Mathematics for Business Administration: Multivariable Optimization

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Chapter Four: Constrained Optimization. The Lagrange Multiplier Method

Outline

- Introduction
- The Lagrange Multiplier Method (the two-variable case)
- The Lagrange Multiplier is a shadow price
- The Lagrange method applied to the general multivariable case

consumer's optimization problem

$$\max U(x, y) \quad \text{subject to } p \cdot x + y = b. \tag{P}$$

Note that:

- The point (x^*, y^*) that solves problem (P) is not necessarily a maximum point (global or local) of the function $U(x, y)$
- In this case $y = b - p \cdot x$, $\Rightarrow$ Max $f(x) = U(x, b - px)$ *unconstrained optimization problem with one variable less*
- If the substitution method is difficult or impossible to carry out in practise $\Rightarrow$ *Lagrange Method*

The two-variables case: Lagrange function

Definition 1

Given the optimization problem

$$\begin{aligned} Opt. &: \quad f(x,y) \\ s.t &: \quad g(x,y) = b \end{aligned}$$

we define the *Lagrange function* $\mathcal{L}$ by

$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - b)$$

where λ is called the Lagrange multiplier.

The two-variables case: Lagrange function

Lagrange function

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - b)$$

Note that the partial derivatives of $\mathcal{L}(x, y, \lambda)$ with respect to x and y are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x}(x, y, \lambda) &= \frac{\partial f}{\partial x}(x, y) - \lambda \frac{\partial g}{\partial x}(x, y), \\ \frac{\partial \mathcal{L}}{\partial y}(x, y, \lambda) &= \frac{\partial f}{\partial y}(x, y) - \lambda \frac{\partial g}{\partial y}(x, y),\end{aligned}$$

respectively. Moreover,

$$\frac{\partial \mathcal{L}}{\partial \lambda}(x, y, \lambda) = -[g(x, y) - b]$$

which must be 0 when the constraint is satisfied. In fact $\mathcal{L}(x, y, \lambda) = f(x, y)$ for all (x, y) that satisfy the constraint $g(x, y) = b$

First-order necessary conditions for optimality

$$\begin{aligned} Opt. &: \quad f(x,y) \\ s.t &: \quad g(x,y) = b \end{aligned} \tag{1}$$

Theorem (Lagrange)

If (x^*, y^*) is a maximum or a minimum point of problem (1) then there exists a Lagrange multiplier λ^* such that (x^*, y^*, λ^*) is a stationary point of the Lagrange function.

First-order necessary conditions for optimality

The following equalities will be then satisfied:

$$\frac{\partial \mathcal{L}}{\partial x}(x^*, y^*, \lambda^*) = \frac{\partial f}{\partial x}(x^*, y^*) - \lambda^* \frac{\partial g}{\partial x}(x^*, y^*) = 0, \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial y}(x^*, y^*, \lambda^*) = \frac{\partial f}{\partial y}(x^*, y^*) - \lambda^* \frac{\partial g}{\partial y}(x^*, y^*) = 0, \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda}(x^*, y^*, \lambda^*) = -[g(x^*, y^*) - b] = 0. \tag{4}$$

The conditions (2)-(4) are called the *first-order necessary conditions*

The two-variable case with lineal constraint

The general utility maximizing problem with two goods:

$$\text{maximize } U(x,y) \quad \text{subject to} \quad p \cdot x + q \cdot y = b$$

where $U(x,y)$ is concave as stated by economic theory.

Follow the following steps:

- Write the Lagrangian

$$\mathcal{L}(x,y,\lambda) = U(x,y) - \lambda\,[p \cdot x + q \cdot y - b]$$

where λ is a constant

- Differentiate $\mathcal{L}$ with respect to x, y and λ and equate the partial derivatives to 0, the first order necessary conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial U}{\partial x}(x,y) - \lambda p = 0 \quad \Rightarrow \lambda = \frac{\partial U(x,y)/\partial x}{p} \\
\frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial U}{\partial y}(x,y) - \lambda q = 0 \quad \Rightarrow \lambda = \frac{\partial U(x,y)/\partial y}{q} \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= g(x,y) - b = 0
\end{aligned}$$

