Mathematics for Business Administration: Multivariable Optimization

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Chapter Three: Classical Optimization

Outline

- Extreme points
- Local extreme points

Let f be defined on a set S in $\mathbb{R}^n$ then

Definition 1

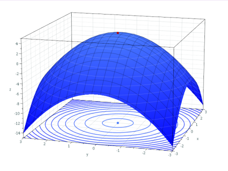
A point $\mathbf{x}^* = (x_1^*, x_2^*, ..., x_n^*)$ is called a *stationary point* of f if all first-order partial derivatives evaluated on $\mathbf{x}^*$ are 0, that is

$$\frac{\partial f}{\partial x_i}(x_1^*, x_2^*, ..., x_n^*) = 0 \text{ for all } i = 1, 2, ..., n.$$

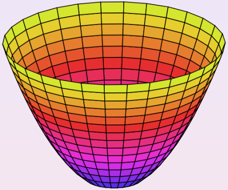
Theorem (Necessary first-order conditions)

Let $\mathbf{x}^* = (x_1^*, x_2^*, ..., x_n^*)$ *be an interior point in* S *at which* f *has partial derivatives, then, a necessary condition for* $\mathbf{x}^*$ *to be a maximum or minimum point for* f *is that* $\mathbf{x}^*$ *is a stationary point for* f*.*

Maxima and minima



Maximum



Minimum

Learn more in http://wikipedia.org

Theorem (Sufficient conditions with concavity/convexity)

Suppose that the function f is $\mathcal{C}^1$,

- *if f is concave in S, then $\mathbf{x}^*$ is a (global) maximum point for f in S if and only if ($\Leftrightarrow$) $\mathbf{x}^*$ is a stationary point for f*
- *if f is convex in S, then $\mathbf{x}^*$ is a (global) manimum point for f in S if and only if ($\Leftrightarrow$) $\mathbf{x}^*$ is a stationary point for f*

If f is strictly concave (convex), the global maximum (minimum) point is unique.

Definition 2

The point $\mathbf{x}^*$ is a *local maximum point* of f in S if

$$f(\mathbf{x}) \leq f(\mathbf{x}^*) \text{ for all } \mathbf{x} \text{ in } S \text{ sufficiently close to } \mathbf{x}^*.$$

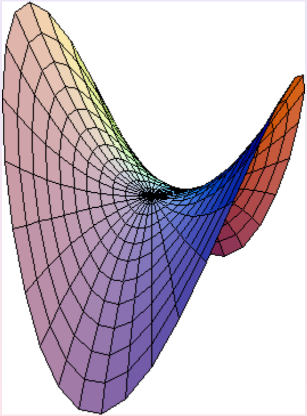
If the inequality is strict then x^* is a *strict local maximum point*.

A (*strict*) *local minimum point* is defined in the obvious way. The first-order necessary conditions for a local maximum (minimum) point remain the same, that is: *a local extreme point in the interior of a domain of a function with partial derivatives must be stationary.*

Definition 3

A stationary point $\mathbf{x}^*$ of f that is neither a local maximum point nor a local minimum point is called a *saddle point* of f.

Saddle Point



Learn more in `http://wikipedia.org`

Theorem (Necessary second-order conditions for local extreme points)

Suppose that f is $\mathcal{C}^2$ and $\mathbf{x}^*$ is an interior stationary point of f, then

- $\mathbf{x}^*$ *is a local minimum point, then ($\Rightarrow$) the Hessian matrix $Hf(\mathbf{x}^*)$ is positive definite or semidefinite*
- $\mathbf{x}^*$ *is a local maximun point, then ($\Rightarrow$) the Hessian matrix $Hf(\mathbf{x}^*)$ is negative definite or semidefinite*

Theorem (Sufficient second-order conditions for local extreme points)

Suppose that the function f is $\mathcal{C}^2$ and $\mathbf{x}^*$ is an interior stationary point of f, then

- *the Hessian matrix $Hf(\mathbf{x}^*)$ is positive definite $\Rightarrow$ $\mathbf{x}^*$ is a local minimum point*
- *the Hessian matrix $Hf(\mathbf{x}^*)$ is negative definite $\Rightarrow$ $\mathbf{x}^*$ is a local maximum point*
- *$|Hf(\mathbf{x}^*)| \neq 0$ but it is not (positive or negative) definite $\Rightarrow$ $\mathbf{x}^*$ is a saddle point*

