x + 3y$ subject to $xy = 6$.

Problem 2 ▸ Answer

Solve the following problems:

- a) Opt. $f(x,y,z) = x^2 + (y-2)^2 + (z-1)^2$ subject to $4x + y + 4z = 39$.
- b) Opt. $f(x,y) = e^{xy}$ subject to $x^2 + y^2 = 8$.
- c) Opt. $f(x,y) = x^{1/4}y^{1/2}$ subject to $x + 2y = 3$.
- d) Opt. $f(x,y) = \ln(xy)$ subject to $x^2 + y^2 = 8$.

Problem 3

Answer

A multinational refreshments firm has 68 monetary units available to produce the maximum possible number of bottles. Its production function is $q(x, y) = 60x + 90y - 2x^2 - 3y^2$ where x and y are the required inputs. The inputs prices are $p_x = 2$ m.u. and $p_y = 4$ m.u. repectively. Given the budget restriction, maximize the production of bottles. By means of the Lagrange multiplier, how will the maximum number of bottles produced be modified if the budget is increased in one unit (or if it is decreased)?

Problem 4 Answer

A worker earns 20 monetary units for each labor hour. The worker's utility, $U(x,y) = x^{1/3}y^{1/3}$, depends on the consumption of goods, x, and also on the free time, y. Knowing that each unit of consumption costs 80 m.u, and that the worker does not save any of the earned money for the future, find the values of x and y that maximize his utility.

Problem 5 Answer

A European research program has 600 thousand euros available to finance research projects on renewable energies. Two teams present their projects and their estimated incomes (derived from the property rights of new discoveries) are given by $I_1(x) = 2x^{1/2}$ and $I_2(y) = \frac{4}{3}y^{3/4}$ where x is the monetary assignation to the first team (in hundreds of thousands of euros) and y is second team s assignation. The program seeks to determine the optimal distribution of quantities x and y to maximize the joint income. Formulate and solve the problem. What happens to the maximum joint income if the budget is increased by 50 thousand euros? Is it worth it?

Problem 6 Answer

A firm's output production function, $f(K, L) = 4(K+1)^{1/2}(L+1)^{1/2}$, depends on the employed capital and labor. Its costs function is $C(K, L) = 2K + 8L$. Find the optimal values for K and L which minimize the cost of producing 32 units of output. If the production increased by one unit, what would be the effect on the cost?

Problem 7 ▸ Answer

The output of an industry depends on a sole resource whose quantity is limited to b and it is mandatory to use it up. There are two production processes available for which the resource must be distributed. The derived incomes from each one of the productions processes are

$$f(x) = 1200 - \left(\frac{x}{2} - 12\right)^2 \quad g(y) = 1400 - (y-1)^2$$

where x and y are the employed resource in each production process.

a. How can the distribution between x and y be done so as to maximize the total income?
b. Assuming that $b = 22$ and that there is the possibility of using one additional unit of the resource with a cost of 0,8 m.u, Is is worth it? And, is it worth it if $b = 28$?

Problem 8 ▸ Answer

The function $U(x, y) = 100x + xy + 100y$ represents a representative consumer's utility depending on the consumption of two goods, x and y. Knowing that the consumer spends her whole income, 336 monetary units, purchasing these goods at prices $p_x = 8$ m.u. and $p_y = 4$ m.u respectively, maximize the consumer's utility.

Problem 9 ▸ Answer

The costs function of a firm is:

$$C(x, y) = (x-1)^2 + 6y + 8$$

where x and y are the quantities of the two productive inputs needed to produce. If $Q(x, y) = (x-1)^2 + 3y^2$ is the output production function, find the input quantities to produce 12 units of product at the minimum cost.

Problem 1 ▸ Answer

Given the following matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} B = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} C = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$$

confirm that the following properties are true

- a $(A+B)C = AC + BC$.
- b $(AB)^t = B^t A^t$.
- c $(A-B)^2 = A^2 + B^2 - AB - BA$.
- d $(AB + A) = A(B+I)$, where I is the identity matrix of order 2.
- e $(BA + A) = (B+I)A$, where I is the identity matrix of order 2.

Problem 2 ▸ Answer

Calculate AB and BA, where

a) $A = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 0 & -5 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 \\ 8 & 4 \\ -2 & 3 \end{pmatrix}$

b) $A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 2 & 4 \\ 1 & 5 & 3 \end{pmatrix}$

c) $A = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 5 & 7 \end{pmatrix}$

Can we say that the product of matrices has the commutative property?

Problem 3 Answer

Let A and B be square matrices. Prove that the property $(A+B)^2 = A^2 + B^2 + 2AB$ is false.