- Solve these equation simultaneously for the three unknowns x, y and $\lambda \Rightarrow (x^*, y^*, \lambda^*)$
- Since function $U(x,y)$ is concave then the obtained point (x^*, y^*) is a global maximum point (Theorem 3).[1]
- Note that the maximum point (x^*, y^*) satisfies that

$$\frac{\partial U(x^*,y^*)/\partial x}{\partial U(x^*,y^*)/\partial y} = \frac{p}{q} \tag{5}$$

[1] In the case of a convex function, the stationary point would be a global minimum point

Geometric interpretation

A geometric interpretation is that the consumer should choose the point on the budget line at which the slope of the level curve of the utility function is equal to the slope of the budget line

$$\frac{\partial U(x^*, y^*)/\partial x}{\partial U(x^*, y^*)/\partial y} = \frac{p}{q} \qquad (6)$$

Thus, at the optimal point the budget line is tangent to a level curve of the utility function

$$\begin{aligned} Opt. &: \quad f(x,y) \\ s.t &: \quad g(x,y)=b \end{aligned} \tag{7}$$

Theorem (Sufficient conditions for Global Optimality when the constraint is lineal)

If (x^*, y^*, λ^*) is a stationary point of the Lagrange function associated to problem (7), then:
If the constraint is lineal and function f is concave in the feasible set, (x^*, y^*) is a global maximum of problem (7).
If the constraint is lineal and function f is convex in the feasible set, (x^*, y^*) is a global minimum of problem (7).

$$\begin{aligned} Opt. &: f(x,y) \\ s.t &: g(x,y) = b \end{aligned} \tag{8}$$

Theorem (Sufficient conditions for Global Optimality)

If (x^*, y^*, λ^*) *is a stationary point of the Lagrange function associated to problem (8), then:*
If the Lagrangian $\mathcal{L}(x, y, \lambda^*)$ *is concave in* (x, y) *then* (x^*, y^*) *is a global maximum of problem (8).*
If the Lagrangian $\mathcal{L}(x, y, \lambda^*)$ *is convex in* (x, y) *then* (x^*, y^*) *is a global minimum of problem (8).*

$$\begin{aligned} Opt. &: \quad f(x,y) \\ s.t &: \quad g(x,y) = b \end{aligned} \tag{9}$$

Theorem (Sufficient conditions for Local Optimality)

Let (x^*, y^*, λ^*) *be a stationary point of the Lagrange function associated to problem (9). Define*

$$D(x,y,\lambda) = \begin{vmatrix} 0 & \frac{\partial g}{\partial x}(x,y) & \frac{\partial g}{\partial y}(x,y) \\ \frac{\partial g}{\partial x}(x,y) & \frac{\partial^2 \mathcal{L}}{\partial x^2}(x,y) & \frac{\partial^2 \mathcal{L}}{\partial y \partial x}(x,y) \\ \frac{\partial g}{\partial y}(x,y) & \frac{\partial^2 \mathcal{L}}{\partial x \partial y}(x,y) & \frac{\partial^2 \mathcal{L}}{\partial y^2}(x,y) \end{vmatrix}$$

If $\mathcal{D}(x^*, y^*, \lambda^*) < 0$ *then* (x^*, y^*) *is a local maximum of problem (9).*
If $\mathcal{D}(x^*, y^*, \lambda^*) > 0$ *then* (x^*, y^*) *is a local maximum of problem (9).*

The Lagrange multiplier is a shadow price

$$\lambda = \frac{\partial f^*}{\partial b}(b)$$

the Lagrange multiplier λ is the rate at which the optimal value of the objective function changes with respect to changes in the constraint constant b.

Suppose, for instance, that $f^*(b)$ is the maximum profit that a firm can obtain from a production process when b is the available quantity of a resource. Then $\partial f^*(b)/\partial b$ is the marginal profit that the firm can earn per extra unit of the resource, which is therefore the firm's marginal willingness to pay for this resource.

In Economics this measure is known as the shadow price.

The Lagrange multiplier is a shadow price

Proof.

