Mathematics for Business Administration: Multivariable Optimization

Universidad de Murcia

María Pilar Martínez-García

These slides are intended for students of Business administration whose mathematical requirements go beyond the calculus for functions of one variable. The material includes a basic course on multivariable optimization problems, with and without constraints, and the tools of linear algebra needed for solving them.

Contents

Chapter One: Convex Sets. Convex and Concave Functions

Return to Contents

Outline

- Convex Sets
- Convex and Concave Functions

Definition 1

Let $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$ be any two points in $\mathbb{R}^n$. The closed line segment between $\mathbf{x}$ and $\mathbf{y}$ is the set

$$[\mathbf{x}, \mathbf{y}] = \{\mathbf{z} \ \ / \ \ \text{there exists } \lambda \in [0,1] \text{ such that } \mathbf{z} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}\}$$

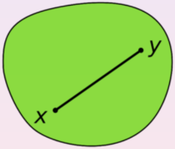
Definition 2

A set S in $\mathbb{R}^n$ is called *convex* if $[\mathbf{x}, \mathbf{y}] \subseteq S$ for all $\mathbf{x}$, $\mathbf{y}$ in S, or equivalently, if

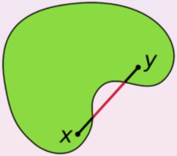
$$\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in S \text{ for all } \mathbf{x},\ \mathbf{y} \text{ in } S \text{ and all } \lambda \in [0,1]$$

Note in particular that the empty set and also any set consisting of one single point are convex.

Intuitively speaking, a convex set must be "connected" without any "holes" and its boundary must not "bend inwards" at any point.



Convex



Not convex

Definition 3

A *hyperplane* in $\mathbb{R}^n$ is the set H of all points $\mathbf{x} = (x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$ that satisfy

$$p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = m$$

where $\mathbf{p} = (p_1, p_2, ..., p_n) \neq \mathbf{0}$.

Proposition 4 A hyperplane in $\mathbb{R}^n$ is a convex set.

Definition 5

A hyperplane H devides $\mathbb{R}^n$ into two sets,

$$\begin{aligned} H_+ &= \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n / \quad p_1x_1 + p_2x_2 + \cdots + p_nx_n \geq m\}, \\ H_- &= \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n / \quad p_1x_1 + p_2x_2 + \cdots + p_nx_n \leq m\}, \end{aligned}$$

which are called half spaces.

Proposition 6 H_+ and H_- are convex sets.

Proposition 7 If S and T are two convex sets in $\mathbb{R}^n$, then their intersection $S \cap T$ is also convex.

The union of convex sets is usually not convex.

Definition 8

A function $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$ defined on a convex set S is *concave* on S if

$$f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) \geq \lambda f(\mathbf{x}) + (1 - \lambda) f(\mathbf{y}) \tag{1}$$

for all $\mathbf{x}$ and $\mathbf{y}$ in S and for all λ in $[0, 1]$.

A function $f(\mathbf{x})$ is *convex* if (1) holds with $\geq$ replaced by $\leq$. Note that (1) holds with equality for $\lambda = 0$ and $\lambda = 1$. If we have *strict* inequality in (1) whenever $\mathbf{x} \neq \mathbf{y}$ and $\lambda \in (0,1)$, then f is *strictly concave*.

Note that a function f is *convex* on S if and only if $-f$ is concave. Furthermore, f is *strictly convex* if and only if $-f$ is strictly concave.

Proposition 9 Let $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$ be defined on a convex set S in $\mathbb{R}^n$. Then

- If f is concave, the set $\{\mathbf{x} \in \mathbf{S} \,/\, f(\mathbf{x}) \geq a\}$ is convex for every number a.
- If f is convex, the set $\{\mathbf{x} \in \mathbf{S} \,/\, f(\mathbf{x}) \leq a\}$ is convex for every number a.

Definition 10

Suppose that $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$ is a $\mathcal{C}^2$ function in an open convex set S in $\mathbb{R}^n$. Then the symmetric matrix

$$\mathcal{H}(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{x}) & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{x}) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_2^2}(\mathbf{x}) & \dots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(\mathbf{x}) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_n \partial x_2}(\mathbf{x}) & \dots & \frac{\partial^2 f}{\partial x_n^2}(\mathbf{x}) \end{pmatrix}$$

is called the *Hessian matrix* of f at $\mathbf{x}$.

Proposition 11 Suppose that $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$ is a $\mathcal{C}^2$ function defined on an open, convex set S in $\mathbb{R}^n$. Then
(i) The *Hessian matrix* is positive definite or semidefinite $\Leftrightarrow$ f is convex.
(ii) The *Hessian matrix* is negative definite or semidefinite $\Leftrightarrow$ f is concave.

Proposition 12 Suppose that $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$ is a $\mathcal{C}^2$ function defined on an open, convex set S in $\mathbb{R}^n$. Then
(i) If the *Hessian matrix* is positive definite $\Rightarrow$ f is strictly convex.
(ii) If the *Hessian matrix* is negative definite $\Rightarrow$ f is strictly concave.

Links to the Wolfram Demostrations Project web page

- Convex sets >>
- Operations on sets >>
- Concavity and convexity in quadratic surfaces >>

Some pictures of convex sets in Wikipedia >>

Bibliography

Sydsaeter,K., Hammond, P.J., Seierstad, A. and Strom, A. Further Mathematics for Economic Analysis. Prentice Hall. New Jersey. pages: 50-62. >>

Return to Contents

Problem 1 Answer

Draw the following sets and say if they are convex, closed and bounded.

a. $A = \{(x,y) \in \mathbb{R}^2 / x^2 + y^2 \leq 4\}$
b. $B = \{(x,y) \in \mathbb{R}^2 / y = 2x + 3\}$
c. $C = \{(x,y) \in \mathbb{R}^2 / (x-1)^2 + (y-3)^2 = 9\}$
d. $D = \{(x,y) \in \mathbb{R}^2 / y > x^2, y \leq 1\}$
e. $E = \{(x,y) \in \mathbb{R}^2 / y \geq x\}$
f. $F = \{(x,y) \in \mathbb{R}^2 / x + y \leq 2, x \geq 0, y \geq 0\}$
g. $G = \{(x,y) \in \mathbb{R}^2 / xy \leq 1\}$
h. $H = \{(x,y) \in \mathbb{R}^2 / xy > 1, x \geq 0, y \geq 0\}$

Problem 2

Investigate the convexity of the following sets

a. $A = \{(x, y) \in \mathbb{R}^2 / 0 \leq x \leq 4, 2 \leq y \leq 6\}$
b. $B = \{(x, y, z) \in \mathbb{R}^3 / x + y + 2z \leq 24\}$
c. $C = \{y \in \mathbb{R}^n / y = \alpha x \quad \text{with } \alpha \in \mathbb{R} \text{ and } \ x \in X \subset \mathbb{R}^n \text{ convex}\}$

Problem 3 ▸ Answer

Investigate the concavity/convexity for the following functions

a. $f(x, y) = 3x^3 - 2y^2$
b. $f(x, y) = (x - 3)^3 + (y + 1)^2$
c. $f(x, y) = (x - 2)^2 + y^4$
d. $f(x, y, z) = x^2 + y^2 + z^3$
e. $f(x, y, z) = x^2 + y^2 + z^2 + yz$
f. $f(x, y, z) = e^x + y^2 + z^2$
g. $f(x, y, z) = e^{2x} + y^2 z$
h. $f(x, y) = xy$

Problem 4

Answer

Check the concavity/convexity of the following functions

a. $f(x,y) = \ln y - e^x$
b. $f(x,y) = \ln xy$ for all $x, y > 0$
c. $f(x,y) = \sqrt{x^2 + y^2}$
d. $f(x,y) = x^{\frac{1}{2}} y^{\frac{1}{3}}$ for all $x, y > 0$

Problem 5 ▸ Answer

Check the concavity/convexity of the following functions for the different values of parameter a.

- a $f(x, y) = x^2 - 2axy$
- b $g(x, y, z) = ax^4 + 8y - z^2$

Problem 6 Answer

Investigate the convexity of the following sets

a. $A = \{(x, y) \in \mathbb{R}^2 / (x-1)^2 + (y-1)^2 \leq 2\}$
b. $B = \{(x, y) \in \mathbb{R}^2 / e^{x+y} \leq 12\}$
c. $C = \{(x, y) \in \mathbb{R}^2 / 3x^2 + 4y^2 \geq 10\}$
d. $D = \{(x, y) \in \mathbb{R}^2 / x + y \leq 2, x \geq 0, y \geq 1\}$
e. $E = \{(x, y) \in \mathbb{R}^2 / x^2 + y^2 - 4x - 2y \leq 3, x \leq 2y\}$
f. $F = \{(x, y) \in \mathbb{R}^2 / x + y \leq 3, 2x + 5y = 10, x \geq 0, y \geq 0\}$

1. Which of the following sets is convex? Answer
   a. $\{(x, y) \in \mathbb{R}^2 / x^2 + y^2 \leq 1\}$
   b. $\{(x, y) \in \mathbb{R}^2 / x^2 + y^2 = 1\}$
   c. $\{(x, y) \in \mathbb{R}^2 / x^2 + y^2 \geq 1\}$

2. The closed line segment between $(1,1)$ and $(-1,-1)$ can be written as the set ▸ Answer

   a. $B = \{(x,y) \in \mathbb{R}^2 / (x,y) = (2\lambda - 1, 2\lambda - 1), \forall \lambda \in [0,1]\}$
   b. $B = \{(x,y) \in \mathbb{R}^2 / (x,y) = (\lambda, 1-\lambda), \forall \lambda \in [0,1]\}$
   c. $B = \{(x,y) \in \mathbb{R}^2 / x = y\}$

3. Given $S \subseteq \mathbb{R}^2$ a convex set, the function $f : S \to \mathbb{R}$ will be convex if Answer

   a. the Hessian matrix $Hf(x, y)$ is negative definite for all (x, y) in S
   b. the sets $\{(x, y) \in S / f(x, y) \leq k\}$ are convex for all k in $\mathbb{R}$
   c. f is a lineal function

4. The set $S = \{(x, y, z) \in \mathbb{R}^3 / x + y^2 + z^2 \leq 1\}$ Answer
   a. is convex because the Hessian matrix of the function $f(x, y) = x + y + z^2$ is positive semidefinite
   b. is convex because the function $f(x, y) = x + y^2 + z^2$ is lineal
   c. in not convex

5. Which of the following sets is not convex? Answer

   a. $\{(x, y) \in \mathbb{R}^2 / x \leq 1, y \leq 1\}$
   b. $\{(x, y) \in \mathbb{R}^2 / x, y \in [0, 1]\}$
   c. $\{(x, y) \in \mathbb{R}^2 / xy \leq 1,\ x, y \geq 0\}$

6. Which of the following Hessian matrices belongs to a concave function? Answer

   a. $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$
   b. $\begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}$
   c. $\begin{pmatrix} -2 & 2 \\ 2 & 2 \end{pmatrix}$

7. The function $f(x,y) = \ln x + \ln y$ is concave on the set

Answer

   a. $S = \{(x,y) \in \mathbb{R}^2 / x, y > 0\}$
   b. $\mathbb{R}^2$
   c. $S = \{(x,y) \in \mathbb{R}^2 / x, y \neq 0\}$

8. Which of the following sets is convex? Answer

   a. $A = \{(x, y) \in \mathbb{R}^2 / xy \geq 1, x \geq 0, y \geq 0\}$
   b. $B = \{(x, y) \in \mathbb{R}^2 / xy \geq 1\}$
   c. $C = \{(x, y) \in \mathbb{R}^2 / xy \leq 1, x \geq 0, y \geq 0\}$

Chapter Two: Multivariate Optimization. The Extreme value theorem

Return to Contents

Outline

- Multivariable optimization. The Extreme Value Theorem (Weierstrass)
- A graphical approach to two-variable optimization problems

An *Optimization Problem* is the problem of finding those points in a domain where a function reaches its largest and its smallest values (referred to as maximum and minimum points):

$$\max \ (\min) \ f(\mathbf{x}) \quad \text{subject to } \mathbf{x} \in \mathbf{S}$$

where $\max$ $(\min)$ indicates that we want to maximize or minimize f and $\mathbf{x} = (x_1, ..., x_n) \in S$ a subset of $\mathbb{R}^n$.

