Mathematics for Business Administration: Multivariable Optimization

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Chapter One: Convex Sets. Convex and Concave Functions

Outline

- Convex Sets
- Convex and Concave Functions

Definition 1

Let $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$ be any two points in $\mathbb{R}^n$. The closed line segment between $\mathbf{x}$ and $\mathbf{y}$ is the set

$$[\mathbf{x}, \mathbf{y}] = \{\mathbf{z} \;\; / \;\; \text{there exists } \lambda \in [0, 1] \text{ such that } \mathbf{z} = \lambda\mathbf{x} + (1 - \lambda)\mathbf{y}\}$$

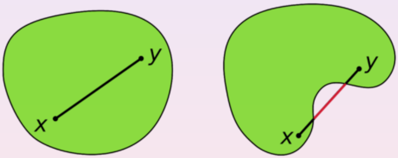
Definition 2

A set S in $\mathbb{R}^n$ is called *convex* if $[\mathbf{x}, \mathbf{y}] \subseteq S$ for all $\mathbf{x}$, $\mathbf{y}$ in S, or equivalently, if

$$\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in S \text{ for all } \mathbf{x},\ \mathbf{y} \text{ in } S \text{ and all } \lambda \in [0, 1]$$

Note in particular that the empty set and also any set consisting of one single point are convex.

Intuitively speaking, a convex set must be "connected" without any "holes" and its boundary must not "bend inwards" at any point.



Convex Not convex

Definition 3

A *hyperplane* in $\mathbb{R}^n$ is the set H of all points $\mathbf{x} = (x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$ that satisfy

$$p_1x_1 + p_2x_2 + \cdots + p_nx_n = m$$

where $\mathbf{p} = (p_1, p_2, ..., p_n) \neq \mathbf{0}$.

Proposition 4 A hyperplane in $\mathbb{R}^n$ is a convex set.

Definition 5

A hyperplane H devides $\mathbb{R}^n$ into two sets,

$$\begin{aligned} H_+ &= \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n / \quad p_1x_1 + p_2x_2 + \cdots + p_nx_n \geq m\}, \\ H_- &= \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n / \quad p_1x_1 + p_2x_2 + \cdots + p_nx_n \leq m\}, \end{aligned}$$

which are called half spaces.

Proposition 6 H_+ and H_- are convex sets.

Proposition 7 If S and T are two convex sets in $\mathbb{R}^n$, then their intersection $S \cap T$ is also convex.
The union of convex sets is usually not convex.

Definition 8

A function $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$ defined on a convex set S is *concave* on S if

$$f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \geq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}) \tag{1}$$

for all $\mathbf{x}$ and $\mathbf{y}$ in S and for all λ in $[0, 1]$.

A function $f(\mathbf{x})$ is *convex* if (1) holds with $\geq$ replaced by $\leq$. Note that (1) holds with equality for $\lambda = 0$ and $\lambda = 1$. If we have *strict* inequality in (1) whenever $\mathbf{x} \neq \mathbf{y}$ and $\lambda \in (0,1)$, then f is *strictly concave*.

Note that a function f is *convex* on S if and only if $-f$ is concave. Furthermore, f is *strictly convex* if and only if $-f$ is strictly concave.

Proposition 9 Let $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$ be defined on a convex set S in $\mathbb{R}^n$. Then

- If f is concave, the set $\{\mathbf{x} \in \mathbf{S} \,/\, f(\mathbf{x}) \geq a\}$ is convex for every number a.
- If f is convex, the set $\{\mathbf{x} \in \mathbf{S} \,/\, f(\mathbf{x}) \leq a\}$ is convex for every number a.

Definition 10

Suppose that $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$ is a $\mathcal{C}^2$ function in an open convex set S in $\mathbb{R}^n$. Then the symmetric matrix

$$\mathcal{H}(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{x}) & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{x}) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_2^2}(\mathbf{x}) & \dots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(\mathbf{x}) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_n \partial x_2}(\mathbf{x}) & \dots & \frac{\partial^2 f}{\partial x_n^2}(\mathbf{x}) \end{pmatrix}$$

is called the *Hessian matrix* of f at $\mathbf{x}$.