Let (x^*, y^*, λ^*) be a stationary point of the Lagrange function, then, the first order necessary conditions must be satisfied, that is

$$\frac{\partial \mathcal{L}}{\partial x}(x^*, y^*, \lambda^*) = \frac{\partial f}{\partial x}(x^*, y^*) - \lambda^* \frac{\partial g}{\partial x}(x^*, y^*) = 0, \quad (10)$$

$$\frac{\partial \mathcal{L}}{\partial y}(x^*, y^*, \lambda^*) = \frac{\partial f}{\partial y}(x^*, y^*) - \lambda^* \frac{\partial g}{\partial y}(x^*, y^*) = 0. \quad (11)$$

Note that $x^* = x(b)$ and $y^* = y(b)$. Let

$$f^*(b) = f(x^*, y^*) = f(x(b), y(b))$$

be the optimum (maximun or minimum) value function, which is a function of b.
Using the change rule and (13)-(14), the following arises

$\square$

Proof.

$$\begin{aligned}
\frac{\partial f^*}{\partial b}(b) &= \frac{\partial f}{\partial x}(x^*, y^*)\frac{\partial x}{\partial b}(b) + \frac{\partial f}{\partial y}(x^*, y^*)\frac{\partial y}{\partial b}(b) \\
&= \lambda^* \frac{\partial g}{\partial x}(x^*, y^*)\frac{\partial x}{\partial b}(b) + \lambda^* \frac{\partial g}{\partial y}(x^*, y^*)\frac{\partial y}{\partial b}(b) \\
&= \lambda^* \left[\frac{\partial g}{\partial x}(x^*, y^*)\frac{\partial x}{\partial b}(b) + \frac{\partial g}{\partial y}(x^*, y^*)\frac{\partial y}{\partial b}(b)\right]
\end{aligned}$$

Moreover,

$$g(x(b), y(b)) = g(x^*, y^*) = b$$

then

$$\frac{\partial g}{\partial x}(x(b), y(b))\frac{\partial x}{\partial b}(b) + \frac{\partial g}{\partial y}(x(b), y(b))\frac{\partial y}{\partial b}(b) = 1.$$

Which implies that

$$\frac{\partial f^*}{\partial b}(b) = \lambda^*$$

The Lagrange method applied to the general multivariable case.

Definition 2

Given the problem

$$\begin{aligned} Opt. &: \quad f(x_1, x_2, \cdots, x_n) \\ s.t. &: \quad g(x_1, x_2, \cdots, x_n) = b \end{aligned} \tag{12}$$

we define the Lagrange function, or Lagrangian, by

$$\mathcal{L}(x_1, x_2, \cdots, x_n, \lambda) = f(x_1, x_2, \cdots, x_n) - \lambda(g(x_1, x_2, \cdots, x_n) - b)$$

where λ is called Lagrange multiplier.

Theorem (Lagrange)

If $\mathbf{x}^* = (x_1^*, x_2^*, \cdots, x_n^*)$ is a maximum (or minimum) point of the problem with one equality restriction (12), then there exists one Lagrange multiplier λ^* such that $(x_1^*, x_2^*, \cdots, x_n^*, \lambda^*)$ is a stationary point of the Lagrange function. That is

$$\frac{\partial \mathcal{L}}{\partial x_j}(x_1^*, x_2^*, \cdots, x_n^*, \lambda^*) = \frac{\partial f}{\partial x_j}(x_1^*, x_2^*, \cdots, x_n^*) - \lambda^* \frac{\partial g}{\partial x_j}(x_1^*, x_2^*, \cdots, x_n^*) = 0 \tag{13}$$

for all $\quad j = 1, .2, ..., n \quad$ *and*

$$\frac{\partial \mathcal{L}}{\partial \lambda}(x_1^*, x_2^*, \cdots, x_n^*, \lambda^*) = g(x_1^*, x_2^*, \cdots, x_n^*) - b = 0 \tag{14}$$

Theorem (Sufficient conditions for Global Optimality when the constraint is lineal)

If $(x_1^*, x_2^*, \cdots, x_n^*, \lambda^*)$ *is a stationary point of the Lagrange function associated to problem (12), then*
If the constraint is lineal and function f *is concave in the feasible set,* $(x_1^*, x_2^*, \cdots, x_n^*)$ *is a global maximum of problem (12).*
If the constraint is lineal and function f *is convex in the feasible set,* $(x_1^*, x_2^*, \cdots, x_n^*)$ *is a global minimum of problem (12).*