In most static optimization problems there are

- an *objective function* $f(\mathbf{x}) = f(x_1, x_2, ..., x_n)$, a real-valued function of n variables whose value is to be optimized, i.e. maximized or minimized.
- an *admissible set* (or *feasible set*) S that is some subset of $\mathbb{R}^n$.

Depending on the set S, several different types of optimization problems can arise:

- *Classical case*: if the optimum occurs at an interior point of S (Chapter 5)
- *Lagrange problem*: if S is the set of all points $\mathbf{x}$ that satisfy a given system of equations (equality constraints) (Chapter 6)
- *Nonlinear programming problem*: if S consists of all points $\mathbf{x}$ that satisfy a system of inequality constraints

Definition 13

The point $\mathbf{x}^* \in S$ is called a *(global) maximum point* for f in S if

$$f(\mathbf{x}^*) \geq f(\mathbf{x}) \qquad \text{for all } \mathbf{x} \text{ in } S,$$

and $f(\mathbf{x}^*)$ is called the *maximum value*.

Definition 14

The point $\mathbf{x}^* \in S$ is called a *(global) minimum point* for f in S if

$$f(\mathbf{x}^*) \leq f(\mathbf{x}) \qquad \text{for all } \mathbf{x} \text{ in } S,$$

and $f(\mathbf{x}^*)$ is called the *minimum value*.

If the inequalities are strict then $\mathbf{x}^*$ is called a *strict maximum (minimum) point* for f in S.

Extreme Value Theorem: Weiestrass

Theorem

Let $f(\mathbf{x})$ be a continuous function on a closed, bounded set S. Then f has both a maximum point and a minimum point in S.

A set S is called *closed* if it contains all its boundary points. Moreover, it is called *bounded* if it is contained in some ball around the origin.

Example

In problems with two variables, if $g(x_1, x_2)$ is a continuous function and c is a real number, the sets

$$\{(x,y)\,/\,g(x_1,x_2) \geq c\} \quad \{(x,y)\,/\,g(x_1,x_2) \leq c\} \quad \{(x,y)\,/\,g(x_1,x_2) = c\}$$

are closed. If $\geq$ is replaced by $>$, or $\leq$ replaced by $<$, or $=$ replaced by $\neq$, then the corresponding set is not closed.

Example

Provided that p, q and m are positive parameters, the (budget) set of points (x, y) that satisfy the inequalities

$$px + qy \leq m, \quad x \geq 0, \quad y \geq 0$$

is closed and bounded.

Given the general maximizing/minimizing problem with two variables

$$\text{maximize } f(x,y) \text{ subject to } (x,y) \text{ in } S$$

a graphical resolution can be done by drawing the feasible set and the level curves of the objective function.

For a graphical resolution follow the following steps:

1. Draw the feasible set
2. Draw the level curves of the objective function which lie in the feasible set
3. In the case of a maximization problem, the maximum point is the feasible point which lies on the highest level curve
   In a minimization problem, the minimum point is the feasible point which lies on the lowest level curve

Links to the Wolfram Demostrations Project web page

- The consumer's optimization problem >>
- Level curves >>
- Surfaces and level curves >>

Bibliography

Sydsaeter,K., Hammond, P.J., Seierstad, A. and Strom, A. Essential Mathematics for Economic Analysis. Prentice Hall. New Jersey. pages: 474-475 and 494 (for a graphical approach) >>

Sydsaeter,K., Hammond, P.J., Seierstad, A. and Strom, A. Further Mathematics for Economic Analysis. Prentice Hall. New Jersey. pages: 103-106 >>

Return to Contents

Problem 1

Provide a graphical resolution of the following optimization problems:

a
$$\begin{cases} \max. : 6x + y \\ s.t. : 2x + y \leq 6 \\ \qquad x + y \geq 1 \\ \qquad y \leq 3 \\ \qquad x, y \geq 0 \end{cases}$$

b
$$\begin{cases} opt. : x + y \\ s.t. : x^2 + y^2 = 1 \\ \qquad x, y \geq 0 \end{cases}$$

Problem 1

Provide a graphical resolution of the following optimization problems:

c $$\begin{cases} opt. : (x-2)^2 + (y-1)^2 \\ s.t. : x^2 - y \leq 0 \\ \qquad x + y \leq 2 \\ \qquad x, y \geq 0 \end{cases}$$

d $$\begin{cases} opt. : x - y^2 \\ s.t. : (x-1)(y-2) \geq 0 \\ \qquad 2 \leq x \leq 4 \end{cases}$$

Problem 1

Provide a graphical resolution of the following optimization problems:

- e $\begin{cases} opt. : 3x + 2y \\ s.t. : -x + y \leq 2 \\ \qquad x - y \leq 2 \end{cases}$
- f $\begin{cases} opt. : (x-2)^2 + (y-2)^2 \\ s.t. : x + y \geq 1 \\ \qquad -x + y \leq 1 \end{cases}$

Problem 1

Provide a graphical resolution of the following optimization problems:

g $$\begin{cases} \min . : x + y \\ s.t. : x^2 + y^2 \geq 4 \\ \qquad x^2 + y^2 \leq 1 \end{cases}$$

h $$\begin{cases} \max . : x + y \\ s.t. : x - y^2 \geq 0 \\ \qquad x + y \leq 2 \end{cases}$$

Problem 2 Answer

A firm produces two goods. The profit obtained after the purchase of each are 10 and 15 monetary units respectively. To produce one unit of good 1 requires 4 hours of man-labor and 3 hours of machine work. Each unit of good 2 needs 7 hours of man-labor and 6 hours of machine work. The maximum man-labor time available is 300 hours and for the machines 500 hours. Find the quantities produced of each good which maximize the profit.

Problem 3 ▸ Answer

Maximize the utility function $U(x, y) = xy$, where x and y are the quantities consumed of two goods. The price of each unit of these goods is 2 and 1 monetary units respectively and the available budget is 100 monetary units. Formulate the optimization problem the consumer must solve in order to achieve the maximum utility. Calculate the optimal consumed quantities of goods x and y.

Problem 4 Answer

Which of the following optimization problems satisfy the Weierstrass' theorem conditions?

a) $\begin{cases} \min.: x^2 + y^2 \\ s.t.: x + y = 3 \end{cases}$

b) $\begin{cases} \text{max.}: x + y^2 \\ s.t.: 3x^2 + 5y \leq 4 \\ \quad x, y \geq 0 \end{cases}$

c) $\begin{cases} opt.: 2x + y \\ s.t.: x + y = 1 \\ \quad x^2 + y^2 \leq 9 \end{cases}$

d) $\begin{cases} opt.: x + \ln y \\ s.t.: x - 5y^2 \geq -1 \\ \quad x + y^2 \leq 1 \end{cases}$

e) $\begin{cases} \max.: x^2 + y^2 \\ s.t.: x + y \geq 4 \\ \quad 2x + y \geq 5 \\ \quad x, y \geq 0 \end{cases}$

f) $\begin{cases} \min.: e^{x+y} \\ s.t.: 0 \leq x \leq 1 \\ \quad 0 \leq y \leq 1 \end{cases}$

1. Which of the following points belongs to the feasible set of the optimization problem ▸ Answer

$$\begin{aligned} opt. &: x^2\sqrt{y} \\ s.t. &: x+y=3 \text{ ?} \end{aligned}$$

a. $(1,2)$
b. $(-1,2)$
c. $(4,-1)$

2. If the feasible set of an optimization problem is unbounded then ▸ Answer

   a. no finite optimum point exists
   b. it has an infinite number of feasible points
   c. the existence of a finite optimum point cannot be assured

3. Given $f(x, y) = ax + by$ with $a, b \in \mathbb{R}$ and the set Answer

$$S = \{(x, y) \in \mathbb{R}^2 / x + y = 2, x \geq 0, y \geq 0\},$$

- a. f has a global maximum point and a global minimum point in S
- b. f has a global maximum point in S if a and b are positive
- c. there is no maximum or minimum point of f in S

4. Which of the following is the feasible set of the optimization problem Answer

$$\begin{aligned} \max. \quad &: \quad 2x + y \\ s.t. \quad &: \quad x + y = 1 \\ &: \quad x^2 + y^2 \leq 5 \text{ ?} \end{aligned}$$

a. $\{(x, y) \in \mathbb{R}^2 / x + y = 1, x \geq 0, y \geq 0\}$
b. $\{(x, y) \in \mathbb{R}^2 / (x, y) = \lambda(5, 0) + (1 - \lambda)(0, 5), \forall \lambda \in [0, 1]\}$
c. $\{(x, y) \in \mathbb{R}^2 / (x, y) = \lambda(2, -1) + (1 - \lambda)(-1, 2), \forall \lambda \in [0, 1]\}$

Chapter Three: Classical Optimization

Return to Contents

Outline

- Extreme points
- Local extreme points

Let f be defined on a set S in $\mathbb{R}^n$ then

Definition 15

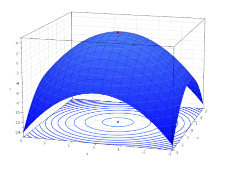
A point $\mathbf{x}^* = (x_1^*, x_2^*, ..., x_n^*)$ is called a *stationary point* of f if all first-order partial derivatives evaluated on $\mathbf{x}^*$ are 0, that is

$$\frac{\partial f}{\partial x_i}(x_1^*, x_2^*, ..., x_n^*) = 0 \text{ for all } i = 1, 2, ..., n.$$

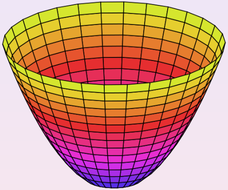
Theorem (Necessary first-order conditions)

Let $\mathbf{x}^* = (x_1^*, x_2^*, ..., x_n^*)$ be an interior point in S at which f has partial derivatives, then, a necessary condition for $\mathbf{x}^*$ to be a maximum or minimum point for f is that $\mathbf{x}^*$ is a stationary point for f.

Maxima and minima



Maximum



Minimum

Learn more in http://wikipedia.org

Theorem (Sufficient conditions with concavity/convexity)

Suppose that the function f is $\mathcal{C}^1$,

- *if f is concave in S, then $\mathbf{x}^*$ is a (global) maximum point for f in S if and only if ($\Leftrightarrow$) $\mathbf{x}^*$ is a stationary point for f*
- *if f is convex in S, then $\mathbf{x}^*$ is a (global) manimum point for f in S if and only if ($\Leftrightarrow$) $\mathbf{x}^*$ is a stationary point for f*

If f is strictly concave (convex), the global maximum (minimum) point is unique.

Definition 16

The point $\mathbf{x}^*$ is a *local maximum point* of f in S if

$$f(\mathbf{x}) \leq f(\mathbf{x}^*) \text{ for all } \mathbf{x} \text{ in } S \text{ sufficiently close to } \mathbf{x}^*.$$

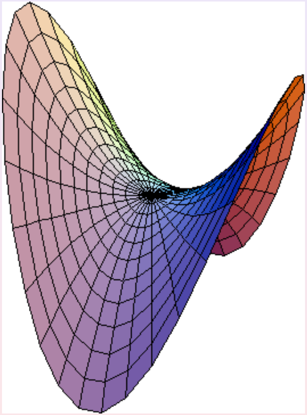
If the inequality is strict then x^* is a *strict local maximum point*.