Problem 4

Answer

Calculate the following determinants:

a) $\begin{vmatrix} 1 & 3 & 2 \\ -1 & 3 & 1 \\ 2 & 0 & -3 \end{vmatrix}$

b) $\begin{vmatrix} 3 & 1 & -1 & 1 \\ -1 & 2 & 0 & 3 \\ 2 & 3 & 4 & 7 \\ -1 & -1 & 2 & -2 \end{vmatrix}$

c) $\begin{vmatrix} 3 & 1 & 2 & -1 \\ -4 & 1 & 0 & 3 \\ 4 & -3 & 0 & -1 \\ -5 & 2 & 0 & -2 \end{vmatrix}$

d) $\begin{vmatrix} 1 & -3 & 5 & -2 \\ 2 & 4 & -1 & 0 \\ 3 & 2 & 1 & -3 \\ -1 & -2 & 3 & 0 \end{vmatrix}$

Problem 5

Without computing the determinants, show that

a $$\begin{vmatrix} 1 & x_1 & x_2 \\ 1 & y_1 & x_2 \\ 1 & y_1 & y_2 \end{vmatrix} = (y_1 - x_1)(y_2 - x_2)$$

b $$\begin{vmatrix} 1 & x_1 & x_2 & x_3 \\ 1 & y_1 & x_2 & x_3 \\ 1 & y_1 & y_2 & x_3 \\ 1 & y_1 & y_2 & y_3 \end{vmatrix} = (y_1 - x_1)(y_2 - x_2)(y_3 - x_3)$$

Problem 6 Answer

Without computing the determinants, find the value of :

a) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1+c \end{vmatrix}$ b) $\begin{vmatrix} 1 & 4 & 4 & 4 \\ 4 & 2 & 4 & 4 \\ 4 & 4 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{vmatrix}$ c) $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ -1 & 0 & 3 & 4 & 5 \\ -1 & -2 & 0 & 4 & 5 \\ -1 & -2 & -3 & 0 & 5 \\ -1 & -2 & -3 & -4 & 0 \end{vmatrix}$

Problem 7

Without computing the determinants, show that

$$\begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} = (a+3)(a-1)^3$$

Problem 8 ▸ Answer

Given the matrices

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 2 & 8 \end{pmatrix}$$

Compute:

a) $|AB|$ b) $\left|(BA)^t\right|$ c) $|2A3B|$ d) $|A+B|$

Problem 9 ▸ Answer

Let A and B be matrices of order n. Knowing that $|A| = 5$ and $|B| = 3$, compute:

a) $\left|BA^t\right|$ b) $|3A|$ c) $\left|(2B)^2\right|$, B of order 3

Problem 10 Answer

Compute $|AB|$, $\left|(BA)^t\right|$, $|2A3B|$, knowing that

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 2 & 8 \end{pmatrix}$$

Problem 11 ▸ Answer

If $A^2 = A$, what values can $|A|$ have?

Problem 12

Let A be a square matrix of order n such that $A^2 = -I$. Prove that $|A| \neq 0$ and that n is an even number.

Problem 13

Let A be a square matrix of order n. Prove that AA^t is a symmetric matrix.

Problem 14 ▸ Answer

Given the following matrices:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \; B = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$
$$C = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$$

solve the matrix equation $3X + 2A = 6B - 4A + 3C$.

Problem 15 ▸ Answer

Suppose that $A.B$ and X are matrices of orden n, solve the following matrix equations:

a. $3(A^t - 2B) + 5X^t = -B.$
b. $(A^t + X)^t - B = 2A.$

Problem 1 Answer

Without computing any principal minor, determine the definiteness of the following quadratic forms:

a. $Q(x, y, z) = (x + 2y)^2 + z^2$
b. $Q(x, y, z) = (x + y + z)^2 - z^2$
c. $Q(x, y, z) = (x - y)^2 + 2y^2 + z^2$
d. $Q(x, y, z) = -(x - y)^2 - (y + z)^2$
e. $Q(x, y, z) = (x - y)^2 + (y - 2z)^2 + (x - 2z)^2$

Problem 2 ▸ Answer

Write the following quadratic forms in matrix form with A symmetric and determine their definiteness.

a. $Q(x, y, z) = -x^2 - y^2 - z^2 + xy + xz + yz$
b. $Q(x, y, z) = y^2 + 2z^2 + 2xz + 4yz$
c. $Q(x, y, z) = 2y^2 + 4z^2 + 2yz$
d. $Q(x, y, z) = -3x^2 - 2y^2 - 3z^2 + 2xz$
e. $Q(x_1, x_2, x_3, x_4) = x_1^2 - 4x_3^2 + 5x_4^2 + 4x_1x_3 + 2x_2x_3 + 2x_2x_4$

Problem 3 Answer

Investigate the definiteness of the following quadratic forms depending on the value of parameter a.

- a $Q(x, y, z) = -5x^2 - 2y^2 + az^2 + 4xy + 2xz + 4yz$
- b $Q(x, y, z) = 2x^2 + ay^2 + z^2 + 2xy + 2xz$
- c $Q(x, y, z) = x^2 + ay^2 + 2z^2 + 2axy + 2xz$

Problem 4

Answer

Find the value of a which makes the quadratic form $\mathrm{Q(x, y, z) = ax^2 + 2y^2 + z^2 + 2xy + 2xz + 2yz}$ be semidefinite. For such a value, determine its definiteness when it is subject to $\mathrm{x} - y - z = 0$?

Problem 5 ▸ Answer

If $A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & -2 \\ -1 & -2 & 3 \end{pmatrix}$ then

a. investigate its definiteness.
b. Write the polynomial and matrix forms of the quadratic form $Q(h_1, h_2, h_3)$ which is associated with matrix *A*.
c. determine its definiteness when it is subject to $h_1 + 2h_2 - h_3 = 0$.

