Mathematics for Business Administration: Multivariable Optimization

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Chapter Two: Multivariate Optimization. The Extreme value theorem

Outline

- Multivariable optimization. The Extreme Value Theorem (Weierstrass)
- A graphical approach to two-variable optimization problems

An *Optimization Problem* is the problem of finding those points in a domain where a function reaches its largest and its smallest values (referred to as maximum and minimum points):

$$\max\ (\min)\ f(\mathbf{x}) \quad \text{subject to } \mathbf{x} \in \mathbf{S}$$

where $\max\ (\min)$ indicates that we want to maximize or minimize f and $\mathbf{x} = (x_1, ..., x_n) \in S$ a subset of $\mathbb{R}^n$.

In most static optimization problems there are

- an *objective function* $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$, a real-valued function of n variables whose value is to be optimized, i.e. maximized or minimized.
- an *admissible set* (or *feasible set*) S that is some subset of $\mathbb{R}^n$.

Depending on the set S, several different types of optimization problems can arise:

- *Classical case*: if the optimum occurs at an interior point of S (Chapter 5)
- *Lagrange problem*: if S is the set of all points $\mathbf{x}$ that satisfy a given system of equations (equality constraints) (Chapter 6)
- *Nonlinear programming problem*: if S consists of all points $\mathbf{x}$ that satisfy a system of inequality constraints

Definition 1

The point $\mathbf{x}^* \in S$ is called a *(global) maximum point* for f in S if

$$f(\mathbf{x}^*) \geq f(\mathbf{x}) \qquad \text{for all } \mathbf{x} \text{ in } S,$$

and $f(\mathbf{x}^*)$ is called the *maximum value*.

Definition 2

The point $\mathbf{x}^* \in S$ is called a *(global) minimum point* for f in S if

$$f(\mathbf{x}^*) \leq f(\mathbf{x}) \qquad \text{for all } \mathbf{x} \text{ in } S,$$

and $f(\mathbf{x}^*)$ is called the *minimum value*.

If the inequalities are strict then $\mathbf{x}^*$ is called a *strict maximum (minimum) point* for f in S.

Extreme Value Theorem: Weiestrass

Theorem

Let $f(\mathbf{x})$ be a continuous function on a closed, bounded set S. Then f has both a maximum point and a minimum point in S.

A set S is called *closed* if it contains all its boundary points. Moreover, it is called *bounded* if it is contained in some ball around the origin.

Example

In problems with two variables, if $g(x_1, x_2)$ is a continuous function and c is a real number, the sets

$$\{(x, y) \,/\, g(x_1, x_2) \geq c\} \quad \{(x, y) \,/\, g(x_1, x_2) \leq c\} \quad \{(x, y) \,/\, g(x_1, x_2) = c\}$$

are closed. If $\geq$ is replaced by $>$, or $\leq$ replaced by $<$, or $=$ replaced by $\neq$, then the corresponding set is not closed.

Example

Provided that p, q and m are positive parameters, the (budget) set of points (x, y) that satisfy the inequalities

$$px + qy \leq m, \quad x \geq 0, \quad y \geq 0$$

is closed and bounded.

Given the general maximizing/minimizing problem with two variables

$$\text{maximize } f(x,y) \text{ subject to } (x,y) \text{ in } S$$

a graphical resolution can be done by drawing the feasible set and the level curves of the objective function.
For a graphical resolution follow the following steps:

1. Draw the feasible set
2. Draw the level curves of the objective function which lie in the feasible set
3. In the case of a maximization problem, the maximum point is the feasible point which lies on the highest level curve
   In a minimization problem, the minimum point is the feasible point which lies on the lowest level curve

Links to the Wolfram Demostrations Project web page

- The consumer's optimization problem >>
- Level curves >>
- Surfaces and level curves >>

Bibliography

Sydsaeter,K., Hammond, P.J., Seierstad, A. and Strom, A. Essential Mathematics for Economic Analysis. Prentice Hall. New Jersey. pages: 474-475 and 494 (for a graphical approach) >>

Sydsaeter,K., Hammond, P.J., Seierstad, A. and Strom, A. Further Mathematics for Economic Analysis. Prentice Hall. New Jersey. pages: 103-106 >>

Problem 1

Provide a graphical resolution of the following optimization problems:

a. $$\begin{cases} \max . : 6x + y \\ s.t. : 2x + y \leq 6 \\ \quad x + y \geq 1 \\ \quad y \leq 3 \\ \quad x, y \geq 0 \end{cases}$$

b. $$\begin{cases} opt. : x + y \\ s.t. : x^2 + y^2 = 1 \\ \quad x, y \geq 0 \end{cases}$$

Problem 1

Provide a graphical resolution of the following optimization problems:

c
$$\begin{cases} opt.: (x-2)^2 + (y-1)^2 \\ s.t.: x^2 - y \leq 0 \\ \qquad x + y \leq 2 \\ \qquad x, y \geq 0 \end{cases}$$

d
$$\begin{cases} opt.: x - y^2 \\ s.t.: (x-1)(y-2) \geq 0 \\ \qquad 2 \leq x \leq 4 \end{cases}$$

Problem 1

Provide a graphical resolution of the following optimization problems:

e) $$\begin{cases} opt.: 3x+2y \\ s.t.: -x+y \leq 2 \\ \qquad x-y \leq 2 \end{cases}$$

f) $$\begin{cases} opt.: (x-2)^2+(y-2)^2 \\ s.t.: x+y \geq 1 \\ \qquad -x+y \leq 1 \end{cases}$$

Problem 1

Provide a graphical resolution of the following optimization problems:

g. $$\begin{cases} \min . : x+y \\ s.t. : x^2+y^2 \geq 4 \\ \qquad x^2+y^2 \leq 1 \end{cases}$$

h. $$\begin{cases} \max . : x+y \\ s.t. : x-y^2 \geq 0 \\ \qquad x+y \leq 2 \end{cases}$$

Problem 2 Answer

A firm produces two goods. The profit obtained after the purchase of each are 10 and 15 monetary units respectively. To produce one unit of good 1 requires 4 hours of man-labor and 3 hours of machine work. Each unit of good 2 needs 7 hours of man-labor and 6 hours of machine work. The maximum man-labor time available is 300 hours and for the machines 500 hours. Find the quantities produced of each good which maximize the profit.