A (*strict*) *local minimum point* is defined in the obvious way.
The first-order necessary conditions for a local maximum (minimum) point remain the same, that is: *a local extreme point in the interior of a domain of a function with partial derivatives must be stationary.*

Definition 17

A stationary point $\mathbf{x}^*$ of f that is neither a local maximum point nor a local minimum point is called a *saddle point* of f.

Saddle Point



Learn more in `http://wikipedia.org`

Theorem (Necessary second-order conditions for local extreme points)

Suppose that f is $\mathcal{C}^2$ and $\mathbf{x}^*$ is an interior stationary point of f, then

- $\mathbf{x}^*$ *is a local minimum point, then ($\Rightarrow$) the Hessian matrix $Hf(\mathbf{x}^*)$ is positive definite or semidefinite*
- $\mathbf{x}^*$ *is a local maximun point, then ($\Rightarrow$) the Hessian matrix $Hf(\mathbf{x}^*)$ is negative definite or semidefinite*

Theorem (Sufficient second-order conditions for local extreme points)

Suppose that the function f is $\mathcal{C}^2$ and $\mathbf{x}^*$ is an interior stationary point of f, then

- *the Hessian matrix $H f(\mathbf{x}^*)$ is positive definite$\Rightarrow$ $\mathbf{x}^*$ is a local minimum point*
- *the Hessian matrix $H f(\mathbf{x}^*)$ is negative definite$\Rightarrow$ $\mathbf{x}^*$ is a local maximum point*
- *$|H f(\mathbf{x}^*)| \neq 0$ but it is not (positive or negative) definite $\Rightarrow$ $\mathbf{x}^*$ is a saddle point*

Example *The two-variables case.*
If $f(x,y)$ is a $\mathcal{C}^2$ function with (x^*,y^*) as an interior stationary point, then

$$\frac{\partial^2 f}{\partial x^2}(x^*,y^*) > 0 \text{ and } \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(x^*,y^*) & \frac{\partial^2 f}{\partial y \partial x}(x^*,y^*) \\ \frac{\partial^2 f}{\partial x \partial y}(x^*,y^*) & \frac{\partial^2 f}{\partial y^2}(x^*,y^*) \end{vmatrix} > 0 \Rightarrow \text{local min. at } (x^*,y^*)$$

$$\frac{\partial^2 f}{\partial x^2}(x^*,y^*) < 0 \text{ and } \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(x^*,y^*) & \frac{\partial^2 f}{\partial y \partial x}(x^*,y^*) \\ \frac{\partial^2 f}{\partial x \partial y}(x^*,y^*) & \frac{\partial^2 f}{\partial y^2}(x^*,y^*) \end{vmatrix} > 0 \Rightarrow \text{local max. at } (x^*,y^*)$$

$$\begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(x^*,y^*) & \frac{\partial^2 f}{\partial y \partial x}(x^*,y^*) \\ \frac{\partial^2 f}{\partial x \partial y}(x^*,y^*) & \frac{\partial^2 f}{\partial y^2}(x^*,y^*) \end{vmatrix} < 0 \Rightarrow (x^*,y^*) \text{ is a saddle point}$$

Links to the Wolfram Demostrations Project web page

- Stationary points (maximun, minimun and saddle points) >> >>

Bibliography

Sydsaeter,K., Hammond, P.J., Seierstad, A. and Strom, A. Essential Mathematics for Economic Analysis. Prentice Hall. New Jersey. pages: 453-466. >>

Return to Contents

Problem 1 ▸ Answer

Classify the stationary points of

a. $f(x, y) = 2x^2 + xy + 2y^2 - 4x - y$
b. $f(x, y) = (x^2 + y^2)\mathrm{e}^{x^2 - y^2}$
c. $f(x, y) = 2x - 2e^x + 3y - y^3 + 4$
d. $f(x, y) = x \ln(y + 1)$
e. $f(x, y, z) = e^{-x^2 - y^2 - x + z^2}$
f. $f(x, y, z) = x^2 - xy^2 + y^4 - 3yz + z^3$
g. $f(x, y) = x^3 + 3x^2 + y^3 + 6y^2$

Problem 2

Answer

A firm produces an output good using two inputs, denoted by x and y, according to the following production function

$$Q = x^{1/2} y^{1/3}.$$

If $\mathrm{p}_1 = 2$, $\mathrm{p}_2 = 1$ and $\mathrm{p}_3 = 1$ are the prices of output and inputs respectively, maximize the firm's profit.

Problem 3 ▸ Answer

The output production function of a firm is

$$Q = x^{1/2}y^{1/3}.$$

where x and y are the units for two different inputs. If p_1, p_2 and p_3 are the prices of output and inputs respectively, and the firm seeks to maximize profits

a. Find the demand of inputs functions.
b. Suppose that p_3 rises while the rest of parameters remain constant; what is the effect upon the demand for input y?
c. If p_1 rises while p_2 and p_3 remain constant; what is the effect upon the demand for x and y?

Problem 4 Answer

Find the maxima and minima point of the function
$f(x,y) = 2x^3 + ay^3 + 6xy$ for different values of parameter $a \in \mathbb{R}$.

Problem 5 Answer

A firm produces three output goods in units x, y and z respectively. If profit is given by

$$B(x,y,z) = -x^2 + 6x - y^2 + 2yz + 4y - 4z^2 + 8z - 14,$$

find the units of each good that maximize profit and find the maximum profit.

Problem 6 ▸ Answer

A monopolistic firm produces two goods whose demand functions are

$$p_1 = 12 - x_1, \quad p_2 = 36 - 5x_2$$

where x_1 and x_2 are the quantities of the two goods produced and p_1 and p_2 the prices of a unit of each good. Knowing that the cost function is $C(x_1, x_2) = 2x_1x_2 + 15$, solve the corresponding profit maximizing problem.

Problem 7 Answer

Solve the output production maximizing problem

$$\max Q(x, y) = -x^3 - 3y^2 + 3x^2 + 24y$$

where x and y are the necessary inputs. Find the maximum production.

Problem 8 ▸ Answer

In a competitive market, a firm produces good Q according to the function

$$Q(K, L) = 8K^{1/2}L^{1/4}$$

where K and L are capital and labor respectively. Given the unitary prices of 5 m.u for output and 2 m.u. and 10 m.u. for inputs, find the maximum profit.

Problem 9 Answer

The output production function of a firm and its cost function are given, respectively, by

$$\begin{aligned} Q(x,y) &= 7x^2+7y^2+6xy \\ C(x,y) &= 4x^3+4y^3 \end{aligned}$$

where x and y are the productive inputs. Knowing that the selling price of a unit of good is 3 m.u., find the maximum point for both productive inputs, x and y, and find the maximum profit.

1. The function $f(x,y) = x^2 + y^2$ Answer
   a. has no stationary point
   b. has a stationary point at $(0,0)$
   c. has a stationary point at $(1,1)$

2. The function $f(x, y, z) = (x-2)^2 + (y-3)^2 + (z-1)^2$ has, at point $(2, 3, 1)$, ▸ Answer

   a. a global maximum point
   b. a global minimum point
   c. a saddle point

3. The function $f(x, y) = xy^2(2 - x - y)$ has, at point $(0, 2)$,

Answer

a. a local maximum point
b. a local minimum point
c. a saddle point

4. The function $f(x, y) = x^2y + y^2 + 2y$ has ▸ Answer

   a. a local maximum point
   b. a local minimum point
   c. a saddle point

5. The function $f(x,y) = \dfrac{\ln(x^3+2)}{y^2+3}$: Answer

   a. has a stationary point at $(1,0)$
   b. has a stationary point at $(0,0)$
   c. has no stationary points

6. If the determinant of the Hessian matrix of $f(x, y)$ on a stationary point is negative, then ▸ Answer
   a. the stationary point is a saddle point
   b. the stationary point is a local minimum point
   c. the stationary point is a local maximum

7. If (a, b) is a stationary point of the function $f(x, y)$ such that

$$\frac{\partial^2 f(a,b)}{\partial x^2} = -2 \text{ and } |Hf(a,b)| = 3$$

then ▸ Answer

a. (a, b) is a local maximum point
b. (a, b) is a local minimum point
c. (a, b) is a saddle point

8. If $(2,1)$ is a stationary point of the function $f(x,y)$ such that

$$\frac{\partial^2 f(2,1)}{\partial x^2} = 3 \text{ and } |Hf(2,1)| = 1$$

then ▸ Answer

a. $(2,1)$ is a local maximum point
b. $(2,1)$ is a local minimum point
c. $(2,1)$ is a saddle point

9. The Hessian matrix of function $f(x, y, z)$ is

$$Hf(x, y, z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

If the function had a stationary point, this would be ▸ Answer

a. a local maximum point
b. a global maximum point
c. a global minimum point

10. Let $B(x, y)$ be the profit function of a firm which produces two output goods in quantities x and y. If (a, b) is a stationary point of function $B(x, y)$, for it to be a global maximum point it must occur that ▸ Answer

   a. the profit function is concave for all (x, y) in $\mathbb{R}^2$
   b. the profit function is convex for all (x, y) in $\mathbb{R}^2$
   c. the profit function is concave in a neighborhood of the point (a, b)

11. The Hessian matrix of function $f(x,y)$ is given by ▸ Answer

$$Hf(x,y) = \begin{pmatrix} x^2+2 & -1 \\ -1 & 1 \end{pmatrix}.$$

If $f(x,y)$ had a stationary point then this point would be

a. a global maximum point
b. a global minimum point
c. a local minimum point that couldn't be global

12. If $(2,1)$ is a stationary point of the function $f(x,y)$, which of the following conditions assures that $(2,1)$ is a global maximum point of the function? Answer

  a. $Hf(2,1)$ is negative definite
  b. $Hf(x,y)$ is negative definite fort all (x,y) in $\mathbb{R}^2$
  c. $Hf(2,1)$ is positive definite

Chapter Four: Constrained Optimization. The Lagrange Multiplier Method

Return to Contents

Outline

- Introduction
- The Lagrange Multiplier Method (the two-variable case)
- The Lagrange Multiplier is a shadow price
- The Lagrange method applied to the general multivariable case

consumer's optimization problem

$$\max U(x,y) \quad \text{subject to } p \cdot x + y = b. \tag{P}$$

Note that:

- The point (x^*, y^*) that solves problem (P) is not necessarily a maximum point (global or local) of the function $U(x,y)$
- In this case $y = b - p \cdot x$, $\Rightarrow$ Max $f(x) = U(x, b - px)$ *unconstrained optimization problem with one variable less*
- If the substitution method is difficult or impossible to carry out in practise $\Rightarrow$ *Lagrange Method*

The two-variables case: Lagrange function

Definition 18

Given the optimization problem

$$\begin{aligned} Opt. &: \quad f(x,y) \\ s.t &: \quad g(x,y) = b \end{aligned}$$

we define the *Lagrange function* $\mathcal{L}$ by

$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - b)$$

where λ is called the Lagrange multiplier.

