Mathematics for Business Administration: Matrix Algebra and Quadratic forms

María Pilar Martínez-García

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Outline

Matrix Algebra

Back

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- Matrices and Matrix Operations
- Determinants

Back

Definition of matrix

A matrix is simply a rectangular array of numbers considered as an entity. When there are m rows and n columns in the array, we have an m-by-n matrix (written as $m \times n$). In general, an $m \times n$ matrix is of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

or, equivalently

$$A = (a_{ij})_{m \times n}.$$

Definition 1

A matrix with only one row is also called a row vector, and a matrix with only one column is called a column vector. We refer to both types as vectors.

Definition 2

If $m = n$, then the matrix has the same number of columns as rows and it is called a square matrix of order n and is denoted by $A = (a_{ij})_n$.

Definition 3

If $A = (a_{ij})_n$ is a square matrix, then the elements a_{11}, a_{22}, a_{33}, ..., a_{nn} constitute the main diagonal.

Definition 4

The identity matrix of order n, denoted by $\mathbf{I}_n$ (or by $\mathbf{I}$), is the $n \times n$ matrix having ones along the main diagonal and zeros elsewhere:

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Addition of Matrices
$A + B = (a_{ij})_{m\times n} + (b_{ij})_{m\times n} = (a_{ij} + b_{ij})_{m\times n}$
Rules
$(A + B) + C = A + (B + C)$ $A + B = B + A$ $A + 0 = A$

Multiplication by a real number
$\alpha A = \alpha \left(a_{ij}\right)_{m\times n} = \left(\alpha a_{ij}\right)_{m\times n}$
Rules
$(\alpha+\beta)A = \alpha A + \beta A$ $\alpha(A+B) = \alpha A + \beta B$ $A + (-A) = 0$

Matrix Multiplication
$A \cdot B = (a_{ij})_{\mathbf{m} \times n} \cdot (b_{ij})_{n \times \mathbf{p}} = (c_{ij})_{\mathbf{m} \times \mathbf{p}}$ such that $c_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$
Rules
$(AB)C = A(BC)$ $A(B + C) = AB + AC$ $(A + B)C = AC + BC$

The transpose

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \Rightarrow A' = A^t = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

Symmetric matrix

Definition 5

Square matrices with the property that they are symmetric with respect the main diagonal are called symmetric.

The matrix $A = (a_{ij})_n$ is symmetric $\Leftrightarrow a_{ij} = a_{ji}\ \forall i, j = 1, 2, ..., n$.
A matrix A is symmetric $\Leftrightarrow A = A'$.

Determinants

Definition 6

Let A be an $n \times n$ matrix. Then $\det(A)$ or $|A|$ is a sum of $n!$ terms where:

1. Each term is the product of n elements of the matrix, with one element (and only one) from each row, and one (and only one) element from each column.
2. The sign of each term is $+$ or $-$ depending on whether the permutation of row subindexes is of the same class as the permutation of the column subindices or not.

Determinants of order 2

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Leftrightarrow |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Determinants of order 3

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

Determinants of diagonal matrices

Diagonal matrix:

$$\left| \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \right| = a_{11}a_{22}\cdots a_{nn}$$

Determinants of triangular matrices

upper triangular matrix:

$$\left|\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}\right| = a_{11}a_{22}\cdots a_{nn}$$

The same occurs to lower triangular matrices.

Rules for determinants

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- If all the elements in a row (or column) of A are 0 then $|A| = 0$.
- If all the elements in a single row (or column) of A are multiplied by a number α, the determinant is multiplied by α.
- If two rows (or two columns) of A are interchanged, the sign of the determinant changes, but the absolute value remains unchanged.

Rules for determinants

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- Is α is a real number

$$|\alpha A| = \alpha^n\,|A|\,.$$

Practical methods to calculate $|A|$

In practice, there are two methods to calculate the determinants of a square matrix (mainly used when its order is higher than 3):

- **Triangularization**: Rule number 6 allows us to convert matrix A into one that is (upper or lower) triangular. Because the determinant will remain unchanged (as is stated by the property) its value will be equal to the product of the elements in the main diagonal of the triangular matrix.

Practical methods to calculate $|A|$

- **Expansion of $|A|$ in terms of the elements of a row:**

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$
$$= a_{i1}(-1)^{i+1} \det(A_{i1}) + a_{i2}(-1)^{i+2} \det(A_{i2}) + \cdots + a_{in}(-1)^{i+n} \det(A_{in})$$

where $a_{i1}, a_{i2}, \cdots a_{in}$ are the elements of the row i and A_{ij} is the determinant of order $n-1$ which results from deleting row i and column j of matrix A (it is called a *minor*).