Proposition 11 Suppose that $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$ is a $\mathcal{C}^2$ function defined on an open, convex set S in $\mathbb{R}^n$. Then
(i) The *Hessian matrix* is positive definite or semidefinite $\Leftrightarrow$ f is convex.
(ii) The *Hessian matrix* is negative definite or semidefinite $\Leftrightarrow$ f is concave.

Proposition 12 Suppose that $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$ is a $\mathcal{C}^2$ function defined on an open, convex set S in $\mathbb{R}^n$. Then
(i) If the *Hessian matrix* is positive definite $\Rightarrow$ f is strictly convex.
(ii) If the *Hessian matrix* is negative definite $\Rightarrow$ f is strictly concave.

Links to the Wolfram Demostrations Project web page

- Convex sets >>
- Operations on sets >>
- Concavity and convexity in quadratic surfaces >>

Some pictures of convex sets in Wikipedia >>

Bibliography

Sydsaeter,K., Hammond, P.J., Seierstad, A. and Strom, A. Further Mathematics for Economic Analysis. Prentice Hall. New Jersey. pages: 50-62. >>

Problem 1 ▸ Answer

Draw the following sets and say if they are convex, closed and bounded.

a. $A = \{(x, y) \in \mathbb{R}^2 / x^2 + y^2 \leq 4\}$
b. $B = \{(x, y) \in \mathbb{R}^2 / y = 2x + 3\}$
c. $C = \{(x, y) \in \mathbb{R}^2 / (x-1)^2 + (y-3)^2 = 9\}$
d. $D = \{(x, y) \in \mathbb{R}^2 / y > x^2, y \leq 1\}$
e. $E = \{(x, y) \in \mathbb{R}^2 / y \geq x\}$
f. $F = \{(x, y) \in \mathbb{R}^2 / x + y \leq 2, x \geq 0, y \geq 0\}$
g. $G = \{(x, y) \in \mathbb{R}^2 / xy \leq 1\}$
h. $H = \{(x, y) \in \mathbb{R}^2 / xy > 1, x \geq 0, y \geq 0\}$

Problem 2

Investigate the convexity of the following sets

a. $A = \{(x, y) \in \mathbb{R}^2 / 0 \leq x \leq 4, 2 \leq y \leq 6\}$
b. $B = \{(x, y, z) \in \mathbb{R}^3 / x + y + 2z \leq 24\}$
c. $C = \{y \in \mathbb{R}^n / y = \alpha x \quad \text{with } \alpha \in \mathbb{R} \text{ and } \; x \in X \subset \mathbb{R}^n \text{ convex}\}$

Problem 3 Answer

Investigate the concavity/convexity for the following functions

a. $f(x, y) = 3x^3 - 2y^2$
b. $f(x, y) = (x-3)^3 + (y+1)^2$
c. $f(x, y) = (x-2)^2 + y^4$
d. $f(x, y, z) = x^2 + y^2 + z^3$
e. $f(x, y, z) = x^2 + y^2 + z^2 + yz$
f. $f(x, y, z) = e^x + y^2 + z^2$
g. $f(x, y, z) = e^{2x} + y^2 z$
h. $f(x, y) = xy$

Problem 4 ▸ Answer

Check the concavity/convexity of the following functions

a. $f(x,y) = \ln y - e^x$
b. $f(x,y) = \ln xy \quad$ for all $x, y > 0$
c. $f(x,y) = \sqrt{x^2 + y^2}$
d. $f(x,y) = x^{\frac{1}{2}} y^{\frac{1}{3}} \quad$ for all $x, y > 0$

Problem 5

Answer

Check the concavity/convexity of the following functions for the different values of parameter a.