The two-variables case: Lagrange function

Lagrange function

$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - b)$$

Note that the partial derivatives of $\mathcal{L}(x,y,\lambda)$ with respect to x and y are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x}(x,y,\lambda) &= \frac{\partial f}{\partial x}(x,y) - \lambda \frac{\partial g}{\partial x}(x,y),\\ \frac{\partial \mathcal{L}}{\partial y}(x,y,\lambda) &= \frac{\partial f}{\partial y}(x,y) - \lambda \frac{\partial g}{\partial y}(x,y),\end{aligned}$$

respectively. Moreover,

$$\frac{\partial \mathcal{L}}{\partial \lambda}(x,y,\lambda) = -[g(x,y) - b]$$

which must be 0 when the constraint is satisfied. In fact $\mathcal{L}(x,y,\lambda) = f(x,y)$ for all (x,y) that satisfy the constraint $g(x,y) = b$

First-order necessary conditions for optimality

$$\begin{aligned} Opt. \quad &: \quad f(x,y) \\ s.t \quad &: \quad g(x,y) = b \end{aligned} \tag{2}$$

Theorem (Lagrange)

If (x^*, y^*) is a maximum or a minimum point of problem (2) then there exists a Lagrange multiplier λ^* such that (x^*, y^*, λ^*) is a stationary point of the Lagrange function.

First-order necessary conditions for optimality

The following equalities will be then satisfied:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x}(x^*, y^*, \lambda^*) &= \frac{\partial f}{\partial x}(x^*, y^*) - \lambda^* \frac{\partial g}{\partial x}(x^*, y^*) = 0, \quad (3)\\
\frac{\partial \mathcal{L}}{\partial y}(x^*, y^*, \lambda^*) &= \frac{\partial f}{\partial y}(x^*, y^*) - \lambda^* \frac{\partial g}{\partial y}(x^*, y^*) = 0, \quad (4)\\
\frac{\partial \mathcal{L}}{\partial \lambda}(x^*, y^*, \lambda^*) &= -[g(x^*, y^*) - b] = 0. \quad (5)
\end{aligned}$$

The conditions (3)-(5) are called the *first-order necessary conditions*

The two-variable case with lineal constraint

The general utility maximizing problem with two goods:

$$\text{maximize } U(x,y) \quad \text{subject to} \quad p \cdot x + q \cdot y = b$$

where $U(x,y)$ is concave as stated by economic theory.

Follow the following steps:

- Write the Lagrangian

$$\mathcal{L}(x,y,\lambda) = U(x,y) - \lambda\left[p \cdot x + q \cdot y - b\right]$$

where λ is a constant

- Differentiate $\mathcal{L}$ with respect to x, y and λ and equate the partial derivatives to 0, the first order necessary conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial U}{\partial x}(x,y) - \lambda p = 0 \quad \Rightarrow \lambda = \frac{\partial U(x,y)/\partial x}{p} \\
\frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial U}{\partial y}(x,y) - \lambda q = 0 \quad \Rightarrow \lambda = \frac{\partial U(x,y)/\partial y}{q} \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= g(x,y) - b = 0
\end{aligned}$$

- Solve these equation simultaneously for the three unknowns x, y and $\lambda \Rightarrow (x^*, y^*, \lambda^*)$
- Since function $U(x,y)$ is concave then the obtained point (x^*, y^*) is a global maximum point (Theorem 3).[1]
- Note that the maximum point (x^*, y^*) satisfies that

$$\frac{\partial U(x^*,y^*)/\partial x}{\partial U(x^*,y^*)/\partial y} = \frac{p}{q} \tag{6}$$

[1] In the case of a convex function, the stationary point would be a global minimum

Geometric interpretation

A geometric interpretation is that the consumer should choose the point on the budget line at which the slope of the level curve of the utility function is equal to the slope of the budget line

$$\frac{\partial U(x^*, y^*)/\partial x}{\partial U(x^*, y^*)/\partial y} = \frac{p}{q} \tag{7}$$

Thus, at the optimal point the budget line is tangent to a level curve of the utility function

$$\begin{aligned} Opt. &: \quad f(x,y) \\ s.t &: \quad g(x,y) = b \end{aligned} \tag{8}$$

Theorem (Sufficient conditions for Global Optimality when the constraint is lineal)

If (x^*, y^*, λ^*) is a stationary point of the Lagrange function associated to problem (8), then:
If the constraint is lineal and function f is concave in the feasible set, (x^*, y^*) is a global maximum of problem (8).
If the constraint is lineal and function f is convex in the feasible set, (x^*, y^*) is a global minimum of problem (8).

$$\begin{aligned} Opt. &: \quad f(x,y) \\ s.t &: \quad g(x,y) = b \end{aligned} \tag{9}$$

Theorem (Sufficient conditions for Global Optimality)

If (x^*, y^*, λ^*) is a stationary point of the Lagrange function associated to problem (9), then:
If the Lagrangian $\mathcal{L}(x, y, \lambda^*)$ is concave in (x, y) then (x^*, y^*) is a global maximum of problem (9).
If the Lagrangian $\mathcal{L}(x, y, \lambda^*)$ is convex in (x, y) then (x^*, y^*) is a global minimum of problem (9).

$$\begin{aligned} Opt. &: \quad f(x,y) \\ s.t &: \quad g(x,y) = b \end{aligned} \tag{10}$$

Theorem (Sufficient conditions for Local Optimality)

Let (x^*, y^*, λ^*) be a stationary point of the Lagrange function associated to problem (10). Define

$$D(x,y,\lambda) = \begin{vmatrix} 0 & \frac{\partial g}{\partial x}(x,y) & \frac{\partial g}{\partial y}(x,y) \\ \frac{\partial g}{\partial x}(x,y) & \frac{\partial^2 \mathcal{L}}{\partial x^2}(x,y) & \frac{\partial^2 \mathcal{L}}{\partial y \partial x}(x,y) \\ \frac{\partial g}{\partial y}(x,y) & \frac{\partial^2 \mathcal{L}}{\partial x \partial y}(x,y) & \frac{\partial^2 \mathcal{L}}{\partial y^2}(x,y) \end{vmatrix}$$

If $\mathcal{D}(x^*, y^*, \lambda^*) < 0$ then (x^*, y^*) is a local maximum of problem (10).
If $\mathcal{D}(x^*, y^*, \lambda^*) > 0$ then (x^*, y^*) is a local maximum of problem (10).

The Lagrange multiplier is a shadow price

$$\lambda = \frac{\partial f^*}{\partial b}(b)$$

the Lagrange multiplier λ is the rate at which the optimal value of the objective function changes with respect to changes in the constraint constant b.

Suppose, for instance, that $f^*(b)$ is the maximum profit that a firm can obtain from a production process when b is the available quantity of a resource. Then $\partial f^*(b)/\partial b$ is the marginal profit that the firm can earn per extra unit of the resource, which is therefore the firm's marginal willingness to pay for this resource.

In Economics this measure is known as the shadow price.

The Lagrange multiplier is a shadow price

Proof.

Let (x^*, y^*, λ^*) be a stationary point of the Lagrange function, then, the first order necessary conditions must be satisfied, that is

$$\frac{\partial \mathcal{L}}{\partial x}(x^*, y^*, \lambda^*) = \frac{\partial f}{\partial x}(x^*, y^*) - \lambda^* \frac{\partial g}{\partial x}(x^*, y^*) = 0, \quad (11)$$
$$\frac{\partial \mathcal{L}}{\partial y}(x^*, y^*, \lambda^*) = \frac{\partial f}{\partial y}(x^*, y^*) - \lambda^* \frac{\partial g}{\partial y}(x^*, y^*) = 0. \quad (12)$$

Note that $x^* = x(b)$ and $y^* = y(b)$. Let

$$f^*(b) = f(x^*, y^*) = f(x(b), y(b))$$

be the optimum (maximun or minimum) value function, which is a function of b.
Using the change rule and (14)-(15), the following arises

Proof.

$$\begin{aligned}\frac{\partial f^*}{\partial b}(b) &= \frac{\partial f}{\partial x}(x^*,y^*)\frac{\partial x}{\partial b}(b)+\frac{\partial f}{\partial y}(x^*,y^*)\frac{\partial y}{\partial b}(b)\\ &= \lambda^*\frac{\partial g}{\partial x}(x^*,y^*)\frac{\partial x}{\partial b}(b)+\lambda^*\frac{\partial g}{\partial y}(x^*,y^*)\frac{\partial y}{\partial b}(b)\\ &= \lambda^*\left[\frac{\partial g}{\partial x}(x^*,y^*)\frac{\partial x}{\partial b}(b)+\frac{\partial g}{\partial y}(x^*,y^*)\frac{\partial y}{\partial b}(b)\right]\end{aligned}$$

Moreover,

$$g(x(b),y(b)) = g(x^*,y^*) = b$$

then

$$\frac{\partial g}{\partial x}(x(b),y(b))\frac{\partial x}{\partial b}(b)+\frac{\partial g}{\partial y}(x(b),y(b))\frac{\partial y}{\partial b}(b) = 1.$$

Which implies that

$$\frac{\partial f^*}{\partial b}(b) = \lambda^*$$

The Lagrange method applied to the general multivariable case.

Definition 19

Given the problem

$$\begin{aligned} Opt. &: \quad f(x_1, x_2, \cdots, x_n) \\ s.t. &: \quad g(x_1, x_2, \cdots, x_n) = b \end{aligned} \tag{13}$$

we define the Lagrange function, or Lagrangian, by

$$\mathcal{L}(x_1, x_2, \cdots, x_n, \lambda) = f(x_1, x_2, \cdots, x_n) - \lambda(g(x_1, x_2, \cdots, x_n) - b)$$

where λ is called Lagrange multiplier.

Theorem (Lagrange)

If $\mathbf{x}^* = (x_1^*, x_2^*, \cdots, x_n^*)$ is a maximum (or minimum) point of the problem with one equality restriction (13), then there exists one Lagrange multiplier λ^* such that $(x_1^*, x_2^*, \cdots, x_n^*, \lambda^*)$ is a stationary point of the Lagrange function. That is

$$\frac{\partial \mathcal{L}}{\partial x_j}(x_1^*, x_2^*, \cdots, x_n^*, \lambda^*) = \frac{\partial f}{\partial x_j}(x_1^*, x_2^*, \cdots, x_n^*) - \lambda^* \frac{\partial g}{\partial x_j}(x_1^*, x_2^*, \cdots, x_n^*) = 0 \tag{14}$$

for all $\quad j = 1, .2, ..., n \quad$ *and*

$$\frac{\partial \mathcal{L}}{\partial \lambda}(x_1^*, x_2^*, \cdots, x_n^*, \lambda^*) = g(x_1^*, x_2^*, \cdots, x_n^*) - b = 0 \tag{15}$$

Theorem (Sufficient conditions for Global Optimality when the constraint is lineal)

If $(x_1^*, x_2^*, \cdots, x_n^*, \lambda^*)$ *is a stationary point of the Lagrange function associated to problem (13), then*
If the constraint is lineal and function f *is concave in the feasible set,* $(x_1^*, x_2^*, \cdots, x_n^*)$ *is a global maximum of problem (13).*
If the constraint is lineal and function f *is convex in the feasible set,* $(x_1^*, x_2^*, \cdots, x_n^*)$ *is a global minimum of problem (13).*

Theorem (Sufficient conditions for Global Optimality)

If $(x_1^*, x_2^*, \cdots, x_n^*, \lambda^*)$ *is a stationary point of the Lagrange function associated to problem (13), then*
If the Lagrangian $\mathcal{L}(x_1, x_2, \cdots, x_n, \lambda^*)$ *is concave in* $(x_1, x_2, \cdots, x_n)$ *then* $(x_1^*, x_2^*, \cdots, x_n^*)$ *is a global maximum of problem (13).*
If the Lagrangian $\mathcal{L}(x_1, x_2, \cdots, x_n, , \lambda^*)$ *is convex in* $(x_1, x_2, \cdots, x_n,)$ *then* $(x_1^*, x_2^*, \cdots, x_n^*)$ *is a global minimum of problem (13).*

Links to the Wolfram Demostrations Project web page

- Constrained Optimization >>

Bibliography

Sydsaeter,K., Hammond, P.J., Seierstad, A. and Strom, A. Essential Mathematics for Economic Analysis. Prentice Hall. New Jersey. pages: 489-494. >>

Return to Contents

Problem 1 ▸ Answer

Solve the following problems using the substitution method and also the Lagrange multipliers method (the understanding of the problems can be improved using a graphical resolution approach).

a. Min. $f(x,y) = (x-1)^2 + y^2$ subject to $y - 2x = 0$.
b. Max. $f(x,y) = xy$ subject to $2x + 3y = 6$.
c. Opt. $f(x,y) = 2x + 3y$ subject to $xy = 6$.