Practical methods to calculate $|A|$

- **Expansion of $|A|$ in terms of the elements of a column:**

$$\begin{aligned} \det(A) &= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}) \\ &= a_{1j}(-1)^{1+j}\det(A_{1j}) + a_{2j}(-1)^{2+j}\det(A_{2j}) + \cdots + a_{nj}(-1)^{n+j}\det(A_n \end{aligned}$$

where $a_{1j}, a_{2j}, \cdots a_{nj}$ are the elements of the column j and A_{ij} is the determinant of order $n-1$ which results from deleting row i and column j of matrix A (it is called a *minor*).

Definition 7

The product $(-1)^{i+j}\det(A_{ij})$ is called the adjoint element of a_{ij}.

Review Problems for Matrix algebra

Problem 1 ▸ Answer

Given the following matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} B = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} C = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$$

confirm that the following properties are true

a. $(A + B)C = AC + BC$.
b. $(AB)^t = B^t A^t$.
c. $(A - B)^2 = A^2 + B^2 - AB - BA$.
d. $(AB + A) = A(B + I)$, where I is the identity matrix of order 2.
e. $(BA + A) = (B + I)A$, where I is the identity matrix of order 2.

Problem 2 Answer

Calculate AB and BA, where

a. $A = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 0 & -5 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 \\ 8 & 4 \\ -2 & 3 \end{pmatrix}$

b. $A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 2 & 4 \\ 1 & 5 & 3 \end{pmatrix}$

c. $A = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 5 & 7 \end{pmatrix}$

Can we say that the product of matrices has the commutative property?

Problem 3 ▸ Answer

Let A and B be square matrices. Prove that the property $(A+B)^2 = A^2 + B^2 + 2AB$ is false.

Problem 4 Answer

Calculate the following determinants:

a) $\begin{vmatrix} 1 & 3 & 2 \\ -1 & 3 & 1 \\ 2 & 0 & -3 \end{vmatrix}$

b) $\begin{vmatrix} 3 & 1 & -1 & 1 \\ -1 & 2 & 0 & 3 \\ 2 & 3 & 4 & 7 \\ -1 & -1 & 2 & -2 \end{vmatrix}$

c) $\begin{vmatrix} 3 & 1 & 2 & -1 \\ -4 & 1 & 0 & 3 \\ 4 & -3 & 0 & -1 \\ -5 & 2 & 0 & -2 \end{vmatrix}$

d) $\begin{vmatrix} 1 & -3 & 5 & -2 \\ 2 & 4 & -1 & 0 \\ 3 & 2 & 1 & -3 \\ -1 & -2 & 3 & 0 \end{vmatrix}$

Problem 5

Without computing the determinants, show that

a. $$\begin{vmatrix} 1 & x_1 & x_2 \\ 1 & y_1 & x_2 \\ 1 & y_1 & y_2 \end{vmatrix} = (y_1 - x_1)(y_2 - x_2)$$

b. $$\begin{vmatrix} 1 & x_1 & x_2 & x_3 \\ 1 & y_1 & x_2 & x_3 \\ 1 & y_1 & y_2 & x_3 \\ 1 & y_1 & y_2 & y_3 \end{vmatrix} = (y_1 - x_1)(y_2 - x_2)(y_3 - x_3)$$

Problem 6

Answer

Without computing the determinants, find the value of :

a) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1+c \end{vmatrix}$ b) $\begin{vmatrix} 1 & 4 & 4 & 4 \\ 4 & 2 & 4 & 4 \\ 4 & 4 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{vmatrix}$ c) $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ -1 & 0 & 3 & 4 & 5 \\ -1 & -2 & 0 & 4 & 5 \\ -1 & -2 & -3 & 0 & 5 \\ -1 & -2 & -3 & -4 & 0 \end{vmatrix}$

Problem 7

Without computing the determinants, show that

$$\begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} = (a+3)(a-1)^3$$

Problem 8 ▸ Answer

Given the matrices

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 2 & 8 \end{pmatrix}$$

Compute:

a) $|AB|$ b) $\left|(BA)^t\right|$ c) $|2A3B|$ d) $|A+B|$

Problem 9 Answer

Let A and B be matrices of order n. Knowing that $|A| = 5$ and $|B| = 3$, compute:

a) $\left|BA^t\right|$ b) $|3A|$ c) $\left|(2B)^2\right|$, B of order 3

Problem 10 ▸ Answer

Compute $|AB|$, $\left|(BA)^t\right|$, $|2A3B|$, knowing that

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 2 & 8 \end{pmatrix}$$

Problem 11 ▸ Answer

If $A^2 = A$, what values can $|A|$ have?