Problem 6 ▸ Answer

Investigate the definiteness of the following matrices. Write the polynomial and matrix form of the associated quadratic forms:

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & -3/2 & 1/2 \\ -3/2 & 1 & -1/2 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 3 \end{pmatrix} \quad D = \begin{pmatrix} -5 & 2 & 1 & 0 \\ 2 & -2 & 0 & 2 \\ 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -5 \end{pmatrix}$$

Problem 7 ▸ Answer

Determine the definiteness of the following constrained quadratic forms.

a. $Q(x,y,z) = (x,y,z)\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ s. t $x + y - 2z = 0$

b. $Q(x,y,z) = (x,y,z)\begin{pmatrix} 2 & -3/2 & 1/2 \\ -3/2 & 1 & -1/2 \\ 1/2 & -1/2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ s. t $x - y = 0$

c. $Q(x,y,z) = (x,y,z)\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ s. t $x + 2y - z = 0$

d. $Q(x_1,x_2,x_3,x_4) = (x_1,x_2,x_3,x_4)\begin{pmatrix} -5 & 2 & 1 & 0 \\ 2 & -2 & 0 & 2 \\ 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ s. t

$2x_1 - 4x_4 = 0$

Problem 8

Answer

Let $Q(x, y, z) = -x^2 - 2y^2 - z^2 + 2xy - 2yz$ be a quadratic form

a. write its matrix form and investigate its definiteness.
b. Investigate its definiteness if it is constrained to $2x - 2y + az = 0$ for the different values parameter a can have.

Problem 9 Answer

Determine the definiteness of the Hessian matrix of the following functions

- a $f(x, y, z) = 2x^2 + y^2 - 2xy + xz - yz + 2x - y + 8$
- b $f(x, y) = x^4 + y^4 + x^2 + y^2 + 2xy$
- c $f(x, y, z) = \ln(x) + \ln(y) + \ln(z)$

Problem 10

Answer

Determine the definiteness of the Hessian matrix of the following production functions when $K, L > 0$.

a) $\mathrm{Q(K, L) = K^{1/2}L^{1/2}}$ b) $\mathrm{Q(K, L) = K^{1/2}L^{2/3}}$

Problem 11 ▸ Answer

The production function $\mathrm{Q(x, y, z){=}ax^2 + 4ay^2 + a^2z^2{-}4axy}$, with $a > 0$, relates the produced quantity of a good to three raw materials (x, y and z) used in the production precess.

a. Determine the definiteness of $Q(x, y, z)$.
b. Knowing that if $x = y = z = 1$ then six units of a good are produced, find the value of parameter a.
c. Using the value of a found in (b), determine the definiteness of $Q(x, y, z)$ when the raw materials x and y are used in the same quantity.

Answers to Multiple Choice Questions

1. Which of the following sets is convex?

   a. $\left\{(x, y) \in \mathbb{R}^2 / x^2 + y^2 \leq 1\right\}$
   b. $\left\{(x, y) \in \mathbb{R}^2 / x^2 + y^2 = 1\right\}$
   c. $\left\{(x, y) \in \mathbb{R}^2 / x^2 + y^2 \geq 1\right\}$

2. The closed line segment between $(1,1)$ and $(-1,-1)$ can be written as the set

   a. $B = \{(x,y) \in \mathbb{R}^2 / (x,y) = (2\lambda - 1, 2\lambda - 1), \forall \lambda \in [0,1]\}$
   b. $B = \{(x,y) \in \mathbb{R}^2 / (x,y) = (\lambda, 1-\lambda), \forall \lambda \in [0,1]\}$
   c. $B = \{(x,y) \in \mathbb{R}^2 / x = y\}$

3. Given $S \subseteq \mathbb{R}^2$ a convex set, the function $f : S \to \mathbb{R}$ will be convex if

   a. the Hessian matrix $Hf(x, y)$ is negative definite for all (x, y) in S
   b. the sets $\{(x, y) \in S / f(x, y) \leq k\}$ are convex for all k in $\mathbb{R}$
   c. f is a lineal function

4. The set $S = \{(x, y, z) \in \mathbb{R}^3 / x + y^2 + z^2 \leq 1\}$
   a. is convex because the Hessian matrix of the function $f(x, y) = x + y + z^2$ is positive semidefinite
   b. is convex because the function $f(x, y) = x + y^2 + z^2$ is lineal
   c. in not convex

5. Which of the following sets is not convex?
   a. $\{(x, y) \in \mathbb{R}^2 / x \leq 1, y \leq 1\}$
   b. $\{(x, y) \in \mathbb{R}^2 / x, y \in [0, 1]\}$
   c. $\{(x, y) \in \mathbb{R}^2 / xy \leq 1,\ x, y \geq 0\}$

6. Which of the following Hessian matrices belongs to a concave function?

   a. $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$
   b. $\begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}$
   c. $\begin{pmatrix} -2 & 2 \\ 2 & 2 \end{pmatrix}$

7. The function $f(x, y) = \ln x + \ln y$ is concave on the set
   a. $S = \{(x, y) \in \mathbb{R}^2 / x, y > 0\}$
   b. $\mathbb{R}^2$
   c. $S = \{(x, y) \in \mathbb{R}^2 / x, y \neq 0\}$