Problem 2 ▸ Answer

Solve the following problems:

- a Opt. $f(x,y,z) = x^2 + (y-2)^2 + (z-1)^2$ subject to $4x + y + 4z = 39$.
- b Opt. $f(x,y) = e^{xy}$ subject to $x^2 + y^2 = 8$.
- c Opt. $f(x,y) = x^{1/4}y^{1/2}$ subject to $x + 2y = 3$.
- d Opt. $f(x,y) = \ln(xy)$ subject to $x^2 + y^2 = 8$.

Problem 3 ▸ Answer

A multinational refreshments firm has 68 monetary units available to produce the maximum possible number of bottles. Its production function is $q(x,y) = 60x + 90y - 2x^2 - 3y^2$ where x and y are the required inputs. The inputs prices are $p_x = 2$ m.u. and $p_y = 4$ m.u. repectively. Given the budget restriction, maximize the production of bottles. By means of the Lagrange multiplier, how will the maximum number of bottles produced be modified if the budget is increased in one unit (or if it is decreased)?

Problem 4 Answer

A worker earns 20 monetary units for each labor hour. The worker's utility, $U(x, y) = x^{1/3}y^{1/3}$, depends on the consumption of goods, x, and also on the free time, y. Knowing that each unit of consumption costs 80 m.u, and that the worker does not save any of the earned money for the future, find the values of x and y that maximize his utility.

Problem 5 Answer

A European research program has 600 thousand euros available to finance research projects on renewable energies. Two teams present their projects and their estimated incomes (derived from the property rights of new discoveries) are given by $I_1(x) = 2x^{1/2}$ and $I_2(y) = \frac{4}{3}y^{3/4}$ where x is the monetary assignation to the first team (in hundreds of thousands of euros) and y is second team s assignation. The program seeks to determine the optimal distribution of quantities x and y to maximize the joint income. Formulate and solve the problem. What happens to the maximum joint income if the budget is increased by 50 thousand euros? Is it worth it?

Problem 6 Answer

A firm's output production function, $f(K,L) = 4(K+1)^{1/2}(L+1)^{1/2}$, depends on the employed capital and labor. Its costs function is $C(K,L) = 2K + 8L$. Find the optimal values for K and L which minimize the cost of producing 32 units of output. If the production increased by one unit, what would be the effect on the cost?

Problem 7 ▸ Answer

The output of an industry depends on a sole resource whose quantity is limited to b and it is mandatory to use it up. There are two production processes available for which the resource must be distributed. The derived incomes from each one of the productions processes are

$$f(x) = 1200 - \left(\frac{x}{2} - 12\right)^2 \quad g(y) = 1400 - (y-1)^2$$

where x and y are the employed resource in each production process.

a. How can the distribution between x and y be done so as to maximize the total income?
b. Assuming that $b = 22$ and that there is the possibility of using one additional unit of the resource with a cost of 0,8 m.u, Is is worth it? And, is it worth it if $b = 28$?

Problem 8 Answer

The function $U(x, y) = 100x + xy + 100y$ represents a representative consumer's utility depending on the consumption of two goods, x and y. Knowing that the consumer spends her whole income, 336 monetary units, purchasing these goods at prices $p_x = 8$ m.u. and $p_y = 4$ m.u respectively, maximize the consumer's utility.

Problem 9 Answer

The costs function of a firm is:

$$C(x,y) = (x-1)^2 + 6y + 8$$

where x and y are the quantities of the two productive inputs needed to produce. If $Q(x,y) = (x-1)^2 + 3y^2$ is the output production function, find the input quantities to produce 12 units of product at the minimum cost.

1. The Lagrange function associated with the problem ▸ Answer

$$Opt. f(x,y,z) \text{ subject to } g(x,y,z) = c \text{ is}$$

a. $\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - b)$
b. $\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda(g(x,y) + b)$
c. $\mathcal{L}(x,y,\lambda) = g(x,y) - \lambda(f(x,y) - b)$

2. The maximum production of a firm is 500 units of a certain good and the shadow price of the available resource is 3. What would be the effect on the maximum production level if the resource were increased by one unit? Answer
   a. The maximum production level would not be affected
   b. The maximum production level would reduce by 3 units
   c. The maximum production level would increase by 3 units

3. Given the optimization problem ▸ Answer

$$\min f(x,y) \text{ subject to } 3x - 6y = 9. \qquad \text{(P)}$$

If $(x,y,\lambda) = (1,-1,3)$ is a stationary point of the associated Lagrange function, it can be assured that $(1,-1)$ is a global minimum of problem (P) when the function $f(x,y)$ is

- a. convex
- b. concave
- c. neither convex nor concave

4. Given the following optimization problem ▸ Answer

$$\min f(x,y) \text{ subject to } x^2+y=5. \tag{P}$$

Let $(x,y,\lambda)=(1,4,3)$ be is a stationary point of the associated Lagrange function $\mathcal{L}(x,y,\lambda)$. Then, if the Hessian matrix of function $\mathcal{L}(x,y,3)$ is positive semidefinite then $(1,4)$ is a

a. is a global maximum point of problem (P)
b. is a global minimum point of problem (P)
c. It can't be assured that it is a global extreme point for problem (P)

5. The Hessian matrix of the Lagrange function $\mathcal{L}(x, y, z, \lambda^*)$ is given by Answer

$$H(\mathcal{L}(x, y, z, \lambda^*)) = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & -4 \end{pmatrix},$$

then, a stationary point is a

a. global maximum
b. global minimum
c. neither of the above

6. Given the optimization problem

$$Opt. f(x, y, z) \text{ subject to } g(x, y, z) = c$$

and given $(1, 2, 3, 4)$, a stationary point of the Lagrange function ($\lambda = 4$ is the Lagrange multiplier) if the Hessian matrix of $\mathcal{L}(x, y, z, 4)$ is

$$H\mathcal{L}(x, y, z, 4) = \begin{pmatrix} x^2 + 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4y^2 + 3 \end{pmatrix}.$$

then ▸ Answer

a. the problem has no solution
b. point $(1, 2, 3)$ is a global minimum point
c. $(1, 2, 3)$ is a global maximum point

7. Given the optimization problem

$$Opt. f(x, y, z) \text{ subject to } g(x, y, z) = c,$$

it is known that the Hessian matrix of the Lagrange function when $\lambda = 4$ is given by

$$H\mathcal{L}(x, y, z, 4) = \begin{pmatrix} -x^2 - 7 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

Then, if $(1, 2, 3, 4)$ is a stationary point of the Lagrange function, ▸ Answer

a. the problem has no solution
b. the problem has a global maximum point
c. the problem has a global minimum point

8. Given the optimization problem

$$Opt. f(x, y) \text{ subject to } g(x, y) = c,$$

it is known that the Hessian matrix of the Lagrange function when $\lambda = 4$ is given by

$$H\mathcal{L}(x, y, 4) = \begin{pmatrix} -x^2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Then, if $(1, 2, 4)$ is a stationary point of the Lagrange function, ▸ Answer

a. the minimum value of the objective function is 4
b. $(1, 2)$ is a global minimum point
c. $(1, 2)$ is a global maximum point

9. In the maximization of profits with a linear constraint on costs $x + y + z = 89$, the Lagrange multiplier is $-0,2$. Is it worth increasing the level of cost? Answer
   a. No. The maximum profit would decrease
   b. Yes, since the maximum profit would increase
   c. Yes, because we would continue with positive profits

Matrix Algebra and Quadratic forms

Return to Contents

Outline

- Matrix Algebra
  - Matrices and Matrix Operations
  - Determinants

  Review problems for Matrix algebra

  Multiple choice questions
- Quadratic forms
  - Definiteness of a quadratic form
  - The sign of a quadratic form attending the principal minors
  - Quadratic forms with linear constraints

  Review problems for Quadratic forms

  Multiple choice questions

Useful Links

Return to Contents

Matrix Algebra

Back

Return to Contents

Outline

- Matrices and Matrix Operations
- Determinants

Back

Return to Contents

Definition of matrix

A matrix is simply a rectangular array of numbers considered as an entity. When there are m rows and n columns in the array, we have an m-by-n matrix (written as $m \times n$). In general, an $m \times n$ matrix is of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

or, equivalently

$$A = (a_{ij})_{m \times n}.$$

Definition 20

A matrix with only one row is also called a row vector, and a matrix with only one column is called a column vector. We refer to both types as vectors.

Definition 21

If $m = n$, then the matrix has the same number of columns as rows and it is called a square matrix of order n and is denoted by $A = (a_{ij})_n$.

Definition 22

If $A = (a_{ij})_n$ is a square matrix, then the elements a_{11}, a_{22}, a_{33}, ..., a_{nn} constitute the main diagonal.

Definition 23

The identity matrix of order n, denoted by $\mathbf{I}_n$ (or by $\mathbf{I}$), is the $n \times n$ matrix having ones along the main diagonal and zeros elsewhere:

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Addition of Matrices
$A+B=(a_{ij})_{m\times n}+(b_{ij})_{m\times n}=(a_{ij}+b_{ij})_{m\times n}$
Rules
$(A+B)+C=A+(B+C)$ $A+B=B+A$ $A+0=A$

Multiplication by a real number
$\alpha A = \alpha\left(a_{ij}\right)_{m\times n} = \left(\alpha a_{ij}\right)_{m\times n}$
Rules
$(\alpha+\beta)A = \alpha A + \beta A$ $\alpha(A+B) = \alpha A + \beta B$ $A + (-A) = 0$

Matrix Multiplication
$A \cdot B = (a_{ij})_{\mathbf{m}\times n} \cdot (b_{ij})_{n\times\mathbf{p}} = (c_{ij})_{\mathbf{m}\times\mathbf{p}}$ such that $c_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$
Rules
$(AB)C = A(BC)$ $A(B+C) = AB + AC$ $(A+B)C = AC + BC$

The transpose

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \Rightarrow A' = A^t = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

Symmetric matrix

Definition 24

Square matrices with the property that they are symmetric with respect the main diagonal are called symmetric.

The matrix $A = \left(a_{ij}\right)_n$ is symmetric $\Leftrightarrow a_{ij} = a_{ji}\ \forall i, j = 1, 2, ..., n.$

A matrix A is symmetric $\Leftrightarrow A = A'$.

Determinants

Definition 25

Let A be an $n \times n$ matrix. Then $\det(A)$ or $|A|$ is a sum of $n!$ terms where:

1. Each term is the product of n elements of the matrix, with one element (and only one) from each row, and one (and only one) element from each column.
2. The sign of each term is $+$ or $-$ depending on whether the permutation of row subindexes is of the same class as the permutation of the column subindices or not.

Determinants of order 2

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Leftrightarrow |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Determinants of order 3

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

Determinants of diagonal matrices

Diagonal matrix:

$$\left|\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}\right| = a_{11}a_{22}\cdots a_{nn}$$

Determinants of triangular matrices

upper triangular matrix:

$$\left|\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}\right| = a_{11}a_{22}\cdots a_{nn}$$

The same occurs to lower triangular matrices.

Rules for determinants

Let A be an $n \times n$ matrix. Then

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- $|A| = |A'|$.