Theorem (Sufficient conditions for Global Optimality)

If $(x_1^*, x_2^*, \cdots, x_n^*, \lambda^*)$ *is a stationary point of the Lagrange function associated to problem (12), then*

If the Lagrangian $\mathcal{L}(x_1, x_2, \cdots, x_n, \lambda^*)$ *is concave in* $(x_1, x_2, \cdots, x_n)$ *then* $(x_1^*, x_2^*, \cdots, x_n^*)$ *is a global maximum of problem (12).*

If the Lagrangian $\mathcal{L}(x_1, x_2, \cdots, x_n, , \lambda^*)$ *is convex in* $(x_1, x_2, \cdots, x_n,)$ *then* $(x_1^*, x_2^*, \cdots, x_n^*)$ *is a global minimum of problem (12).*

Links to the Wolfram Demostrations Project web page

- Constrained Optimization >>

Bibliography

Sydsaeter,K., Hammond, P.J., Seierstad, A. and Strom, A. Essential Mathematics for Economic Analysis. Prentice Hall. New Jersey. pages: 489-494. >>

Problem 1 Answer

Solve the following problems using the substitution method and also the Lagrange multipliers method (the understanding of the problems can be improved using a graphical resolution approach).

a. Min. $f(x, y) = (x-1)^2 + y^2$ subject to $y - 2x = 0$.
b. Max. $f(x, y) = xy$ subject to $2x + 3y = 6$.
c. Opt. $f(x, y) = 2x + 3y$ subject to $xy = 6$.

Problem 2 Answer

Solve the following problems:

a. Opt. $f(x,y,z) = x^2 + (y-2)^2 + (z-1)^2$ subject to $4x + y + 4z = 39$.
b. Opt. $f(x,y) = e^{xy}$ subject to $x^2 + y^2 = 8$.
c. Opt. $f(x,y) = x^{1/4}y^{1/2}$ subject to $x + 2y = 3$.
d. Opt. $f(x,y) = \ln(xy)$ subject to $x^2 + y^2 = 8$.

Problem 3 ▸ Answer

A multinational refreshments firm has 68 monetary units available to produce the maximum possible number of bottles. Its production function is $q(x, y) = 60x + 90y - 2x^2 - 3y^2$ where x and y are the required inputs. The inputs prices are $p_x = 2$ m.u. and $p_y = 4$ m.u. repectively. Given the budget restriction, maximize the production of bottles. By means of the Lagrange multiplier, how will the maximum number of bottles produced be modified if the budget is increased in one unit (or if it is decreased)?

Problem 4

Answer

A worker earns 20 monetary units for each labor hour. The worker's utility, $U(x, y) = x^{1/3}y^{1/3}$, depends on the consumption of goods, x, and also on the free time, y. Knowing that each unit of consumption costs 80 m.u, and that the worker does not save any of the earned money for the future, find the values of x and y that maximize his utility.

Problem 5 Answer

A European research program has 600 thousand euros available to finance research projects on renewable energies. Two teams present their projects and their estimated incomes (derived from the property rights of new discoveries) are given by $I_1(x) = 2x^{1/2}$ and $I_2(y) = \frac{4}{3}y^{3/4}$ where x is the monetary assignation to the first team (in hundreds of thousands of euros) and y is second team s assignation. The program seeks to determine the optimal distribution of quantities x and y to maximize the joint income. Formulate and solve the problem. What happens to the maximum joint income if the budget is increased by 50 thousand euros? Is it worth it?

Problem 6

Answer

A firm's output production function, $f(K, L) = 4(K+1)^{1/2}(L+1)^{1/2}$, depends on the employed capital and labor. Its costs function is $C(K, L) = 2K + 8L$. Find the optimal values for K and L which minimize the cost of producing 32 units of output. If the production increased by one unit, what would be the effect on the cost?

Problem 7 ▸ Answer

The output of an industry depends on a sole resource whose quantity is limited to b and it is mandatory to use it up. There are two production processes available for which the resource must be distributed. The derived incomes from each one of the productions processes are

$$f(x) = 1200 - \left(\frac{x}{2} - 12\right)^2 \quad g(y) = 1400 - (y-1)^2$$

where x and y are the employed resource in each production process.

a. How can the distribution between x and y be done so as to maximize the total income?
b. Assuming that $b = 22$ and that there is the possibility of using one additional unit of the resource with a cost of 0,8 m.u, Is is worth it? And, is it worth it if $b = 28$?