8. Which of the following sets is convex?

   a. $A = \{(x, y) \in \mathbb{R}^2 / xy \geq 1, x \geq 0, y \geq 0\}$
   b. $B = \{(x, y) \in \mathbb{R}^2 / xy \geq 1\}$
   c. $C = \{(x, y) \in \mathbb{R}^2 / xy \leq 1, x \geq 0, y \geq 0\}$

1. Which of the following points belongs to the feasible set of the optimization problem

$$\begin{aligned} opt. &: \quad x^2\sqrt{y} \\ s.t. &: \quad x+y=3 \ ? \end{aligned}$$

   a. $(1,2)$
   b. $(-1,2)$
   c. $(4,-1)$

2. If the feasible set of an optimization problem is unbounded then
   a. no finite optimum point exists
   b. it has an infinite number of feasible points
   c. the existence of a finite optimum point cannot be assured

3. Given $f(x,y) = ax + by$ with $a, b \in \mathbb{R}$ and the set

$$S = \{(x,y) \in \mathbb{R}^2 / x + y = 2, x \geq 0, y \geq 0\},$$

- a. f has a global maximum point and a global minimum point in S
- b. f has a global maximum point in S if a and b are positive
- c. there is no maximum or minimum point of f in S

4. Which of the following is the feasible set of the optimization problem

$$\begin{aligned} \max. \quad &: \quad 2x + y \\ s.t. \quad &: \quad x + y = 1 \\ &: \quad x^2 + y^2 \leq 5 \text{ ?} \end{aligned}$$

a. $\{(x, y) \in \mathbb{R}^2 / x + y = 1, x \geq 0, y \geq 0\}$
b. $\{(x, y) \in \mathbb{R}^2 / (x, y) = \lambda(5, 0) + (1 - \lambda)(0, 5), \forall \lambda \in [0, 1]\}$
c. $\{(x, y) \in \mathbb{R}^2 / (x, y) = \lambda(2, -1) + (1 - \lambda)(-1, 2), \forall \lambda \in [0, 1]\}$

1. The function $f(x, y) = x^2 + y^2$
   a. has no stationary point
   b. has a stationary point at $(0, 0)$
   c. has a stationary point at $(1, 1)$

2. The function $f(x, y, z) = (x-2)^2 + (y-3)^2 + (z-1)^2$ has, at point $(2, 3, 1)$,
   a. a global maximum point
   b. a global minimum point
   c. a saddle point

3. The function $f(x, y) = xy^2(2 - x - y)$ has, at point $(0, 2)$,
   a. a local maximum point
   b. a local minimum point
   c. a saddle point

4. The function $f(x, y) = x^2y + y^2 + 2y$ has
   a. a local maximum point
   b. a local minimum point
   c. a saddle point

5. The function $f(x, y) = \dfrac{\ln(x^3 + 2)}{y^2 + 3}$:
   a. has a stationary point at $(1, 0)$
   b. has a stationary point at $(0, 0)$
   c. has no stationary points

6. If the determinant of the Hessian matrix of $f(x, y)$ on a stationary point is negative, then
   a. the stationary point is a saddle point
   b. the stationary point is a local minimum point
   c. the stationary point is a local maximum

7. If (a, b) is a stationary point of the function $f(x, y)$ such that

$$\frac{\partial^2 f(a,b)}{\partial x^2} = -2 \text{ and } |Hf(a,b)| = 3$$

then

a. (a, b) is a local maximum point
b. (a, b) is a local minimum point
c. (a, b) is a saddle point

8. If $(2,1)$ is a stationary point of the function $f(x,y)$ such that

$$\frac{\partial^2 f(2,1)}{\partial x^2} = 3 \text{ and } |Hf(2,1)| = 1$$

then

a. $(2,1)$ is a local maximum point
b. $(2,1)$ is a local minimum point
c. $(2,1)$ is a saddle point

9. The Hessian matrix of function $f(x, y, z)$ is

$$Hf(x,y,z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

If the function had a stationary point, this would be

a. a local maximum point
b. a global maximum point
c. a global minimum point

10. Let $B(x, y)$ be the profit function of a firm which produces two output goods in quantities x and y. If (a, b) is a stationary point of function $B(x, y)$, for it to be a global maximum point it must occur that

    a. the profit function is concave for all (x, y) in $\mathbb{R}^2$
    b. the profit function is convex for all (x, y) in $\mathbb{R}^2$
    c. the profit function is concave in a neighborhood of the point (a, b)

11. The Hessian matrix of function $f(x,y)$ is given by

$$Hf(x,y) = \begin{pmatrix} x^2+2 & -1 \\ -1 & 1 \end{pmatrix}.$$

If $f(x,y)$ had a stationary point then this point would be

a. a global maximum point
b. a global minimum point
c. a local minimum point that couldn't be global

12. If $(2,1)$ is a stationary point of the function $f(x,y)$, which of the following conditions assures that $(2,1)$ is a global maximum point of the function?

    a. $Hf(2,1)$ is negative definite
    b. $Hf(x,y)$ is negative definite fort all (x,y) in $\mathbb{R}^2$
    c. $Hf(2,1)$ is positive definite