Problem 12

Let A be a square matrix of order n such that $A^2 = -I$. Prove that $|A| \neq 0$ and that n is an even number.

Problem 13

Let A be a square matrix of order n. Prove that AA^t is a symmetric matrix.

Problem 14 Answer

Given the following matrices:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \; B = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$
$$C = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$$

solve the matrix equation $3X + 2A = 6B - 4A + 3C$.

Problem 15 ▸ Answer

Suppose that $A.B$ and X are matrices of orden n, solve the following matrix equations:

a. $3(A^t - 2B) + 5X^t = -B.$
b. $(A^t + X)^t - B = 2A.$

Multiple choice questions

1. The matrix $A = \begin{pmatrix} 4 & 1 & -3 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \end{pmatrix}$ is ▸ Answer

   a. a lower triangular matrix
   b. an upper triangular matrix
   c. a diagonal matrix

2. Which of the following is the true definition of a symmetric matrix? Answer
   a. A square matrix A is said to be symmetric if $A = -A$
   b. A square matrix A is said to be symmetric if $A = -A^t$
   c. A square matrix A is said to be symmetric if $A = A^t$

3. Given A and B matrices of order $m \times n$ and $n \times p$, $(AB)^t$ equals to: ▸ Answer
   a. B^tA^t
   b. A^tB^t
   c. AB

4. Let A be a matrix such that $A^2 = A$ then, if $B = A - I$, then: Answer
   a. $B^2 = B$
   b. $B^2 = I$
   c. $B^2 = -B$

5. Let A and B be square matrices of order 3. If $|A| = 3|$ and $|B| = -1$ then: Answer
   a. $|2A \,.\, 4B| = (-4)2^3$
   b. $|2A \,.\, 4B| = (-3)2^3$
   c. $|2A \,.\, 4B| = (-3)2^9$

6. Which of the following properties is NOT always true? Answer

a. $\left|A^2\right| = |A|^2$
b. $|A + B| = |A| + |B|$
c. $|A^t B| = |A|\,|B|$

7. Given the 3 by 3 matrices A, B, C such that $|A| = 2$, $|B| = 4$ and $|C| = 3$, compute $\left|\frac{1}{|A|} B^t C^{-1}\right|$: ▸ Answer

   a. $\frac{2}{3}$
   b. $\frac{1}{6}$
   c. 6

8. Let A and B be matrices of the same order, which of the following properties is always true? Answer

   a. $(A-B)(A+B)=A^2-B^2$
   b. $(A-B)^2=A^2-2AB+B^2$
   c. $A(A+B)=A^2+AB$

9. Which of the following properties is NOT true? Answer

   a. $\left|A^2\right| = |A|^2$
   b. $|-A| = |A|$
   c. $|A^t| = |A|$

10. Let A and B be symmetric matrices, then which of the following is also a symmetric matrix: ▸ Answer

   a. BA
   b. $A + B$
   c. AB

11. Let A be an n by n real matrix. Then, if $k \in \mathbb{R}$ one has that

Answer

a. $|kA| = k\,|A|$
b. $|kA| = |k|\,|A|$, being $|k|$ the absolute value of the real number k
c. $|kA| = k^n\,|A|$

12. Given the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} B = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} C = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$$

the solution to the matrix equation
$3X + 2A = 6B - 4A + 3C$ is ▸ Answer

a. $X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
b. $X = \begin{pmatrix} -4 & 1 \\ -4 & 0 \end{pmatrix}$
c. $X = \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}$

13. Let A, B be matrices of order n. Then: ▸ Answer

a. $(AB)^2 = A^2B^2$
b. $(AB)^2 = B^2A^2$
c. $(AB)^2 = A(BA)B$

14. Given the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$, the adjoint element a_{12} is:

Answer

a. -1
b. 1
c. -2

Answers to Problems

Problem 1 ▸ Return

a. $\begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$

b. $2\begin{pmatrix} -1/2 & -1/2 \\ 1 & 1 \end{pmatrix}$

c. $\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$

d. $3\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

e. $\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$

Problem 2 ▸ Return

a. $AB = \begin{pmatrix} 16 & 10 \\ 13 & -15 \end{pmatrix}$ and $BA = \begin{pmatrix} 12 & 3 & 6 \\ 36 & 8 & -4 \\ -5 & -2 & -19 \end{pmatrix}$

b. $AB = \begin{pmatrix} -2 & 1 & 7 \\ 5 & 11 & 5 \\ 3 & 10 & 8 \end{pmatrix}$ and $BA = \begin{pmatrix} 5 & 5 & 4 \\ 8 & 4 & 6 \\ 12 & 5 & 8 \end{pmatrix}$

c. $AB = 3 \begin{pmatrix} 2 & 3 \end{pmatrix}$ and BA is not possible

Problem 3 Return

It is false because the multiplication of matrices does not verify the conmutative property. To probe its falsity the students must provide a counterexample.