Problem 8 Answer

The function $U(x, y) = 100x + xy + 100y$ represents a representative consumer's utility depending on the consumption of two goods, x and y. Knowing that the consumer spends her whole income, 336 monetary units, purchasing these goods at prices $p_x = 8$ m.u. and $p_y = 4$ m.u respectively, maximize the consumer's utility.

Problem 9 Answer

The costs function of a firm is:

$$C(x,y) = (x-1)^2 + 6y + 8$$

where x and y are the quantities of the two productive inputs needed to produce. If $Q(x,y) = (x-1)^2 + 3y^2$ is the output production function, find the input quantities to produce 12 units of product at the minimum cost.

1. The Lagrange function associated with the problem Answer

$$Opt. f(x, y, z) \text{ subject to } g(x, y, z) = c \text{ is}$$

a. $\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - b)$
b. $\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda(g(x, y) + b)$
c. $\mathcal{L}(x, y, \lambda) = g(x, y) - \lambda(f(x, y) - b)$

2. The maximum production of a firm is 500 units of a certain good and the shadow price of the available resource is 3. What would be the effect on the maximum production level if the resource were increased by one unit? Answer

   a. The maximum production level would not be affected
   b. The maximum production level would reduce by 3 units
   c. The maximum production level would increase by 3 units

3. Given the optimization problem Answer

$$\min f(x,y) \text{ subject to } 3x-6y=9. \qquad \text{(P)}$$

If $(x,y,\lambda)=(1,-1,3)$ is a stationary point of the associated Lagrange function, it can be assured that $(1,-1)$ is a global minimum of problem (P) when the function $f(x,y)$ is

- a. convex
- b. concave
- c. neither convex nor concave

4. Given the following optimization problem ▸ Answer

$$\min f(x,y) \text{ subject to } x^2 + y = 5. \tag{P}$$

Let $(x, y, \lambda) = (1, 4, 3)$ be is a stationary point of the associated Lagrange function $\mathcal{L}(x, y, \lambda)$. Then, if the Hessian matrix of function $\mathcal{L}(x, y, 3)$ is positive semidefinite then $(1, 4)$ is a

a. is a global maximum point of problem (P)
b. is a global minimum point of problem (P)
c. It can't be assured that it is a global extreme point for problem (P)

5. The Hessian matrix of the Lagrange function $\mathcal{L}(x, y, z, \lambda^*)$ is given by Answer

$$H(\mathcal{L}(x, y, z, \lambda^*)) = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & -4 \end{pmatrix},$$

then, a stationary point is a

a. global maximum
b. global minimum
c. neither of the above

6. Given the optimization problem

$$Opt. f(x,y,z) \text{ subject to } g(x,y,z) = c$$

and given $(1,2,3,4)$, a stationary point of the Lagrange function ($\lambda = 4$ is the Lagrange multiplier) if the Hessian matrix of $\mathcal{L}(x,y,z,4)$ is

$$H\mathcal{L}(x,y,z,4) = \begin{pmatrix} x^2+1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4y^2+3 \end{pmatrix}.$$

then ▸ Answer

a. the problem has no solution
b. point $(1,2,3)$ is a global minimum point
c. $(1,2,3)$ is a global maximum point

7. Given the optimization problem

$$Opt. f(x, y, z) \text{ subject to } g(x, y, z) = c,$$

it is known that the Hessian matrix of the Lagrange function when $\lambda = 4$ is given by

$$H\mathcal{L}(x, y, z, 4) = \begin{pmatrix} -x^2 - 7 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

Then, if $(1, 2, 3, 4)$ is a stationary point of the Lagrange function, Answer

a. the problem has no solution
b. the problem has a global maximum point
c. the problem has a global minimum point

8. Given the optimization problem

$$Opt. f(x,y) \text{ subject to } g(x,y) = c,$$

it is known that the Hessian matrix of the Lagrange function when $\lambda = 4$ is given by