1. The Lagrange function associated with the problem

$$Opt. f(x,y,z) \text{ subject to } g(x,y,z) = c \text{ is}$$

   a. $\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - b)$
   b. $\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda(g(x,y) + b)$
   c. $\mathcal{L}(x,y,\lambda) = g(x,y) - \lambda(f(x,y) - b)$

2. The maximum production of a firm is 500 units of a certain good and the shadow price of the available resource is 3. What would be the effect on the maximum production level if the resource were increased by one unit?
    a. The maximum production level would not be affected
    b. The maximum production level would reduce by 3 units
    c. The maximum production level would increase by 3 units

3. Given the optimization problem

$$\min f(x,y) \text{ subject to } 3x - 6y = 9. \qquad \text{(P)}$$

If $(x, y, \lambda) = (1, -1, 3)$ is a stationary point of the associated Lagrange function, it can be assured that $(1, -1)$ is a global minimum of problem (P) when the function $f(x, y)$ is

a. convex
b. concave
c. neither convex nor concave

4. Given the following optimization problem

$$\min f(x,y) \text{ subject to } x^2 + y = 5. \qquad \text{(P)}$$

Let $(x, y, \lambda) = (1, 4, 3)$ be is a stationary point of the associated Lagrange function $\mathcal{L}(x, y, \lambda)$. Then, if the Hessian matrix of function $\mathcal{L}(x, y, 3)$ is positive semidefinite then $(1, 4)$ is a

a. is a global maximum point of problem (P)
b. is a global minimum point of problem (P)
c. It can't be assured that it is a global extreme point for problem (P)

5. The Hessian matrix of the Lagrange function $\mathcal{L}(x, y, z, \lambda^*)$ is given by

$$H(\mathcal{L}(x, y, z, \lambda^*)) = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & -4 \end{pmatrix},$$

then, a stationary point is a

a. global maximum
b. global minimum
c. neither of the above

6. Given the optimization problem

$$Opt. f(x,y,z) \text{ subject to } g(x,y,z) = c$$

and given $(1,2,3,4)$, a stationary point of the Lagrange function ($\lambda = 4$ is the Lagrange multiplier) if the Hessian matrix of $\mathcal{L}(x,y,z,4)$ is

$$H\mathcal{L}(x,y,z,4) = \begin{pmatrix} x^2+1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4y^2+3 \end{pmatrix}.$$

then

- a. the problem has no solution
- b. point $(1,2,3)$ is a global minimum point
- c. $(1,2,3)$ is a global maximum point

7. Given the optimization problem

$$Opt. f(x, y, z) \text{ subject to } g(x, y, z) = c,$$

it is known that the Hessian matrix of the Lagrange function when $\lambda = 4$ is given by

$$H\mathcal{L}(x, y, z, 4) = \begin{pmatrix} -x^2 - 7 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

Then, if $(1, 2, 3, 4)$ is a stationary point of the Lagrange function,

a. the problem has no solution
b. the problem has a global maximum point
c. the problem has a global minimum point

8. Given the optimization problem

$$Opt. f(x,y) \text{ subject to } g(x,y) = c,$$

it is known that the Hessian matrix of the Lagrange function when $\lambda = 4$ is given by

$$H\mathcal{L}(x,y,4) = \begin{pmatrix} -x^2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Then, if $(1,2,4)$ is a stationary point of the Lagrange function,

a. the minimum value of the objective function is 4
b. $(1,2)$ is a global minimum point
c. $(1,2)$ is a global maximum point

9. In the maximization of profits with a linear constraint on costs $x + y + z = 89$, the Lagrange multiplier is $-0,2$. Is it worth increasing the level of cost?
   a. No. The maximum profit would decrease
   b. Yes, since the maximum profit would increase
   c. Yes, because we would continue with positive profits

1. The matrix $A = \begin{pmatrix} 4 & 1 & -3 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \end{pmatrix}$ is
   - a. a lower triangular matrix
   - b. an upper triangular matrix
   - c. a diagonal matrix

2. Which of the following is the true definition of a symmetric matrix?
   a. A square matrix A is said to be symmetric if $A = -A$
   b. A square matrix A is said to be symmetric if $A = -A^t$
   c. A square matrix A is said to be symmetric if $A = A^t$

3. Given A and B matrices of order $m \times n$ and $n \times p$, $(AB)^t$ equals to:
   a. $B^t A^t$
   b. $A^t B^t$
   c. AB

4. Let A be a matrix such that $A^2 = A$ then, if $B = A - I$, then:
   a. $B^2 = B$
   b. $B^2 = I$
   c. $B^2 = -B$

5. Let A and B be square matrices of order 3. If $|A| = 3|$ and $|B| = -1$ then:

   a. $|2A \,.\, 4B| = (-4)2^3$
   b. $|2A \,.\, 4B| = (-3)2^3$
   c. $|2A \,.\, 4B| = (-3)2^9$

6. Which of the following properties is NOT always true?
   a. $\left|A^2\right| = |A|^2$
   b. $|A + B| = |A| + |B|$
   c. $|A^t B| = |A|\,|B|$