Problem 4 ▸ Return

(a)-24 (b)-58 (c)-80 (d)35

Problem 6 Return

(a) abc (b) -24 (c) $5!$

Problem 8 ▸ Return

(a)16 (b) 16 (c) $2^8 3^4$ (d) 130

Problem 9 Return

(a)15 (b) $3^n \cdot 5$ (c) $2^6 \cdot 3^2$

Problem 10 ▸ Return

(a)$3^n \cdot 5$ (b) 5^{n-1} (c) 15

Problem 11 ▸ Return

$|A| = 0$ or $|A| = 1$.

Problem 14 ▸ Return

$$X = \begin{pmatrix} -4 & 1 \\ -4 & 0 \end{pmatrix}$$

Problem 15 ▸ Return

(a)$X = B^t - \frac{3}{5}A$ (b)$X = A^t + B^t$.

Answers to Multiple choice questions

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5. Let A and B be square matrices of order 3. If $|A| = 3|$ and $|B| = -1$ then: Back
   a. $|2A \cdot 4B| = (-4)2^3$
   b. $|2A \cdot 4B| = (-3)2^3$
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Quadratic forms

Back

Outline

- Definiteness of a quadratic form
- The sign of a quadratic form attending the principal minors
- Quadratic forms with linear constraints

Back

Quadratic forms: The general case

Definition 8

A *quadratic form* in n variables is a function Q of the form

$$Q(x_1, x_2, ..., x_n) = \mathbf{x}' A \mathbf{x} = (x_1, x_2, ..., x_n) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

where $\mathbf{x}' = (x_1, x_2, ..., x_n)$ is a vector and $A = (a_{ij})_{n \times n}$ is a symmetric matrix of real numbers.

Then A is called the symmetric matrix associated with Q.

Matrix and Polynomial form

1. *Matrix form* of a quadratic form.

$$Q(x_1, x_2, ..., x_n) = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

2. *Polynomial form* of a quadratic form. Expanding the matrix multiplication we obtain a double sum such as

$$Q(x_1, x_2, ..., x_n) = \mathbf{x}' A \mathbf{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

Function Q is a homogeneous polynomial of degree two where each term contains either the square of a variable or a product of exactly two of the variables. The terms can be grouped as follows

$$Q(\mathbf{x}) = \sum_{i=1}^{n} b_{ii} x_i^2 + \sum_{i,j=1, i<j}^{n} b_{ij} x_i x_j$$

where $b_{ii} = a_{ii}$ and, since A is a symmetric matrix with

$$Q(x_1,x_2,\dots,x_n)=(x_1,x_2,\cdots,x_n)\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix}\begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix}=\sum_{i=1}^{n} b_{ii}x_i^2+\sum_{i,j=1,i<j}^{n} b_{ij}$$

There exits a relationship between the elements in the symmetric matrix associated with Q and the coefficients of the polynomial. Note that

The elements in the main diagonal of matrix A are the coefficients of the quadratic terms of the polynomial.

The elements outside the main diagonal of matrix A $(a_{ij}=a_{ji}\ i\neq j)$ are half of the coefficients of the non quadratic terms of the polynomial.

Definition 9

A quadratic form $Q(\mathbf{x}) = \mathbf{x}' A\mathbf{x}$ (as well as its associated symmetric matrix A) is said to be

1. *Positive definite* if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$.
2. *Negative definite* if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq 0$
3. *Positive semidefinite* if $Q(\mathbf{x}) \geq 0$ for all $\mathbf{x} \neq 0$
4. *Negative semidefinite* if $Q(x) \leq 0$ for all $\mathbf{x} \neq 0$
5. *Indefinite* if there exist vectors $\mathbf{x}$ and $\mathbf{y}$ such that $Q(\mathbf{x}) < 0$ and $Q(\mathbf{y}) > 0$. Thus, an indefinite quadratic form assumes both negative and positive values.

Definition 10

A principal minor of order r of an $n \times n$ matrix $A = (a_{ij})$ is the determinant of a matrix obtained by deleting $n - r$ rows and $n - r$ columns such that if the ith row (column) is selected, then so is the ith column (row).