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Then, if $(1,2,4)$ is a stationary point of the Lagrange function, ▸ Answer

a. the minimum value of the objective function is 4
b. $(1,2)$ is a global minimum point
c. $(1,2)$ is a global maximum point

9. In the maximization of profits with a linear constraint on costs $x + y + z = 89$, the Lagrange multiplier is $-0,2$. Is it worth increasing the level of cost? Answer
   a. No. The maximum profit would decrease
   b. Yes, since the maximum profit would increase
   c. Yes, because we would continue with positive profits

Answers to Problems

Problem 1 Return

a. The problem has a global minimum at $x^* = 1/5$ and $y^* = 2/5$ whose value is $20/25$.
b. $(3/2, 1)$ is a global maximum of value $3/2$.
c. $(3, 2)$ is a local minimum point of value 12 and $(-3, -2)$ is a local maximum point of value -12.

Problem 2 Return

a. Global minimum at $x = 4$, $y = 3$ and $z = 5$ of value 33.
b. $(2, 2)$ and $(-2, -2)$ are two global maximum points of value e^4. $(2, -2)$ and $(-2, 2)$ are two global minimum of value e^{-4}.
c. Global maximum at $x = y = 1$ of value 1.
d. Two local maximum points at $(2, 2)$ and $(-2, -2)$ of value $\ln 4$.

Problem 3 Return

The global maximum point is at $x = 12$ and $y = 11$ with a maximum quantity of 1059 bottles.
If the budget is increased by 1 monetary unit, the maximum production would increase by 6 units (approximately). Similarly, if the budget is reduced, the production would reduce by 6 units (approximately)

Problem 4 ▸ Return

$x = 3$ and $y = 12$.

Problem 5 Return

The global maximum is obtained when the first project is assigned 200 thousand euro and the second project with 400 thousand euro. The maximum income will be of 659.970 euro.
An increase of 50 thousand euro will increase the maximum income by 35.355 approximately. It is not worth it.

Problem 6 Return

$K = 15$ and $L = 3$. An increase of one unit in production would increase the minimum cost by 2 units (approximately)

Problem 7 Return

a. $x = \dfrac{4b}{5} + 4$ and $y = \dfrac{b}{5} - 4$.

b. It is worth it in the first case but not in the second.

Problem 8 Return

$y = 84$ and $x = 0$.

Problem 9 Return

$(1 + 2\sqrt{3}, 0)$ and $(1, 2)$ are two minimum points of value 20..

Answers to Multiple choice questions

1. The Lagrange function associated with the problem Back

$$Opt. f(x,y,z) \text{ subject to } g(x,y,z) = c \text{ is}$$

   a. $\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - b)$
   b. $\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda(g(x,y) + b)$
   c. $\mathcal{L}(x,y,\lambda) = g(x,y) - \lambda(f(x,y) - b)$

2. The maximum production of a firm is 500 units of a certain good and the shadow price of the available resource is 3. What would be the effect on the maximum production level if the resource were increased by one unit? Back
    a. The maximum production level would not be affected
    b. The maximum production level would reduce by 3 units
    c. The maximum production level would increase by 3 units

3. Given the optimization problem Back

$$\min f(x,y) \text{ subject to } 3x - 6y = 9. \qquad \text{(P)}$$

If $(x, y, \lambda) = (1, -1, 3)$ is a stationary point of the associated Lagrange function, it can be assured that $(1, -1)$ is a global minimum of problem (P) when the function $f(x, y)$ is

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b. concave
c. neither convex nor concave

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then Back

- a. the problem has no solution
- b. point $(1,2,3)$ is a global minimum point
- c. $(1,2,3)$ is a global maximum point

7. Given the optimization problem

$$Opt. f(x, y, z) \text{ subject to } g(x, y, z) = c,$$

it is known that the Hessian matrix of the Lagrange function when $\lambda = 4$ is given by

$$H\mathcal{L}(x, y, z, 4) = \begin{pmatrix} -x^2 - 7 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

Then, if $(1, 2, 3, 4)$ is a stationary point of the Lagrange function, Back

a. the problem has no solution
b. the problem has a global maximum point
c. the problem has a global minimum point

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it is known that the Hessian matrix of the Lagrange function when $\lambda = 4$ is given by

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Then, if $(1, 2, 4)$ is a stationary point of the Lagrange function, Back

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