- a $f(x, y) = x^2 - 2axy$
- b $g(x, y, z) = ax^4 + 8y - z^2$

Problem 6 Answer

Investigate the convexity of the following sets

a. $A = \{(x, y) \in \mathbb{R}^2 / (x-1)^2 + (y-1)^2 \leq 2\}$
b. $B = \{(x, y) \in \mathbb{R}^2 / e^{x+y} \leq 12\}$
c. $C = \{(x, y) \in \mathbb{R}^2 / 3x^2 + 4y^2 \geq 10\}$
d. $D = \{(x, y) \in \mathbb{R}^2 / x + y \leq 2, x \geq 0, y \geq 1\}$
e. $E = \{(x, y) \in \mathbb{R}^2 / x^2 + y^2 - 4x - 2y \leq 3, x \leq 2y\}$
f. $F = \{(x, y) \in \mathbb{R}^2 / x + y \leq 3, 2x + 5y = 10, x \geq 0, y \geq 0\}$

1. Which of the following sets is convex? Answer
   a. $\{(x, y) \in \mathbb{R}^2 / x^2 + y^2 \leq 1\}$
   b. $\{(x, y) \in \mathbb{R}^2 / x^2 + y^2 = 1\}$
   c. $\{(x, y) \in \mathbb{R}^2 / x^2 + y^2 \geq 1\}$

2. The closed line segment between $(1,1)$ and $(-1,-1)$ can be written as the set ▸ Answer

   a. $B = \{(x,y) \in \mathbb{R}^2 / (x,y) = (2\lambda - 1, 2\lambda - 1), \forall \lambda \in [0,1]\}$
   b. $B = \{(x,y) \in \mathbb{R}^2 / (x,y) = (\lambda, 1-\lambda), \forall \lambda \in [0,1]\}$
   c. $B = \{(x,y) \in \mathbb{R}^2 / x = y\}$

3. Given $S \subseteq \mathbb{R}^2$ a convex set, the function $f : S \to \mathbb{R}$ will be convex if ▸ Answer

   a. the Hessian matrix $Hf(x, y)$ is negative definite for all (x, y) in S
   b. the sets $\{(x, y) \in S / f(x, y) \leq k\}$ are convex for all k in $\mathbb{R}$
   c. f is a lineal function

4. The set $S = \{(x, y, z) \in \mathbb{R}^3 / x + y^2 + z^2 \leq 1\}$ Answer
   a. is convex because the Hessian matrix of the function $f(x, y) = x + y + z^2$ is positive semidefinite
   b. is convex because the function $f(x, y) = x + y^2 + z^2$ is lineal
   c. in not convex

5. Which of the following sets is not convex? ▸ Answer

   a. $\{(x, y) \in \mathbb{R}^2 / x \leq 1, y \leq 1\}$
   b. $\{(x, y) \in \mathbb{R}^2 / x, y \in [0, 1]\}$
   c. $\{(x, y) \in \mathbb{R}^2 / xy \leq 1,\ x, y \geq 0\}$

6. Which of the following Hessian matrices belongs to a concave function? Answer

a. $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$

b. $\begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}$

c. $\begin{pmatrix} -2 & 2 \\ 2 & 2 \end{pmatrix}$

7. The function $f(x,y) = \ln x + \ln y$ is concave on the set

Answer

a. $S = \{(x,y) \in \mathbb{R}^2 / x, y > 0\}$
b. $\mathbb{R}^2$
c. $S = \{(x,y) \in \mathbb{R}^2 / x, y \neq 0\}$

8. Which of the following sets is convex? ▸ Answer

   a. $A = \{(x, y) \in \mathbb{R}^2 / xy \geq 1, x \geq 0, y \geq 0\}$
   b. $B = \{(x, y) \in \mathbb{R}^2 / xy \geq 1\}$
   c. $C = \{(x, y) \in \mathbb{R}^2 / xy \leq 1, x \geq 0, y \geq 0\}$