In particular, a principal minor of order r always includes exactly r elements of the main (principal) diagonal. Also, if matrix A is symmetric, then so is each matrix whose determinant is a principal minor. The determinant of A itself, $|A|$, is also a principal minor (No rows or colmns are deleted)

Definition 11

A principal minor is called a *leading principal minors* of order r ($1 \leq r \leq n$) if it consists of the first ("leading") r rows and columns of $|A|$.

Theorem

Let $Q(\mathbf{x}) = \mathbf{x}' A \mathbf{x}$ be a quadratic form of n variables and let

$$|A_1| = a_{11}, |A_2| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, |A_3| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, ..., |A_n| = |A|$$

be the leading principal minors of matrix A. Then

- $Q(\mathbf{x})$ positive definite $\Leftrightarrow |A_1| > 0, |A_2| > 0, ..., |A_n| > 0$
- $Q(\mathbf{x})$ negative definite $\Leftrightarrow$ the leading principal minors of even order are positive and those of odd order are negative..
- If $|A| = 0$ and the remaining leading principal minors are positive $\Rightarrow Q$ is positive semidefinite.
- If $|A| = 0$ and the remaining leading principal minors of even order are positive and those of odd order are negative $\Longrightarrow Q$ is negative semidefinite.
- If $|A| \neq 0$ and the leading principal minors do not behave as in $a)$ or $b) \Rightarrow Q$ is indefinite.
- If $|A| = 0$ and $|A_i| \neq 0$ $i = 1, 2, ..., n-1$ and the leading principal minors do not behave as in $c)$ or $d) \Rightarrow Q$ is indefinite.

Theorem

Let $Q(\mathbf{x}) = \mathbf{x}' A \mathbf{x}$ be a quadratic form of n variables such that $|A| = 0$. Then

- *All the principal minors are positive or zero $\Leftrightarrow$ Q is positive semidefinite*
- *All the principal minors are of even order are positive or zero and those of odd order are negative or zero $\Leftrightarrow$ Q is negative semidefinite.*

Definition 12

It is said that the quadratic form $Q(x) = x^t Ax$ is constrained to a linear constraint when

$$(x_1, x_2, ..., x_n) \in \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n / b_1x_1 + b_2x_2 + ... + b_nx_n = 0\}$$

To find the sign of a constrained quadratic form, follow the following steps:

1. Analyze the sign of $Q(x) = x^t Ax$ without any constraint. If it is definite (positive or negative), then the constrained quadratic form is of the same sign.
2. If the unconstrained quadratic form is not definite, we solve the linear constraint for one variable and substitute it into the quadratic form. The result is an unconstrained quadratic form with $n - 1$ variables. We study the sign with the principal minors.

Review Problems for Matrix algebra

Problem 1 ▸ Answer

Without computing any principal minor, determine the definiteness of the following quadratic forms:

- a. $Q(x, y, z) = (x + 2y)^2 + z^2$
- b. $Q(x, y, z) = (x + y + z)^2 - z^2$
- c. $Q(x, y, z) = (x - y)^2 + 2y^2 + z^2$
- d. $Q(x, y, z) = -(x - y)^2 - (y + z)^2$
- e. $Q(x, y, z) = (x - y)^2 + (y - 2z)^2 + (x - 2z)^2$

Problem 2 Answer

Write the following quadratic forms in matrix form with A symmetric and determine their definiteness.

a. $Q(x,y,z) = -x^2 - y^2 - z^2 + xy + xz + yz$
b. $Q(x,y,z) = y^2 + 2z^2 + 2xz + 4yz$
c. $Q(x,y,z) = 2y^2 + 4z^2 + 2yz$
d. $Q(x,y,z) = -3x^2 - 2y^2 - 3z^2 + 2xz$
e. $Q(x_1,x_2,x_3,x_4) = x_1^2 - 4x_3^2 + 5x_4^2 + 4x_1x_3 + 2x_2x_3 + 2x_2x_4$

Problem 3 ▸ Answer

Investigate the definiteness of the following quadratic forms depending on the value of parameter a.

- a. $Q(x, y, z) = -5x^2 - 2y^2 + az^2 + 4xy + 2xz + 4yz$
- b. $Q(x, y, z) = 2x^2 + ay^2 + z^2 + 2xy + 2xz$
- c. $Q(x, y, z) = x^2 + ay^2 + 2z^2 + 2axy + 2xz$

Problem 4 ▸ Answer

Find the value of a which makes the quadratic form $\mathrm{Q(x, y, z) = ax^2 + 2y^2 + z^2 + 2xy + 2xz + 2yz}$ be semidefinite. For such a value, determine its definiteness when it is subject to $\mathrm{x} - y - z = 0$?