7. Given the 3 by 3 matrices A, B, C such that $|A| = 2$, $|B| = 4$ and $|C| = 3$, compute $\left|\frac{1}{|A|} B^t C^{-1}\right|$:

   a. $\frac{2}{3}$
   b. $\frac{1}{6}$
   c. 6

8. Let A and B be matrices of the same order, which of the following properties is always true?
   a. $(A-B)(A+B)=A^2-B^2$
   b. $(A-B)^2=A^2-2AB+B^2$
   c. $A(A+B)=A^2+AB$

9. Which of the following properties is NOT true?
   a. $|A^2| = |A|^2$
   b. $|-A| = |A|$
   c. $|A^t| = |A|$

10. Let A and B be symmetric matrices, then which of the following is also a symmetric matrix:
    - a. BA
    - b. $A + B$
    - c. AB

11. Let A be an n by n real matrix. Then, if $k \in \mathbb{R}$ one has that

    a. $|kA| = k\,|A|$
    b. $|kA| = |k|\,|A|$, being $|k|$ the absolute value of the real number k
    c. $|kA| = k^n\,|A|$

12. Given the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} B = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} yC = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$$

the solution to the matrix equation
$3X + 2A = 6B - 4A + 3C$ is

a. $X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
b. $X = \begin{pmatrix} -4 & 1 \\ -4 & 0 \end{pmatrix}$
c. $X = \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}$

13. Let A, B be matrices of order n. Then:
    a. $(AB)^2 = A^2B^2$
    b. $(AB)^2 = B^2A^2$
    c. $(AB)^2 = A(BA)B$

14. Given the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$, the adjoint element a_{12} is:

a. -1
b. 1
c. -2

1. Which of the following is a quadratic form?
   a. $Q(x, y, z) = x^2 + 3z^2 + 6xy + 2z$
   b. $Q(x, y, z) = 2xy^2 + 3z^2 + 6xy$
   c. $Q(x, y, z) = 3xy + 3xz + 6yz$

2. Let $Q(x, y, z)$ be a quadratic form such that $Q(1, 1, 0) = 2$ and $Q(5, 0, 0) = 0$, then
   a. $Q(x, y, z)$ could be indefinite
   b. $Q(x, y, z)$ is positive definite
   c. $Q(x, y, z)$ could be negative semidefinite

3. The quadratic form $Q(x, y, z) = (x - y)^2 + 3z^2$ is
   a. positive definite
   b. positive semidefinite
   c. indefinite

4. The quadratic form $Q(x, y, z) = -y^2 - 2z^2$ is
   a. negative definite
   b. negative semidefinite
   c. indefinite

5. The quadratic form $Q(x, y, z) = x^2 + 2xz + 2y^2 - z^2$ is
   a. positive definite
   b. positive semidefinite
   c. indefinite

6. A 2 by 2 matrix has a negative determinant, then the matrix is
   a. negative definite
   b. negative semidefinite
   c. indefinite

7. The matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ is

   a. indefinite
   b. positive semidefinite
   c. positive definite

8. The leading principal minors of a 4 by 4 matrix are $|\mathrm{A}_1| = -1$, $|\mathrm{A}_2| = 1$, $|\mathrm{A}_3| = 2$, and $|\mathrm{A}_4| = |A| = 0$.Then,
   a. the matrix is negative semidefinite
   b. its definiteness cannot be determined with this information
   c. the matrix is indefinite

9. The leading principal minors of a 4 by 4 matrix are $|\mathrm{A}_1| = -1$, $|\mathrm{A}_2| = 1$, $|\mathrm{A}_3| = -2$ and $|\mathrm{A}_4| = |A| = 0$. Then,

   a. the matrix is negative semidefinite
   b. its definiteness cannot be determined from this information
   c. the matrix is indefinite

10. The leading principal minors of a 4 by 4 matrix are $|A_1| = -1, |A_2| = 1$, $|A_3| = -2$, and $|A_4| = |A| = 1$. Then, the matrix is

    a. negative definite
    b. indefinite
    c. positive definite and negative definite

11. The quadratic form in three variables $Q(x, y, z)$, subject to $x + 2y - z = 0$, is positive semidefinite. Then, the unconstrained quadratic form is:
    a. positive semidefinite or indefinite
    b. positive semidefinite
    c. positive definite or positive semidefinite

12. If $Q(x, y, z)$ is a negative semidefinite quadratic form such that $Q(-1, 1, 1) = 0$, then $Q(x, y, z)$ subject to the constraint $x + 2y - z = 0$
   a. is negative semidefinite
   b. cannot be classified with this information
   c. is negative definite or negative semidefinite

13. The quadratic form $Q(x, y, z) = (x - y)^2 + 3z^2$ subject to the constraint $x = 0$ is:

    a. indefinite
    b. positive semidefinite
    c. positive definite

14. The quadratic form $Q(x, y, z) = (x - y)^2 + 3z^2$ subject to the constraint $z = 0$ is:

   a. indefinite
   b. positive semidefinite
   c. positive definite

Answers to Problems

Return to Contents

Problem 1 ▸ Return

a. Convex, closed and bounded
b. Convex and closed
c. Closed and bounded
d. Convex and bounded
e. Convex and closed
f. Convex, closed and bounded
g. Closed
h. Convex

Problem 3

Return

- a. Concave on the convex set of $\mathbb{R}^2$: $\{(x, y) \in \mathbb{R}^2 / x \leq 0\}$
- b. Convex on the convex set of $\mathbb{R}^2$: $\{(x, y) \in \mathbb{R}^2 / x \geq 3\}$
- c. Convex
- d. Convex if $z \geq 0$
- e. Convex
- f. Convex
- g. Neither concave nor convex
- h. Neither concave nor convex

Problem 4 Return

a. Concave
b. Concave
c. Convex
d. Concave

Problem 5 ▸ Return

- (a) It is convex if $a = 0$
- (b) It is concave if $a < 0$

Problem 6 Return

All of them are convex except for C.