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Rules for determinants

Let A be an $n \times n$ matrix. Then

- $|A| = |A'|$.
- If all the elements in a row (or column) of A are 0 then $|A| = 0$.
- If all the elements in a single row (or column) of A are multiplied by a number α, the determinant is multiplied by α.
- If two rows (or two columns) of A are interchanged, the sign of the determinant changes, but the absolute value remains unchanged.

Rules for determinants

- If two rows (or columns) of A are equal or proportional, then $|A| = 0$.

Rules for determinants

- If two rows (or columns) of A are equal or proportional, then $|A| = 0$.
- The value of the determinant of A is unchanged if a multiple of one row (or column) is added to a different row (or column) of A.

Rules for determinants

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- The determinant of the product of two matrices A and B is the product of the determinants of each of the factors:

$$|AB| = |A|\,|B|\,.$$

Rules for determinants

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$$|AB| = |A|\,|B|\,.$$

- Is α is a real number

$$|\alpha A| = \alpha^n\,|A|\,.$$

Practical methods to calculate $|A|$

In practice, there are two methods to calculate the determinants of a square matrix (mainly used when its order is higher than 3):

- **Triangularization**: Rule number 6 allows us to convert matrix A into one that is (upper or lower) triangular. Because the determinant will remain unchanged (as is stated by the property) its value will be equal to the product of the elements in the main diagonal of the triangular matrix.

Practical methods to calculate $|A|$

- **Expansion of $|A|$ in terms of the elements of a row:**

$$
\begin{aligned}
\det(A) &= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}) \\
&= a_{i1}(-1)^{i+1} \det(A_{i1}) + a_{i2}(-1)^{i+2} \det(A_{i2}) + \cdots + a_{in}(-1)^{i+n} \det(A_{in})
\end{aligned}
$$

where $a_{i1}, a_{i2}, \cdots a_{in}$ are the elements of the row i and A_{ij} is the determinant of order $n-1$ which results from deleting row i and column j of matrix A (it is called a *minor*).

Practical methods to calculate $|A|$

- **Expansion of $|A|$ in terms of the elements of a column:**

$$\begin{aligned}\det(A) &= \sum_{i=1}^{n}(-1)^{i+j}a_{ij}\det(A_{ij})\\ &= a_{1j}(-1)^{1+j}\det(A_{1j})+a_{2j}(-1)^{2+j}\det(A_{2j})+\cdots+a_{nj}(-1)^{n+j}\det(A_{n\end{aligned}$$

where $a_{1j}, a_{2j}, \cdots a_{nj}$ are the elements of the column j and A_{ij} is the determinant of order $n-1$ which results from deleting row i and column j of matrix A (it is called a *minor*).

Definition 26

The product $(-1)^{i+j}\det(A_{ij})$ is called the adjoint element of a_{ij}.

Problem 1 Answer

Given the following matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} B = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} C = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$$

confirm that the following properties are true

a. $(A+B)C = AC + BC$.
b. $(AB)^t = B^t A^t$.
c. $(A-B)^2 = A^2 + B^2 - AB - BA$.
d. $(AB + A) = A(B+I)$, where I is the identity matrix of order 2.
e. $(BA + A) = (B+I)A$, where I is the identity matrix of order 2.

Problem 2 Answer

Calculate AB and BA, where

a. $A = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 0 & -5 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 \\ 8 & 4 \\ -2 & 3 \end{pmatrix}$

b. $A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 2 & 4 \\ 1 & 5 & 3 \end{pmatrix}$

c. $A = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 5 & 7 \end{pmatrix}$

Can we say that the product of matrices has the commutative property?

Problem 3

Answer

Let A and B be square matrices. Prove that the property $(A+B)^2 = A^2 + B^2 + 2AB$ is false.

Problem 4 Answer

Calculate the following determinants:

a) $\begin{vmatrix} 1 & 3 & 2 \\ -1 & 3 & 1 \\ 2 & 0 & -3 \end{vmatrix}$

b) $\begin{vmatrix} 3 & 1 & -1 & 1 \\ -1 & 2 & 0 & 3 \\ 2 & 3 & 4 & 7 \\ -1 & -1 & 2 & -2 \end{vmatrix}$

c) $\begin{vmatrix} 3 & 1 & 2 & -1 \\ -4 & 1 & 0 & 3 \\ 4 & -3 & 0 & -1 \\ -5 & 2 & 0 & -2 \end{vmatrix}$

d) $\begin{vmatrix} 1 & -3 & 5 & -2 \\ 2 & 4 & -1 & 0 \\ 3 & 2 & 1 & -3 \\ -1 & -2 & 3 & 0 \end{vmatrix}$

Problem 5

Without computing the determinants, show that

a) $$\begin{vmatrix} 1 & x_1 & x_2 \\ 1 & y_1 & x_2 \\ 1 & y_1 & y_2 \end{vmatrix} = (y_1 - x_1)(y_2 - x_2)$$

b) $$\begin{vmatrix} 1 & x_1 & x_2 & x_3 \\ 1 & y_1 & x_2 & x_3 \\ 1 & y_1 & y_2 & x_3 \\ 1 & y_1 & y_2 & y_3 \end{vmatrix} = (y_1 - x_1)(y_2 - x_2)(y_3 - x_3)$$

Problem 6

Answer

Without computing the determinants, find the value of :

a) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1+c \end{vmatrix}$ b) $\begin{vmatrix} 1 & 4 & 4 & 4 \\ 4 & 2 & 4 & 4 \\ 4 & 4 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{vmatrix}$ c) $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ -1 & 0 & 3 & 4 & 5 \\ -1 & -2 & 0 & 4 & 5 \\ -1 & -2 & -3 & 0 & 5 \\ -1 & -2 & -3 & -4 & 0 \end{vmatrix}$

Problem 7

Without computing the determinants, show that

$$\begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} = (a+3)(a-1)^3$$

Problem 8 Answer

Given the matrices

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 2 & 8 \end{pmatrix}$$

Compute:

a) $|AB|$ b) $\left|(BA)^t\right|$ c) $|2A3B|$ d) $|A+B|$

Problem 9 ▸ Answer

Let A and B be matrices of order n. Knowing that $|A| = 5$ and $|B| = 3$, compute:

a) $\left|BA^t\right|$ b) $|3A|$ c) $\left|(2B)^2\right|$, B of order 3

Problem 10 Answer

Compute $|AB|$, $\left|(BA)^t\right|$, $|2A3B|$, knowing that

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 2 & 8 \end{pmatrix}$$

Problem 11 ▸ Answer

If $A^2 = A$, what values can $|A|$ have?

Problem 12

Let A be a square matrix of order n such that $A^2 = -I$. Prove that $|A| \neq 0$ and that n is an even number.

Problem 13

Let A be a square matrix of order n. Prove that AA^t is a symmetric matrix.

Problem 14 Answer

Given the following matrices:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$
$$C = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$$

solve the matrix equation $3X + 2A = 6B - 4A + 3C$.

Problem 15 Answer

Suppose that $A.B$ and X are matrices of orden n, solve the following matrix equations:

- a $3(A^t - 2B) + 5X^t = -B.$
- b $(A^t + X)^t - B = 2A.$

1. The matrix $A = \begin{pmatrix} 4 & 1 & -3 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \end{pmatrix}$ is ▸ Answer

   a. a lower triangular matrix
   b. an upper triangular matrix
   c. a diagonal matrix

2. Which of the following is the true definition of a symmetric matrix? Answer
   a. A square matrix A is said to be symmetric if $A = -A$
   b. A square matrix A is said to be symmetric if $A = -A^t$
   c. A square matrix A is said to be symmetric if $A = A^t$

3. Given A and B matrices of order $m \times n$ and $n \times p$, $(AB)^t$ equals to: ▸ Answer

   a. $B^t A^t$
   b. $A^t B^t$
   c. AB

4. Let A be a matrix such that $A^2 = A$ then, if $B = A - I$, then: Answer

   a. $B^2 = B$
   b. $B^2 = I$
   c. $B^2 = -B$

5. Let A and B be square matrices of order 3. If $|A| = 3|$ and $|B| = -1$ then: Answer
   a. $|2A \centerdot 4B| = (-4)2^3$
   b. $|2A \centerdot 4B| = (-3)2^3$
   c. $|2A \centerdot 4B| = (-3)2^9$

6. Which of the following properties is $\boxed{\text{NOT}}$ always true? Answer

   a. $\left|A^2\right| = |A|^2$
   b. $|A+B| = |A| + |B|$.
   c. $|A^t B| = |A|\,|B|$

7. Given the 3 by 3 matrices A, B, C such that $|A| = 2$, $|B| = 4$ and $|C| = 3$, compute $\left|\frac{1}{|A|}B^tC^{-1}\right|$: Answer

   a. $\frac{2}{3}$
   b. $\frac{1}{6}$
   c. 6

8. Let A and B be matrices of the same order, which of the following properties is always true? Answer

   a. $(A-B)(A+B) = A^2 - B^2$
   b. $(A-B)^2 = A^2 - 2AB + B^2$
   c. $A(A+B) = A^2 + AB$

9. Which of the following properties is $\boxed{\text{NOT}}$ true? ▸ Answer
   a. $|A^2| = |A|^2$
   b. $|-A| = |A|$
   c. $|A^t| = |A|$

10. Let A and B be symmetric matrices, then which of the following is also a symmetric matrix: Answer
    a. BA
    b. $A + B$
    c. AB

11. Let A be an n by n real matrix. Then, if $k \in \mathbb{R}$ one has that

Answer

a. $|kA| = k\,|A|$
b. $|kA| = |k|\,|A|$, being $|k|$ the absolute value of the real number k
c. $|kA| = k^n\,|A|$

12. Given the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} B = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} C = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$$

the solution to the matrix equation
$3X + 2A = 6B - 4A + 3C$ is ▸ Answer

a. $X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
b. $X = \begin{pmatrix} -4 & 1 \\ -4 & 0 \end{pmatrix}$
c. $X = \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}$

13. Let A, B be matrices of order n. Then: Answer
    a. $(AB)^2 = A^2B^2$
    b. $(AB)^2 = B^2A^2$
    c. $(AB)^2 = A(BA)B$

14. Given the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$, the adjoint element a_{12} is:

Answer

a. -1
b. 1
c. -2

Quadratic forms

Back

Return to Contents

Outline

- Definiteness of a quadratic form
- The sign of a quadratic form attending the principal minors
- Quadratic forms with linear constraints

Back

Return to Contents

Quadratic forms: The general case

Definition 27

A *quadratic form* in n variables is a function Q of the form

$$Q(x_1, x_2, ..., x_n) = \mathbf{x}' A \mathbf{x} = (x_1, x_2, ..., x_n) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

where $\mathbf{x}' = (x_1, x_2, ..., x_n)$ is a vector and $A = (a_{ij})_{n \times n}$ is a symmetric matrix of real numbers.

Then A is called the symmetric matrix associated with Q.

Matrix and Polynomial form

1. *Matrix form* of a quadratic form.

$$Q(x_1, x_2, ..., x_n) = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

2. *Polynomial form* of a quadratic form. Expanding the matrix multiplication we obtain a double sum such as

$$Q(x_1, x_2, ..., x_n) = \mathbf{x}' A \mathbf{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

Function Q is a homogeneous polynomial of degree two where each term contains either the square of a variable or a product of exactly two of the variables. The terms can be grouped as follows

$$Q(\mathbf{x}) = \sum^{n} b_{ii} x_i^2 + \sum^{n} b_{ij} x_i x_j$$

$$Q(x_1, x_2, ..., x_n) = (x_1, x_2, \cdots, x_n) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} b_{ii} x_i^2 + \sum_{i,j=1,i<j}^{n} b_{ij}$$

There exits a relationship between the elements in the symmetric matrix associated with Q and the coefficients of the polynomial. Note that

The elements in the main diagonal of matrix A are the coefficients of the quadratic terms of the polynomial.