Problem 5

Answer

If $A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & -2 \\ -1 & -2 & 3 \end{pmatrix}$ then

a. investigate its definiteness.
b. Write the polynomial and matrix forms of the quadratic form $Q(h_1, h_2, h_3)$ which is associated with matrix *A*.
c. determine its definiteness when it is subject to $h_1 + 2h_2 - h_3 = 0$.

Problem 6 ▸ Answer

Investigate the definiteness of the following matrices. Write the polynomial and matrix form of the associated quadratic forms:

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & -3/2 & 1/2 \\ -3/2 & 1 & -1/2 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 3 \end{pmatrix} \quad D = \begin{pmatrix} -5 & 2 & 1 & 0 \\ 2 & -2 & 0 & 2 \\ 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -5 \end{pmatrix}$$

Problem 7 ▸ Answer

Determine the definiteness of the following constrained quadratic forms.

a. $Q(x,y,z)=(x,y,z)\begin{pmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$ s. t $x+y-2z=0$

b. $Q(x,y,z)=(x,y,z)\begin{pmatrix}2&-3/2&1/2\\-3/2&1&-1/2\\1/2&-1/2&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$ s. t $x-y=0$

c. $Q(x,y,z)=(x,y,z)\begin{pmatrix}1&-1&1\\-1&1&-1\\1&-1&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$ s. t $x+2y-z=0$

d. $Q(x_1,x_2,x_3,x_4)=(x_1,x_2,x_3,x_4)\begin{pmatrix}-5&2&1&0\\2&-2&0&2\\1&0&-1&0\\0&2&0&-5\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$ s. t $2x_1-4x_4=0$

Problem 8 ▸ Answer

Let $Q(x, y, z) = -x^2 - 2y^2 - z^2 + 2xy - 2yz$ be a quadratic form

a. write its matrix form and investigate its definiteness.
b. Investigate its definiteness if it is constrained to $2x - 2y + az = 0$ for the different values parameter a can have.

Problem 9 ▸ Answer

Determine the definiteness of the Hessian matrix of the following functions

a. $f(x, y, z) = 2x^2 + y^2 - 2xy + xz - yz + 2x - y + 8$
b. $f(x, y) = x^4 + y^4 + x^2 + y^2 + 2xy$
c. $f(x, y, z) = \ln(x) + \ln(y) + \ln(z)$

Problem 10

▸ Answer

Determine the definiteness of the Hessian matrix of the following production functions when $K, L > 0$.

a) $\mathrm{Q(K,L) = K^{1/2}L^{1/2}}$ b) $\mathrm{Q(K,L) = K^{1/2}L^{2/3}}$

Problem 11 Answer

The production function $\mathrm{Q(x, y, z) = ax^2 + 4ay^2 + a^2z^2 - 4axy}$, with $a > 0$, relates the produced quantity of a good to three raw materials (x, y and z) used in the production precess.

- a Determine the definiteness of $Q(x, y, z)$.
- b Knowing that if $x = y = z = 1$ then six units of a good are produced, find the value of parameter a.
- c Using the value of a found in (b), determine the definiteness of $Q(x, y, z)$ when the raw materials x and y are used in the same quantity.

Multiple choice questions

1. Which of the following is a quadratic form? ▸ Answer
   a. $Q(x,y,z) = x^2 + 3z^2 + 6xy + 2z$
   b. $Q(x,y,z) = 2xy^2 + 3z^2 + 6xy$
   c. $Q(x,y,z) = 3xy + 3xz + 6yz$

2. Let $Q(x, y, z)$ be a quadratic form such that $Q(1, 1, 0) = 2$ and $Q(5, 0, 0) = 0$, then

▸ Answer

a. $Q(x, y, z)$ could be indefinite
b. $Q(x, y, z)$ is positive definite
c. $Q(x, y, z)$ could be negative semidefinite

3. The quadratic form $Q(x, y, z) = (x - y)^2 + 3z^2$ is

Answer

a. positive definite
b. positive semidefinite
c. indefinite

4. The quadratic form $Q(x, y, z) = -y^2 - 2z^2$ is

Answer

   a. negative definite
   b. negative semidefinite
   c. indefinite

5. The quadratic form $Q(x, y, z) = x^2 + 2xz + 2y^2 - z^2$ is

Answer

a. positive definite
b. positive semidefinite
c. indefinite

6. A 2 by 2 matrix has a negative determinant, then the matrix is

   ▸ Answer

   a. negative definite
   b. negative semidefinite
   c. indefinite

7. The matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ is

Answer

a. indefinite
b. positive semidefinite
c. positive definite

8. The leading principal minors of a 4 by 4 matrix are $|A_1| = -1$, $|A_2| = 1$, $|A_3| = 2$, and $|A_4| = |A| = 0$. Then,