Problem 2 Return

75 units of good 1 and none unit of good 2.

Problem 3 ▸ Return

$$x^* = 25, y^* = 50 \Rightarrow u^* = 1250$$

Problem 4 Return

The problems that verify the hypothesis of the Extreme Value Theorem are b), c) and f).

Problem 1 Return

a. $(1, 0)$ is a local minimum point.
b. $(0, 0)$ is a local minimum point and $(0, 1)$ and $(0, -1)$ are saddle points.
c. $(0, 1)$ is a local maximum point and $(0, -1)$ is a saddle point.
d. $(0, 0)$ is a saddle point.
e. $(-1/2, 0, 0)$ is a saddle point.
f. $(1/2, 1, 1)$ is a local minimum and $(0, 0, 0)$ is a saddle point.
g. $(-2, -4)$ is a local maximum.

Problem 2 ▸ Return

The maximum point is $x = 4/9$ and $y = 8/27$.

Problem 3

Return

- (a) The maximum point is $x = \left(\frac{p_1^3}{12p_2^2p_3}\right)^2$ and $y = \left(\frac{p_1^2}{6p_2p_3}\right)^3$.
- (b) If the price of y rises with other parameters remaining constant, the quantity demanded of input y will decrease in order to maximize profits. By contrast, if the selling price of output rises, the quantity demanded of input y will increase.

Problem 4 Return

$(0,0)$ is a saddle point for all of the values of parameter a.
$\left(-\sqrt[3]{\frac{2}{a}}, -\sqrt[3]{\frac{4}{a^2}}\right)$ is a local minimum point if $a < 0$ and, a local maximum point if $a > 0$.

Problem 5

Return

The maximum point is $x = 3, \; y = 4, \; z = 2$ and the maximum profit is $B_{\max} = 11$ m.u.

Problem 6

Return

The maximum point is $x_1 = x_2 = 3$, whose prices are, respectively, $p_1 = 9,\ p_2 = 21$.

Problem 7 ▸ Return

The maximum produced quantity is $Q_{\max}(2,4) = 52$ units.

Problem 8 Return

The maximum profit is $B_{\max}(1.000, 100) = 1.000$ m.u.

Problem 9 ▸ Return

The maximum point is $x = y = 5$ and the maximum profit $B_{\max}(5,5) = 500$ m.u.

Problem 1 ▸ Return

- (a) The problem has a global minimum at $x^* = 1/5$ and $y^* = 2/5$ whose value is $20/25$.
- (b) $(3/2, 1)$ is a global maximum of value $3/2$.
- (c) $(3, 2)$ is a local minimum point of value 12 and $(-3, -2)$ is a local maximum point of value -12.

Problem 2 Return

a. Global minimum at $x = 4$, $y = 3$ and $z = 5$ of value 33.
b. $(2, 2)$ and $(-2, -2)$ are two global maximum points of value e^4. $(2, -2)$ and $(-2, 2)$ are two global minimum of value e^{-4}.
c. Global maximum at $x = y = 1$ of value 1.
d. Two local maximum points at $(2, 2)$ and $(-2, -2)$ of value $\ln 4$.

Problem 3 ▸ Return

The global maximum point is at $x = 12$ and $y = 11$ with a maximum quantity of 1059 bottles.
If the budget is increased by 1 monetary unit, the maximum production would increase by 6 units (approximately). Similarly, if the budget is reduced, the production would reduce by 6 units (approximately)

Problem 4 ▸ Return

$x = 3$ and $y = 12$.

Problem 5 ▸ Return

The global maximum is obtained when the first project is assigned 200 thousand euro and the second project with 400 thousand euro. The maximum income will be of 659.970 euro.
An increase of 50 thousand euro will increase the maximum income by 35.355 approximately. It is not worth it.

Problem 6 Return

$K = 15$ and $L = 3$. An increase of one unit in production would increase the minimum cost by 2 units (approximately)

Problem 7 ▸ Return

a. $x = \frac{4b}{5} + 4$ and $y = \frac{b}{5} - 4$.

b. It is worth it in the first case but not in the second.

Problem 8 ▸ Return

$y = 84$ and $x = 0$.

Problem 9 Return

$(1+2\sqrt{3}, 0)$ and $(1, 2)$ are two minimum points of value 20..