Example *The two-variables case.*
If $f(x,y)$ is a $\mathcal{C}^2$ function with (x^*,y^*) as an interior stationary point, then

$$\frac{\partial^2 f}{\partial x^2}(x^*,y^*) > 0 \text{ and } \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(x^*,y^*) & \frac{\partial^2 f}{\partial y \partial x}(x^*,y^*) \\ \frac{\partial^2 f}{\partial x \partial y}(x^*,y^*) & \frac{\partial^2 f}{\partial y^2}(x^*,y^*) \end{vmatrix} > 0 \Rightarrow \text{local min. at } (x^*,y^*)$$

$$\frac{\partial^2 f}{\partial x^2}(x^*,y^*) < 0 \text{ and } \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(x^*,y^*) & \frac{\partial^2 f}{\partial y \partial x}(x^*,y^*) \\ \frac{\partial^2 f}{\partial x \partial y}(x^*,y^*) & \frac{\partial^2 f}{\partial y^2}(x^*,y^*) \end{vmatrix} > 0 \Rightarrow \text{local max. at } (x^*,y^*)$$

$$\begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(x^*,y^*) & \frac{\partial^2 f}{\partial y \partial x}(x^*,y^*) \\ \frac{\partial^2 f}{\partial x \partial y}(x^*,y^*) & \frac{\partial^2 f}{\partial y^2}(x^*,y^*) \end{vmatrix} < 0 \Rightarrow (x^*,y^*) \text{ is a saddle point}$$

Links to the Wolfram Demostrations Project web page

- Stationary points (maximun, minimun and saddle points) >> >>

Bibliography

Sydsaeter,K., Hammond, P.J., Seierstad, A. and Strom, A. Essential Mathematics for Economic Analysis. Prentice Hall. New Jersey. pages: 453-466. >>

Problem 1 Answer

Classify the stationary points of

a. $f(x,y)=2x^2+xy+2y^2-4x-y$
b. $f(x,y)=(x^2+y^2)\mathrm{e}^{x^2-y^2}$
c. $f(x,y)=2x-2e^x+3y-y^3+4$
d. $f(x,y)=x\ln(y+1)$
e. $f(x,y,z)=e^{-x^2-y^2-x+z^2}$
f. $f(x,y,z)=x^2-xy^2+y^4-3yz+z^3$
g. $f(x,y)=x^3+3x^2+y^3+6y^2$

Problem 2 ▸ Answer

A firm produces an output good using two inputs, denoted by x and y, according to the following production function

$$Q = x^{1/2}y^{1/3}.$$

If $\mathrm{p}_1 = 2$, $\mathrm{p}_2 = 1$ and $\mathrm{p}_3 = 1$ are the prices of output and inputs respectively, maximize the firm's profit.

Problem 3 Answer

The output production function of a firm is

$$Q = x^{1/2}y^{1/3}.$$

where x and y are the units for two different inputs. If p_1, p_2 and p_3 are the prices of output and inputs respectively, and the firm seeks to maximize profits

a. Find the demand of inputs functions.
b. Suppose that p_3 rises while the rest of parameters remain constant; what is the effect upon the demand for input y?
c. If p_1 rises while p_2 and p_3 remain constant; what is the effect upon the demand for x and y?

Problem 4 Answer

Find the maxima and minima point of the function $f(x, y) = 2x^3 + ay^3 + 6xy$ for different values of parameter $a \in \mathbb{R}$.

Problem 5 ▸ Answer

A firm produces three output goods in units x, y and z respectively. If profit is given by

$$B(x,y,z) = -x^2 + 6x - y^2 + 2yz + 4y - 4z^2 + 8z - 14,$$

find the units of each good that maximize profit and find the maximum profit.

Problem 6 ▸ Answer

A monopolistic firm produces two goods whose demand functions are

$$p_1 = 12 - x_1, \quad p_2 = 36 - 5x_2$$

where x_1 and x_2 are the quantities of the two goods produced and p_1 and p_2 the prices of a unit of each good. Knowing that the cost function is $C(x_1, x_2) = 2x_1x_2 + 15$, solve the corresponding profit maximizing problem.

Problem 7 Answer

Solve the output production maximizing problem

$$\max Q(x, y) = -x^3 - 3y^2 + 3x^2 + 24y$$

where x and y are the necessary inputs. Find the maximum production.

Problem 8 ▸ Answer

In a competitive market, a firm produces good Q according to the function

$$Q(K, L) = 8K^{1/2}L^{1/4}$$

where K and L are capital and labor respectively. Given the unitary prices of 5 m.u for output and 2 m.u. and 10 m.u. for inputs, find the maximum profit.

Problem 9 Answer

The output production function of a firm and its cost function are given, respectively, by

$$\begin{aligned} Q(x,y) &= 7x^2 + 7y^2 + 6xy \\ C(x,y) &= 4x^3 + 4y^3 \end{aligned}$$

where x and y are the productive inputs. Knowing that the selling price of a unit of good is 3 m.u., find the maximum point for both productive inputs, x and y, and find the maximum profit.