Answer

   a. the matrix is negative semidefinite
   b. its definiteness cannot be determined with this information
   c. the matrix is indefinite

9. The leading principal minors of a 4 by 4 matrix are $|\mathrm{A}_1| = -1$, $|\mathrm{A}_2| = 1$, $|\mathrm{A}_3| = -2$ and $|\mathrm{A}_4| = |A| = 0$. Then,

Answer

   a. the matrix is negative semidefinite
   b. its definiteness cannot be determined from this information
   c. the matrix is indefinite

10. The leading principal minors of a 4 by 4 matrix are $|A_1| = -1, |A_2| = 1,\ |A_3| = -2$, and $|A_4| = |A| = 1$.Then, the matrix is

Answer

   a. negative definite
   b. indefinite
   c. positive definite and negative definite

11. The quadratic form in three variables $Q(x, y, z)$, subject to $x + 2y - z = 0$, is positive semidefinite. Then, the unconstrained quadratic form is:

    ▸ Answer

    a. positive semidefinite or indefinite
    b. positive semidefinite
    c. positive definite or positive semidefinite

12. If $Q(x,y,z)$ is a negative semidefinite quadratic form such that $Q(-1,1,1)=0$, then $Q(x,y,z)$ subject to the constraint $x+2y-z=0$

Answer

a. is negative semidefinite
b. cannot be classified with this information
c. is negative definite or negative semidefinite

13. The quadratic form $Q(x, y, z) = (x - y)^2 + 3z^2$ subject to the constraint $x = 0$ is:

Answer

a. indefinite
b. positive semidefinite
c. positive definite

14. The quadratic form $\mathrm{Q}(\mathrm{x}, \mathrm{y}, \mathrm{z}) = (\mathrm{x} - y)^2 + 3\mathrm{z}^2$ subject to the constraint $z = 0$ is:

Answer

a. indefinite
b. positive semidefinite
c. positive definite

Solutions

Answers to the problems

Problem 1 ▸ Return

- (a) positive semidefinite.
- (b) indefinite.
- (c) positive definite.
- (d) negative semidefinite.
- (e) positive semidefinite.

Problem 2 ▸ Return

(a) $Q(x,y,z) = (x,y,z)\begin{pmatrix} -1 & 1/2 & 1/2 \\ 1/2 & -1 & 1/2 \\ 1/2 & 1/2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ negative semidefinite.

(b) $Q(x,y,z) = (x,y,z)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ indefinite.

(c) $Q(x,y,z) = (x,y,z)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ positive semidefinite.

(d) $Q(x,y,z) = (x,y,z)\begin{pmatrix} -3 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ negative definite.

(e) $Q(x_1,x_2,x_3,x_4) = (x_1,x_2,x_3,x_4)\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & -4 & 0 \\ 0 & 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ indefinite.

Problem 3 ▸ Return

(a) negative definite for $a < -5$, negative semidefinite for $a = -5$ and indefinite when $a > -5$.

(b) positive definite if $a < 1$, positive semidefinite if $a = 1$ and indefinite when $a < 1$.

(c) Indefinite when $a < 0$ or $a > 1/2$, positive semidefinite for $a = 0$ or $a = 1/2$ and positive definite.if $0 < a < 1/2$.

Problem 4 Return

$a = 1$. The constrained quadratic form is positive definite.

Problem 5 ▸ Return

(a) indefinite.

(b) $Q(h_1, h_2, h_3) = (h_1, h_2, h_3) \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & -2 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} =$

$h_2^2 + 3h_3^2 - 2h_1h_3 - 4h_2h_3$

(c) positive definite.

Problem 6 ▸ Return

(a) negative semidefinite,
$Q(x, y, z) = -2x^2 - 2y^2 - 2z^2 + 2xy + 2xz + 2yz.$

(b) indefinite, $Q(x, y, z) = 2x^2 + y^2 - 3xy + xz - yz.$

(c) positive semidefinite,
$Q(x, y, z) = x^2 + y^2 + 3z^2 - 2xy + 2xz - 2yz.$

(d) negative definite, $Q(x_1, x_2, x_3, x_4) =$
$-5x_1^2 - 2x_2^2 - x_3^2 - 5x_4^2 + 4x_1x_2 + 2x_1x_3 + 4x_2x_4.$

Problem 7 ▸ Return

(a) negative semidefinite. (b) The constrained quadratic form is null.

(c) positive definite. (d) negative definite (since the unconstrained quadratic form is negative definite).

Problem 8 Return

(a) $Q(x, y, z) = (x, y, z) \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ negative semidefinite.

(b) negative semidefinite if $a = 0$, negative definite if $a \neq 0$.