Problem 1 Return

a $\begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$

b $2\begin{pmatrix} -1/2 & -1/2 \\ 1 & 1 \end{pmatrix}$

c $\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$

d $3\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

e $\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$

Problem 2 ▸ Return

a. $AB = \begin{pmatrix} 16 & 10 \\ 13 & -15 \end{pmatrix}$ and $BA = \begin{pmatrix} 12 & 3 & 6 \\ 36 & 8 & -4 \\ -5 & -2 & -19 \end{pmatrix}$

b. $AB = \begin{pmatrix} -2 & 1 & 7 \\ 5 & 11 & 5 \\ 3 & 10 & 8 \end{pmatrix}$ and $BA = \begin{pmatrix} 5 & 5 & 4 \\ 8 & 4 & 6 \\ 12 & 5 & 8 \end{pmatrix}$

c. $AB = 3\begin{pmatrix} 2 & 3 \end{pmatrix}$ and BA is not possible.

Problem 3 Return

It is false because the multiplication of matrices does not verify the conmutative property. To probe its falsity the students must provide a counterexample.

Problem 4 ▸ Return

(a)-24 (b)-58 (c)-80 (d)35

Problem 6 ▸ Return

(a) abc (b) -24 (c) $5!$

Problem 8 Return

(a)16 (b) 16 (c) $2^8 3^4$ (d) 130.

Problem 9 ▸ Return

(a) 15 (b) $3^n \cdot 5$ (c) $2^6 \cdot 3^2$

Problem 10 Return

(a)$3^n \cdot 5$ (b) 5^{n-1} (c) 15

Problem 11 Return

$|A| = 0$ or $|A| = 1$.

Problem 14 Return

$$X = \begin{pmatrix} -4 & 1 \\ -4 & 0 \end{pmatrix}$$

Problem 15 ▸ Return

(a) $X = B^t - \frac{3}{5}A$ (b) $X = A^t + B^t$.

Problem 1 ▸ Return

a. positive semidefinite.
b. indefinite.
c. positive definite.
d. negative semidefinite.
e. positive semidefinite.

Problem 2 ▸ Return

(a) $Q(x,y,z) = (x,y,z)\begin{pmatrix} -1 & 1/2 & 1/2 \\ 1/2 & -1 & 1/2 \\ 1/2 & 1/2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ negative semidefinite.

(b) $Q(x,y,z) = (x,y,z)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ indefinite.

(c) $Q(x,y,z) = (x,y,z)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ positive semidefinite.

(d) $Q(x,y,z) = (x,y,z)\begin{pmatrix} -3 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ negative definite.

(e) $Q(x_1,x_2,x_3,x_4) = (x_1,x_2,x_3,x_4)\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & -4 & 0 \\ 0 & 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ indefinite.

Problem 3 ▸ Return

(a) negative definite for $a < -5$, negative semidefinite for $a = -5$ and indefinite when $a > -5$.

(b) positive definite if $a < 1$, positive semidefinite if $a = 1$ and indefinite when $a < 1$.

(c) Indefinite when $a < 0$ or $a > 1/2$, positive semidefinite for $a = 0$ or $a = 1/2$ and positive definite.if $0 < a < 1/2$.

Problem 4 Return

$a = 1$. The constrained quadratic form is positive definite.

Problem 5 Return

(a) indefinite.

(b) $Q(h_1, h_2, h_3) = (h_1, h_2, h_3) \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & -2 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} =$

$h_2^2 + 3h_3^2 - 2h_1h_3 - 4h_2h_3$

(c) positive definite.

Problem 6 Return

(a) negative semidefinite,
$Q(x,y,z) = -2x^2 - 2y^2 - 2z^2 + 2xy + 2xz + 2yz.$
(b) indefinite, $Q(x,y,z) = 2x^2 + y^2 - 3xy + xz - yz.$
(c) positive semidefinite,
$Q(x,y,z) = x^2 + y^2 + 3z^2 - 2xy + 2xz - 2yz.$
(d) negative definite, $Q(x_1,x_2,x_3,x_4) =$
$-5x_1^2 - 2x_2^2 - x_3^2 - 5x_4^2 + 4x_1x_2 + 2x_1x_3 + 4x_2x_4.$

Problem 7 Return

(a) negative semidefinite. (b) The constrained quadratic form is null.

(c) positive definite. (d) negative definite (since the unconstrained quadratic form is negative definite).

Problem 8 Return

(a) $Q(x, y, z) = (x, y, z) \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ negative semidefinite.

(b) negative semidefinite if $a = 0$, negative definite if $a \neq 0$.

Problem 9 Return

(a) $Hf(x,y,z)$ is indefinite$\forall(\mathrm{x,y,z}) \in \mathbb{R}^3$.
(b) $Hf(x,y)$ is positive definite$\forall(\mathrm{x,y}) \in \mathbb{R}^2 - \{(0,0)\}$. H f(0,0) is positive semidefinite.
(c) $Hf(x,y,z)$ is negative definite$\forall(\mathrm{x,y,z}) \in \mathrm{Dom}(f) = \{(\mathrm{x,y,z}) \in \mathbb{R}^3/\mathrm{x} > 0, \mathrm{y} > 0, \mathrm{z} > 0\}$.

Problem 10 ▸ Return

(a) negative semidefinite. (b) indefinite.

Problem 11 Return

(a) Positive Semidefinite.

(b) $a = 2$.

(c) Positive Definite.