Problem 3 Answer

Maximize the utility function $U(x, y) = xy$, where x and y are the quantities consumed of two goods. The price of each unit of these goods is 2 and 1 monetary units respectively and the available budget is 100 monetary units. Formulate the optimization problem the consumer must solve in order to achieve the maximum utility. Calculate the optimal consumed quantities of goods x and y.

Problem 4 ▸ Answer

Which of the following optimization problems satisfy the Weierstrass' theorem conditions?

a) $\begin{cases} \min. : x^2 + y^2 \\ s.t. : x + y = 3 \end{cases}$

b) $\begin{cases} \text{max.} : x + y^2 \\ s.t. : 3x^2 + 5y \leq 4 \\ \quad x, y \geq 0 \end{cases}$

c) $\begin{cases} opt. : 2x + y \\ s.t. : x + y = 1 \\ \quad x^2 + y^2 \leq 9 \end{cases}$

d) $\begin{cases} opt. : x + \ln y \\ s.t. : x - 5y^2 \geq -1 \\ \quad x + y^2 \leq 1 \end{cases}$

e) $\begin{cases} \max. : x^2 + y^2 \\ s.t. : x + y \geq 4 \\ \quad 2x + y \geq 5 \\ \quad x, y \geq 0 \end{cases}$

f) $\begin{cases} \min. : e^{x+y} \\ s.t. : 0 \leq x \leq 1 \\ \quad 0 \leq y \leq 1 \end{cases}$

1. Which of the following points belongs to the feasible set of the optimization problem ▸ Answer

$$\begin{aligned} opt. &\ :\ x^2\sqrt{y} \\ s.t. &\ :\ x+y=3\ ? \end{aligned}$$

a. $(1,2)$
b. $(-1,2)$
c. $(4,-1)$

2. If the feasible set of an optimization problem is unbounded then ▸ Answer
   a. no finite optimum point exists
   b. it has an infinite number of feasible points
   c. the existence of a finite optimum point cannot be assured

3. Given $f(x,y) = ax + by$ with $a, b \in \mathbb{R}$ and the set ▸ Answer

$$S = \{(x,y) \in \mathbb{R}^2 / x + y = 2, x \geq 0, y \geq 0\},$$

a. f has a global maximum point and a global minimum point in S
b. f has a global maximum point in S if a and b are positive
c. there is no maximum or minimum point of f in S

4. Which of the following is the feasible set of the optimization problem ▸ Answer

$$\begin{aligned} \max. \quad &: \quad 2x + y \\ s.t. \quad &: \quad x + y = 1 \\ &: \quad x^2 + y^2 \leq 5 \text{ ?} \end{aligned}$$

- a $\{(x, y) \in \mathbb{R}^2 / x + y = 1, x \geq 0, y \geq 0\}$
- b $\{(x, y) \in \mathbb{R}^2 / (x, y) = \lambda(5, 0) + (1 - \lambda)(0, 5), \forall \lambda \in [0, 1]\}$
- c $\{(x, y) \in \mathbb{R}^2 / (x, y) = \lambda(2, -1) + (1 - \lambda)(-1, 2), \forall \lambda \in [0, 1]\}$

Answers to Problems

Problem 2 ▸ Return

75 units of good 1 and none unit of good 2.

Problem 3 Return

$$x^* = 25, y^* = 50 \Rightarrow u^* = 1250$$

Problem 4 Return

The problems that verify the hypothesis of the Extreme Value Theorem are b, c and f.

Answers to Multiple choice questions

1. Which of the following points belongs to the feasible set of the optimization problem Back

$$\begin{array}{rcl} opt. & : & x^2\sqrt{y} \\ s.t. & : & x+y=3 \ ? \end{array}$$

   a. $(1,2)$
   b. $(-1,2)$
   c. $(4,-1)$

2. If the feasible set of an optimization problem is unbounded then Back

   a. no finite optimum point exists
   b. it has an infinite number of feasible points
   c. the existence of a finite optimum point cannot be assured

3. Given $f(x,y) = ax + by$ with $a, b \in \mathbb{R}$ and the set Back

$$S = \{(x,y) \in \mathbb{R}^2 / x + y = 2, x \geq 0, y \geq 0\},$$

   a. f has a global maximum point and a global minimum point in S
   b. f has a global maximum point in S if a and b are positive
   c. there is no maximum or minimum point of f in S

4. Which of the following is the feasible set of the optimization problem Back

$$\begin{aligned} \max. &\quad : \quad 2x + y \\ s.t. &\quad : \quad x + y = 1 \\ &\quad : \quad x^2 + y^2 \leq 5 \ ? \end{aligned}$$

a. $\{(x, y) \in \mathbb{R}^2 / x + y = 1, x \geq 0, y \geq 0\}$
b. $\{(x, y) \in \mathbb{R}^2 / (x, y) = \lambda(5, 0) + (1 - \lambda)(0, 5), \forall \lambda \in [0, 1]\}$
c. $\{(x, y) \in \mathbb{R}^2 / (x, y) = \lambda(2, -1) + (1 - \lambda)(-1, 2), \forall \lambda \in [0, 1]\}$