Problem 9 ▸ Return

(a) $Hf(x,y,z)$ is indefinite$\forall(\mathrm{x,y,z}) \in \mathbb{R}^3$.

(b) $Hf(x,y)$ is positive definite$\forall(\mathrm{x,y}) \in \mathbb{R}^2 - \{(0,0)\}$. H f(0,0) is positive semidefinite.

(c) $Hf(x,y,z)$ is negative definite$\forall(\mathrm{x,y,z}) \in \mathrm{Dom}(f) = \{(\mathrm{x,y,z}) \in \mathbb{R}^3/\mathrm{x} > 0, \mathrm{y} > 0, \mathrm{z} > 0\}$.

Problem 10 Return

(a) negative semidefinite. (b) indefinite.

Problem 11 Return

(a) Positive Semidefinite.

(b) $a = 2$.

(c) Positive Definite.

Answers to Multiple choice questions

1. Which of the following is a quadratic form? Back
   a. $Q(x, y, z) = x^2 + 3z^2 + 6xy + 2z$
   b. $Q(x, y, z) = 2xy^2 + 3z^2 + 6xy$
   c. $Q(x, y, z) = 3xy + 3xz + 6yz$

2. Let $Q(x, y, z)$ be a quadratic form such that $Q(1, 1, 0) = 2$ and $Q(5, 0, 0) = 0$, then

Back

   a. $Q(x, y, z)$ could be indefinite
   b. $Q(x, y, z)$ is positive definite
   c. $Q(x, y, z)$ could be negative semidefinite

3. The quadratic form $Q(x, y, z) = (x - y)^2 + 3z^2$ is

Back

a. positive definite
b. positive semidefinite
c. indefinite

4. The quadratic form $Q(x, y, z) = -y^2 - 2z^2$ is

◂ Back

   a. negative definite
   b. negative semidefinite
   c. indefinite

5. The quadratic form $Q(x, y, z) = x^2 + 2xz + 2y^2 - z^2$ is

◂ Back

a. positive definite
b. positive semidefinite
c. indefinite

6. A 2 by 2 matrix has a negative determinant, then the matrix is

Back

   a. negative definite
   b. negative semidefinite
   c. indefinite

7. The matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ is

Back

- a. indefinite
- b. positive semidefinite
- c. positive definite

8. The leading principal minors of a 4 by 4 matrix are $|\mathrm{A}_1| = -1$, $|\mathrm{A}_2| = 1$, $|\mathrm{A}_3| = 2$, and $|\mathrm{A}_4| = |A| = 0$. Then,

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a. the matrix is negative semidefinite
b. its definiteness cannot be determined with this information
c. the matrix is indefinite

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◂ Back

   a. the matrix is negative semidefinite
   b. its definiteness cannot be determined from this information
   c. the matrix is indefinite

10. The leading principal minors of a 4 by 4 matrix are $|A_1| = -1, |A_2| = 1,\ |A_3| = -2$, and $|A_4| = |A| = 1$. Then, the matrix is

◂ Back

a. negative definite
b. indefinite
c. positive definite and negative definite

11. The quadratic form in three variables $Q(x, y, z)$, subject to $x + 2y - z = 0$, is positive semidefinite. Then, the unconstrained quadratic form is:

◂ Back

a. positive semidefinite or indefinite
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c. positive definite or positive semidefinite

12. If $Q(x, y, z)$ is a negative semidefinite quadratic form such that $Q(-1, 1, 1) = 0$, then $Q(x, y, z)$ subject to the constraint $x + 2y - z = 0$

Back

a. is negative semidefinite
b. cannot be classified with this information
c. is negative definite or negative semidefinite

13. The quadratic form $Q(x, y, z) = (x - y)^2 + 3z^2$ subject to the constraint $x = 0$ is:

Back

a. indefinite
b. positive semidefinite
c. positive definite

14. The quadratic form $Q(x, y, z) = (x - y)^2 + 3z^2$ subject to the constraint $z = 0$ is:

Back

- a. indefinite
- b. positive semidefinite
- c. positive definite

Links to the Wolfram Demostrations Project web page

- Matrix Multiplication >>
- Matrix Transposition >>
- Determinants by expansion >>
- Determinants using diagonals >>

Bibliography

- Matrix algebra:

  Sydsaeter,K. and Hammond,P.J. Essential Mathematics for Economic Analysis. Prentice Hall. New Jersey. Pages: 537-554 and 573-591. >>
- Quadratic forms

  Sydsaeter,K. and Hammond,P.J. Further Mathematics for Economic Analysis. Prentice Hall. New Jersey. Pages: 28-37. >>

Back