1. The function $f(x, y) = x^2 + y^2$ ▸ Answer
   a. has no stationary point
   b. has a stationary point at $(0, 0)$
   c. has a stationary point at $(1, 1)$

2. The function $f(x, y, z) = (x-2)^2 + (y-3)^2 + (z-1)^2$ has, at point $(2, 3, 1)$, ▸ Answer

   a. a global maximum point
   b. a global minimum point
   c. a saddle point

3. The function $f(x, y) = xy^2(2 - x - y)$ has, at point $(0, 2)$,

Answer

a. a local maximum point
b. a local minimum point
c. a saddle point

4. The function $f(x,y) = x^2y + y^2 + 2y$ has ▸ Answer
   a. a local maximum point
   b. a local minimum point
   c. a saddle point

5. The function $f(x,y) = \dfrac{\ln(x^3+2)}{y^2+3}$: Answer

   a. has a stationary point at $(1,0)$
   b. has a stationary point at $(0,0)$
   c. has no stationary points

6. If the determinant of the Hessian matrix of $f(x, y)$ on a stationary point is negative, then ▸ Answer

   a. the stationary point is a saddle point
   b. the stationary point is a local minimum point
   c. the stationary point is a local maximum

7. If (a, b) is a stationary point of the function $f(x, y)$ such that

$$\frac{\partial^2 f(a,b)}{\partial x^2} = -2 \text{ and } |Hf(a,b)| = 3$$

then ▸ Answer

- a. (a, b) is a local maximum point
- b. (a, b) is a local minimum point
- c. (a, b) is a saddle point

8. If $(2,1)$ is a stationary point of the function $f(x,y)$ such that

$$\frac{\partial^2 f(2,1)}{\partial x^2} = 3 \text{ and } |Hf(2,1)| = 1$$

then Answer

a. $(2,1)$ is a local maximum point
b. $(2,1)$ is a local minimum point
c. $(2,1)$ is a saddle point

9. The Hessian matrix of function $f(x,y,z)$ is

$$Hf(x,y,z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

If the function had a stationary point, this would be ▸ Answer

a. a local maximum point
b. a global maximum point
c. a global minimum point

10. Let $B(x, y)$ be the profit function of a firm which produces two output goods in quantities x and y. If (a, b) is a stationary point of function $B(x, y)$, for it to be a global maximum point it must occur that ▸ Answer

   a. the profit function is concave for all (x, y) in $\mathbb{R}^2$
   b. the profit function is convex for all (x, y) in $\mathbb{R}^2$
   c. the profit function is concave in a neighborhood of the point (a, b)

11. The Hessian matrix of function $f(x, y)$ is given by ▸ Answer

$$Hf(x, y) = \begin{pmatrix} x^2 + 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

If $f(x, y)$ had a stationary point then this point would be

a. a global maximum point
b. a global minimum point
c. a local minimum point that couldn't be global

12. If $(2,1)$ is a stationary point of the function $f(x,y)$, which of the following conditions assures that $(2,1)$ is a global maximum point of the function? Answer

   a. $Hf(2,1)$ is negative definite
   b. $Hf(x,y)$ is negative definite fort all (x,y) in $\mathbb{R}^2$
   c. $Hf(2,1)$ is positive definite

Answers to Problems

Problem 1 Return

- a. $(1, 0)$ is a local minimum point.
- b. $(0, 0)$ is a local minimum point and $(0, 1)$ and $(0, -1)$ are saddle points.
- c. $(0, 1)$ is a local maximum point and $(0, -1)$ is a saddle point.
- d. $(0, 0)$ is a saddle point.
- e. $(-1/2, 0, 0)$ is a saddle point.
- f. $(1/2, 1, 1)$ is a local minimum and $(0, 0, 0)$ is a saddle point.
- g. $(-2, -4)$ is a local maximum.

Problem 2 ▸ Return

The maximum point is $x = 4/9$ and $y = 8/27$.

Problem 3

Return

- a. The maximum point is $x = \left(\frac{p_1^3}{12p_2^2p_3}\right)^2$ and $y = \left(\frac{p_1^2}{6p_2p_3}\right)^3$.
- b. If the price of y rises with other parameters remaining constant, the quantity demanded of input y will decrease in order to maximize profits. By contrast, if the selling price of output rises, the quantity demanded of input y will increase.

Problem 4

Return

$(0,0)$ is a saddle point for all of the values of parameter a.
$\left(-\sqrt[3]{\frac{2}{a}}, -\sqrt[3]{\frac{4}{a^2}}\right)$ is a local minimum point if $a<0$ and, a local maximum point if $a>0$.

Problem 5 ▸ Return

The maximum point is $x = 3,\ y = 4,\ z = 2$ and the maximum profit is $B_{\max} = 11$ m.u.

Problem 6 Return

The maximum point is $x_1 = x_2 = 3$, whose prices are, respectively, $p_1 = 9,\ p_2 = 21$.

Problem 7 Return

The maximum produced quantity is $Q_{\max}(2,4) = 52$ units.

Problem 8 ▸ Return

The maximum profit is $B_{\max}(1.000, 100) = 1.000$ m.u.

Problem 9 ▸ Return

The maximum point is $x = y = 5$ and the maximum profit $B_{\max}(5, 5) = 500$ m.u.

Answers to Multiple choice questions

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