Answers to Problems

Problem 1 Return

- a Convex, closed and bounded
- b Convex and closed
- c Closed and bounded
- d Convex and bounded
- e Convex and closed
- f Convex, closed and bounded
- g Closed
- h Convex

Problem 3 ▸ Return

- a. Concave on the convex set of $\mathbb{R}^2$: $\{(x, y) \in \mathbb{R}^2 / x \leq 0\}$
- b. Convex on the convex set of $\mathbb{R}^2$: $\{(x, y) \in \mathbb{R}^2 / x \geq 3\}$
- c. Convex
- d. Convex if $z \geq 0$
- e. Convex
- f. Convex
- g. Neither concave nor convex
- h. Neither concave nor convex

Problem 4 ▸ Return

a. Concave
b. Concave
c. Convex
d. Concave

Problem 5 Return

a. It is convex if $a = 0$
b. It is concave if $a < 0$

Problem 6 Return

All of them are convex except for C.

Answers to Multiple choice questions

1. Which of the following sets is convex? Back

   a. $\{(x, y) \in \mathbb{R}^2 / x^2 + y^2 \leq 1\}$
   b. $\{(x, y) \in \mathbb{R}^2 / x^2 + y^2 = 1\}$
   c. $\{(x, y) \in \mathbb{R}^2 / x^2 + y^2 \geq 1\}$

2. The closed line segment between $(1,1)$ and $(-1,-1)$ can be written as the set Back

   a. $B = \{(x, y) \in \mathbb{R}^2 / (x, y) = (2\lambda - 1, 2\lambda - 1), \forall\lambda \in [0, 1]\}$
   b. $B = \{(x, y) \in \mathbb{R}^2 / (x, y) = (\lambda, 1 - \lambda), \forall\lambda \in [0, 1]\}$
   c. $B = \{(x, y) \in \mathbb{R}^2 / x = y\}$

3. Given $S \subseteq \mathbb{R}^2$ a convex set, the function $f : S \to \mathbb{R}$ will be convex if Back

   a. the Hessian matrix $Hf(x, y)$ is negative definite for all (x, y) in S
   b. the sets $\{(x, y) \in S / f(x, y) \leq k\}$ are convex for all k in $\mathbb{R}$
   c. f is a lineal function

4. The set $S = \{(x, y, z) \in \mathbb{R}^3 / x + y^2 + z^2 \leq 1\}$ Back

   a. is convex because the Hessian matrix of the function $f(x, y) = x + y + z^2$ is positive semidefinite
   b. is convex because the function $f(x, y) = x + y^2 + z^2$ is lineal
   c. in not convex

5. Which of the following sets is not convex? Back

   a. $\{(x, y) \in \mathbb{R}^2 / x \leq 1, y \leq 1\}$
   b. $\{(x, y) \in \mathbb{R}^2 / x, y \in [0, 1]\}$
   c. $\{(x, y) \in \mathbb{R}^2 / xy \leq 1,\ x, y \geq 0\}$

6. Which of the following Hessian matrices belongs to a concave function? Back

   a. $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$
   b. $\begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}$
   c. $\begin{pmatrix} -2 & 2 \\ 2 & 2 \end{pmatrix}$

7. The function $f(x,y) = \ln x + \ln y$ is concave on the set Back
   - a. $S = \{(x,y) \in \mathbb{R}^2 / x, y > 0\}$
   - b. $\mathbb{R}^2$
   - c. $S = \{(x,y) \in \mathbb{R}^2 / x, y \neq 0\}$

8. Which of the following sets is convex? Back

   a. $A = \{(x, y) \in \mathbb{R}^2 / xy \geq 1, x \geq 0, y \geq 0\}$
   b. $B = \{(x, y) \in \mathbb{R}^2 / xy \geq 1\}$
   c. $C = \{(x, y) \in \mathbb{R}^2 / xy \leq 1, x \geq 0, y \geq 0\}$