The elements outside the main diagonal of matrix A ($a_{ij} = a_{ji}\ i \neq j$) are half of the coefficients of the non quadratic terms of the polynomial.

Definition 28

A quadratic form $Q(\mathbf{x}) = \mathbf{x}' A\mathbf{x}$ (as well as its associated symmetric matrix A) is said to be

1. *Positive definite* if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$.
2. *Negative definite* if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq 0$
3. *Positive semidefinite* if $Q(\mathbf{x}) \geq 0$ for all $\mathbf{x} \neq 0$
4. *Negative semidefinite* if $Q(x) \leq 0$ for all $\mathbf{x} \neq 0$
5. *Indefinite* if there exist vectors $\mathbf{x}$ and $\mathbf{y}$ such that $Q(\mathbf{x}) < 0$ and $Q(\mathbf{y}) > 0$. Thus, an indefinite quadratic form assumes both negative and positive values.

Definition 29

A principal minor of order r of an $n \times n$ matrix $A = (a_{ij})$ is the determinant of a matrix obtained by deleting $n - r$ rows and $n - r$ columns such that if the ith row (column) is selected, then so is the ith column (row).

In particular, a principal minor of order r always includes exactly r elements of the main (principal) diagonal. Also, if matrix A is symmetric, then so is each matrix whose determinant is a principal minor. The determinant of A itself, $|A|$, is also a principal minor (No rows or colmns are deleted)

Definition 30

A principal minor is called a *leading principal minors* of order r ($1 \leq r \leq n$) if it consists of the first ("leading") r rows and columns of $|A|$.

Theorem

Let $Q(\mathbf{x}) = \mathbf{x}'A\mathbf{x}$ be a quadratic form of n variables and let

$$|A_1| = a_{11}, |A_2| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, |A_3| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} ..., |A_n| = |A|$$

be the leading principal minors of matrix A. Then

- $Q(\mathbf{x})$ positive definite $\Leftrightarrow |A_1| > 0, |A_2| > 0, ..., |A_n| > 0$
- $Q(\mathbf{x})$ negative definite $\Leftrightarrow$ the leading principal minors of even order are positive and those of odd order are negative..
- If $|A| = 0$ and the remaining leading principal minors are positive $\Rightarrow Q$ is positive semidefinite.
- If $|A| = 0$ and the remaining leading principal minors of even order are positive and those of odd order are negative $\Longrightarrow Q$ is negative semidefinite.
- If $|A| \neq 0$ and the leading principal minors do not behave as in $a)$ or $b)$ $\Rightarrow Q$ is indefinite.
- If $|A| = 0$ and $|A_i| \neq 0$ $i = 1, 2, ..., n-1$ and the leading principal

Theorem

Let $Q(\mathbf{x}) = \mathbf{x}' A \mathbf{x}$ be a quadratic form of n variables such that $|A| = 0$. Then

- *All the principal minors are positive or zero $\Leftrightarrow$ Q is positive semidefinite*
- *All the principal minors are of even order are positive or zero and those of odd order are negative or zero $\Leftrightarrow$ Q is negative semidefinite.*

Definition 31

It is said that the quadratic form $Q(x) = x^t Ax$ is constrained to a linear constraint when

$$(x_1, x_2, ..., x_n) \in \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n / b_1x_1 + b_2x_2 + ... + b_nx_n = 0\}$$

To find the sign of a constrained quadratic form, follow the following steps:

1. Analyze the sign of $Q(x) = x^t Ax$ without any constraint. If it is definite (positive or negative), then the constrained quadratic form is of the same sign.
2. If the unconstrained quadratic form is not definite, we solve the linear constraint for one variable and substitute it into the quadratic form. The result is an unconstrained quadratic form with $n-1$ variables. We study the sign with the principal minors

Problem 1 ▸ Answer

Without computing any principal minor, determine the definiteness of the following quadratic forms:

a. $Q(x, y, z) = (x + 2y)^2 + z^2$
b. $Q(x, y, z) = (x + y + z)^2 - z^2$
c. $Q(x, y, z) = (x - y)^2 + 2y^2 + z^2$
d. $Q(x, y, z) = -(x - y)^2 - (y + z)^2$
e. $Q(x, y, z) = (x - y)^2 + (y - 2z)^2 + (x - 2z)^2$

Problem 2 ▸ Answer

Write the following quadratic forms in matrix form with A symmetric and determine their definiteness.

- a $Q(x, y, z) = -x^2 - y^2 - z^2 + xy + xz + yz$
- b $Q(x, y, z) = y^2 + 2z^2 + 2xz + 4yz$
- c $Q(x, y, z) = 2y^2 + 4z^2 + 2yz$
- d $Q(x, y, z) = -3x^2 - 2y^2 - 3z^2 + 2xz$
- e $Q(x_1, x_2, x_3, x_4) = x_1^2 - 4x_3^2 + 5x_4^2 + 4x_1x_3 + 2x_2x_3 + 2x_2x_4$

Problem 3 ▸ Answer

Investigate the definiteness of the following quadratic forms depending on the value of parameter a.

a. $Q(x, y, z) = -5x^2 - 2y^2 + az^2 + 4xy + 2xz + 4yz$
b. $Q(x, y, z) = 2x^2 + ay^2 + z^2 + 2xy + 2xz$
c. $Q(x, y, z) = x^2 + ay^2 + 2z^2 + 2axy + 2xz$

Problem 4 ▸ Answer

Find the value of a which makes the quadratic form $\mathrm{Q}(\mathrm{x}, \mathrm{y}, \mathrm{z}) = \mathrm{ax}^2 + 2\mathrm{y}^2 + \mathrm{z}^2 + 2\mathrm{xy} + 2\mathrm{xz} + 2\mathrm{yz}$ be semidefinite. For such a value, determine its definiteness when it is subject to $\mathrm{x} - y - z = 0$?

Problem 5

Answer

If $A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & -2 \\ -1 & -2 & 3 \end{pmatrix}$ then

a. investigate its definiteness.
b. Write the polynomial and matrix forms of the quadratic form $Q(h_1, h_2, h_3)$ which is associated with matrix A.
c. determine its definiteness when it is subject to $h_1 + 2h_2 - h_3 = 0$.

Problem 6 Answer

Investigate the definiteness of the following matrices. Write the polynomial and matrix form of the associated quadratic forms:

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & -3/2 & 1/2 \\ -3/2 & 1 & -1/2 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 3 \end{pmatrix} \quad D = \begin{pmatrix} -5 & 2 & 1 & 0 \\ 2 & -2 & 0 & 2 \\ 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -5 \end{pmatrix}$$

Problem 7 ▸ Answer

Determine the definiteness of the following constrained quadratic forms.

a) $$Q(x,y,z) = (x,y,z)\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ s. t } x + y - 2z = 0$$

b) $$Q(x,y,z) = (x,y,z)\begin{pmatrix} 2 & -3/2 & 1/2 \\ -3/2 & 1 & -1/2 \\ 1/2 & -1/2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ s. t } x - y = 0$$

c) $$Q(x,y,z) = (x,y,z)\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ s. t } x + 2y - z = 0$$

d) $$Q(x_1,x_2,x_3,x_4) = (x_1,x_2,x_3,x_4)\begin{pmatrix} -5 & 2 & 1 & 0 \\ 2 & -2 & 0 & 2 \\ 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{ s. t}$$
$$2x_1 - 4x_4 = 0$$

Problem 8 ▸ Answer

Let $Q(x, y, z) = -x^2 - 2y^2 - z^2 + 2xy - 2yz$ be a quadratic form

a. write its matrix form and investigate its definiteness.
b. Investigate its definiteness if it is constrained to $2x - 2y + az = 0$ for the different values parameter a can have.

Problem 9 Answer

Determine the definiteness of the Hessian matrix of the following functions

a. $\mathrm{f(x, y, z) = 2x^2 + y^2 - 2xy + xz - yz + 2x} - y + 8$
b. $\mathrm{f(x, y) = x^4 + y^4 + x^2 + y^2 + 2xy}$
c. $\mathrm{f(x, y, z) = \ln}(x) + \ln(y) + \ln(z)$

Problem 10 ▸ Answer

Determine the definiteness of the Hessian matrix of the following production functions when $K, L > 0$.

a) $\mathrm{Q}(\mathrm{K},\mathrm{L}) = \mathrm{K}^{1/2}\mathrm{L}^{1/2}$ b) $\mathrm{Q}(\mathrm{K},\mathrm{L}) = \mathrm{K}^{1/2}\mathrm{L}^{2/3}$

Problem 11 ▸ Answer

The production function $\mathrm{Q(x, y, z){=}ax^2 + 4ay^2 + a^2z^2{-}4axy}$, with $a > 0$, relates the produced quantity of a good to three raw materials (x, y and z) used in the production precess.

a. Determine the definiteness of $Q(x, y, z)$.
b. Knowing that if $x = y = z = 1$ then six units of a good are produced, find the value of parameter a.
c. Using the value of a found in (b), determine the definiteness of $Q(x, y, z)$ when the raw materials x and y are used in the same quantity.

1. Which of the following is a quadratic form? Answer
   a. $Q(x, y, z) = x^2 + 3z^2 + 6xy + 2z$
   b. $Q(x, y, z) = 2xy^2 + 3z^2 + 6xy$
   c. $Q(x, y, z) = 3xy + 3xz + 6yz$

2. Let $Q(x, y, z)$ be a quadratic form such that $Q(1, 1, 0) = 2$ and $Q(5, 0, 0) = 0$, then

Answer

   a. $Q(x, y, z)$ could be indefinite
   b. $Q(x, y, z)$ is positive definite
   c. $Q(x, y, z)$ could be negative semidefinite

3. The quadratic form $Q(x, y, z) = (x - y)^2 + 3z^2$ is

Answer

   a. positive definite
   b. positive semidefinite
   c. indefinite

4. The quadratic form $Q(x, y, z) = -y^2 - 2z^2$ is
   Answer
   a. negative definite
   b. negative semidefinite
   c. indefinite

5. The quadratic form $Q(x, y, z) = x^2 + 2xz + 2y^2 - z^2$ is

Answer

a. positive definite
b. positive semidefinite
c. indefinite

6. A 2 by 2 matrix has a negative determinant, then the matrix is

   Answer

   a. negative definite
   b. negative semidefinite
   c. indefinite

7. The matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ is

Answer

a. indefinite
b. positive semidefinite
c. positive definite

8. The leading principal minors of a 4 by 4 matrix are $|\mathrm{A}_1| = -1$, $|\mathrm{A}_2| = 1$, $|\mathrm{A}_3| = 2$, and $|\mathrm{A}_4| = |A| = 0$. Then,

   ▸ Answer

   a. the matrix is negative semidefinite
   b. its definiteness cannot be determined with this information
   c. the matrix is indefinite

9. The leading principal minors of a 4 by 4 matrix are $|\mathrm{A}_1| = -1$, $|\mathrm{A}_2| = 1$, $|\mathrm{A}_3| = -2$ and $|\mathrm{A}_4| = |A| = 0$.Then,

   Answer

   a. the matrix is negative semidefinite
   b. its definiteness cannot be determined from this information
   c. the matrix is indefinite

10. The leading principal minors of a 4 by 4 matrix are $|A_1| = -1, |A_2| = 1$, $|A_3| = -2$, and $|A_4| = |A| = 1$. Then, the matrix is

Answer

a. negative definite
b. indefinite
c. positive definite and negative definite

11. The quadratic form in three variables $Q(x, y, z)$, subject to $x + 2y - z = 0$, is positive semidefinite. Then, the unconstrained quadratic form is:

Answer

   a. positive semidefinite or indefinite
   b. positive semidefinite
   c. positive definite or positive semidefinite

12. If $Q(x, y, z)$ is a negative semidefinite quadratic form such that $Q(-1, 1, 1) = 0$, then $Q(x, y, z)$ subject to the constraint $x + 2y - z = 0$

    Answer

    a. is negative semidefinite
    b. cannot be clasified with this information
    c. is negative definite or negative semidefinite

13. The quadratic form $\mathrm{Q}(\mathrm{x}, \mathrm{y}, \mathrm{z}) = (\mathrm{x} - y)^2 + 3\mathrm{z}^2$ subject to the constraint $x = 0$ is:

Answer

a. indefinite
b. positive semidefinite
c. positive definite

14. The quadratic form $\mathrm{Q}(\mathrm{x},\mathrm{y},\mathrm{z}) = (\mathrm{x}-y)^2 + 3\mathrm{z}^2$ subject to the constraint $z = 0$ is:

Answer

a. indefinite
b. positive semidefinite
c. positive definite

Links to the Wolfram Demostrations Project web page

- Matrix Multiplication >>
- Matrix Transposition >>
- Determinants by expansion >>
- Determinants using diagonals >>

Bibliography

- Matrix algebra:

  Sydsaeter,K. and Hammond,P.J. Essential Mathematics for Economic Analysis. Prentice Hall. New Jersey. Pages: 537-554 and 573-591. >>
- Quadratic forms

  Sydsaeter,K. and Hammond,P.J. Further Mathematics for Economic Analysis. Prentice Hall. New Jersey. Pages: 28-37. >>

Answers to Problems

Return to Contents

Problem 1

Return

- a. Convex, closed and bounded
- b. Convex and closed
- c. Closed and bounded
- d. Convex and bounded
- e. Convex and closed
- f. Convex, closed and bounded
- g. Closed
- h. Convex

Problem 3 Return

a. Concave on the convex set of $\mathbb{R}^2$: $\{(x, y) \in \mathbb{R}^2 / x \leq 0\}$
b. Convex on the convex set of $\mathbb{R}^2$: $\{(x, y) \in \mathbb{R}^2 / x \geq 3\}$
c. Convex
d. Convex if $z \geq 0$
e. Convex
f. Convex
g. Neither concave nor convex
h. Neither concave nor convex

Problem 4 Return

a. Concave
b. Concave
c. Convex
d. Concave

Problem 5 Return

- (a) It is convex if $a = 0$
- (b) It is concave if $a < 0$

Problem 6 Return

All of them are convex except for C.

Problem 2 ▸ Return

75 units of good 1 and none unit of good 2.

Problem 3 Return

$$x^* = 25, y^* = 50 \Rightarrow u^* = 1250$$

Problem 4 ▸ Return

The problems that verify the hypothesis of the Extreme Value Theorem are b), c) and f).

Problem 1 ▸ Return

a) $(1, 0)$ is a local minimum point.
b) $(0, 0)$ is a local minimum point and $(0, 1)$ and $(0, -1)$ are saddle points.
c) $(0, 1)$ is a local maximum point and $(0, -1)$ is a saddle point.
d) $(0, 0)$ is a saddle point.
e) $(-1/2, 0, 0)$ is a saddle point.
f) $(1/2, 1, 1)$ is a local minimum and $(0, 0, 0)$ is a saddle point.
g) $(-2, -4)$ is a local maximum.

Problem 2

Return

The maximum point is $x = 4/9$ and $y = 8/27$.

Problem 3 ▸ Return

- a) The maximum point is $x = \left(\frac{p_1^3}{12p_2^2p_3}\right)^2$ and $y = \left(\frac{p_1^2}{6p_2p_3}\right)^3$.
- b) If the price of y rises with other parameters remaining constant, the quantity demanded of input y will decrease in order to maximize profits. By contrast, if the selling price of output rises, the quantity demanded of input y will increase.

Problem 4

Return

$(0, 0)$ is a saddle point for all of the values of parameter a.
$\left(-\sqrt[3]{\frac{2}{a}}, -\sqrt[3]{\frac{4}{a^2}}\right)$ is a local minimum point if $a < 0$ and, a local maximum point if $a > 0$.

Problem 5 Return

The maximum point is $x = 3,\ y = 4,\ z = 2$ and the maximum profit is $B_{\max} = 11$ m.u.

Problem 6 ▸ Return

The maximum point is $x_1 = x_2 = 3$, whose prices are, respectively, $p_1 = 9,\ p_2 = 21$.

Problem 7 Return

The maximum produced quantity is $Q_{\max}(2,4) = 52$ units.

Problem 8 Return

The maximum profit is $B_{\max}(1.000, 100) = 1.000$ m.u.

Problem 9 ▸ Return

The maximum point is $x = y = 5$ and the maximum profit $B_{\max}(5, 5) = 500$ m.u.

Problem 1

Return

- (a) The problem has a global minimum at $x^* = 1/5$ and $y^* = 2/5$ whose value is $20/25$.
- (b) $(3/2, 1)$ is a global maximum of value $3/2$.
- (c) $(3, 2)$ is a local minimum point of value 12 and $(-3, -2)$ is a local maximum point of value -12.

Problem 2 ▸ Return

- a) Global minimum at $x = 4$, $y = 3$ and $z = 5$ of value 33.
- b) $(2, 2)$ and $(-2, -2)$ are two global maximum points of value e^4. $(2, -2)$ and $(-2, 2)$ are two global minimum of value e^{-4}.
- c) Global maximum at $x = y = 1$ of value 1.
- d) Two local maximum points at $(2, 2)$ and $(-2, -2)$ of value $\ln 4$.

Problem 3

Return

The global maximum point is at $x = 12$ and $y = 11$ with a maximum quantity of 1059 bottles.

If the budget is increased by 1 monetary unit, the maximum production would increase by 6 units (approximately). Similarly, if the budget is reduced, the production would reduce by 6 units (approximately)

Problem 4 ▸ Return

$x = 3$ and $y = 12$.

Problem 5 Return

The global maximum is obtained when the first project is assigned 200 thousand euro and the second project with 400 thousand euro. The maximum income will be of 659.970 euro.
An increase of 50 thousand euro will increase the maximum income by 35.355 approximately. It is not worth it.

Problem 6 ▸ Return

$K = 15$ and $L = 3$. An increase of one unit in production would increase the minimum cost by 2 units (approximately)

Problem 7

Return

(a) $x = \dfrac{4b}{5} + 4$ and $y = \dfrac{b}{5} - 4$.

(b) It is worth it in the first case but not in the second.

Problem 8

Return

$y = 84$ and $x = 0$.

Problem 9 Return

$(1 + 2\sqrt{3}, 0)$ and $(1, 2)$ are two minimum points of value 20..

Problem 1 ▸ Return

a $\begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$

b $2\begin{pmatrix} -1/2 & -1/2 \\ 1 & 1 \end{pmatrix}$

c $\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$

d $3\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

e $\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$

Problem 2

Return

- a $AB = \begin{pmatrix} 16 & 10 \\ 13 & -15 \end{pmatrix}$ and $BA = \begin{pmatrix} 12 & 3 & 6 \\ 36 & 8 & -4 \\ -5 & -2 & -19 \end{pmatrix}$
- b $AB = \begin{pmatrix} -2 & 1 & 7 \\ 5 & 11 & 5 \\ 3 & 10 & 8 \end{pmatrix}$ and $BA = \begin{pmatrix} 5 & 5 & 4 \\ 8 & 4 & 6 \\ 12 & 5 & 8 \end{pmatrix}$
- c $AB = 3\begin{pmatrix} 2 & 3 \end{pmatrix}$ and BA is not possible.

Problem 3 Return

It is false because the multiplication of matrices does not verify the commutative property. To probe its falsity the students must provide a counterexample.

Problem 4

Return

(a)-24 (b)-58 (c)-80 (d)35

Problem 6 ▸ Return

(a) abc (b) -24 (c) $5!$

Problem 8 Return

(a) 16 (b) 16 (c) $2^8 3^4$ (d) 130.

Problem 9 Return

(a) 15 (b) $3^n \cdot 5$ (c) $2^6 \cdot 3^2$

Problem 10 Return

(a)$3^n \cdot 5$ (b) 5^{n-1} (c) 15

Problem 11 Return

$|A| = 0$ or $|A| = 1$.

Problem 14 Return

$$X = \begin{pmatrix} -4 & 1 \\ -4 & 0 \end{pmatrix}$$

Problem 15 ▸ Return

(a) $X = B^t - \frac{3}{5}A$ (b) $X = A^t + B^t.$

Problem 1 Return

- (a) positive semidefinite.
- (b) indefinite.
- (c) positive definite.
- (d) negative semidefinite.
- (e) positive semidefinite.

Problem 2 Return

(a) $Q(x,y,z) = (x,y,z)\begin{pmatrix} -1 & 1/2 & 1/2 \\ 1/2 & -1 & 1/2 \\ 1/2 & 1/2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ negative semidefinite.

(b) $Q(x,y,z) = (x,y,z)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ indefinite.

(c) $Q(x,y,z) = (x,y,z)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ positive semidefinite.

(d) $Q(x,y,z) = (x,y,z)\begin{pmatrix} -3 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ negative definite.

(e) $Q(x_1,x_2,x_3,x_4) = (x_1,x_2,x_3,x_4)\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & -4 & 0 \\ 0 & 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ indefinite.

Problem 3 ▸ Return

(a) negative definite for $a < -5$, negative semidefinite for $a = -5$ and indefinite when $a > -5$.
(b) positive definite if $a < 1$, positive semidefinite if $a = 1$ and indefinite when $a < 1$.
(c) Indefinite when $a < 0$ or $a > 1/2$, positive semidefinite for $a = 0$ or $a = 1/2$ and positive definite.if $0 < a < 1/2$.

Problem 4 Return

$a = 1$. The constrained quadratic form is positive definite.

Problem 5 ▸ Return

(a) indefinite.

(b) $Q(h_1, h_2, h_3) = (h_1, h_2, h_3) \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & -2 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} =$
$h_2^2 + 3h_3^2 - 2h_1h_3 - 4h_2h_3$

(c) positive definite.

Problem 6 ▸ Return

(a) negative semidefinite,
$Q(x, y, z) = -2x^2 - 2y^2 - 2z^2 + 2xy + 2xz + 2yz.$

(b) indefinite, $Q(x, y, z) = 2x^2 + y^2 - 3xy + xz - yz.$

(c) positive semidefinite,
$Q(x, y, z) = x^2 + y^2 + 3z^2 - 2xy + 2xz - 2yz.$

(d) negative definite, $Q(x_1, x_2, x_3, x_4) =$
$-5x_1^2 - 2x_2^2 - x_3^2 - 5x_4^2 + 4x_1x_2 + 2x_1x_3 + 4x_2x_4.$

Problem 7

Return

(a) negative semidefinite. (b) The constrained quadratic form is null.
(c) positive definite. (d) negative definite (since the unconstrained quadratic form is negative definite).

Problem 8 Return

(a) $Q(\mathrm{x},\mathrm{y},\mathrm{z}) = (\mathrm{x},\mathrm{y},\mathrm{z}) \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{pmatrix}$ negative semidefinite.

(b) negative semidefinite if $a = 0$, negative definite if $a \neq 0$.

Problem 9 Return

(a) $Hf(x,y,z)$ is indefinite$\forall(\mathrm{x},\mathrm{y},\mathrm{z}) \in \mathbb{R}^3$.
(b) $Hf(x,y)$ is positive definite$\forall(\mathrm{x},\mathrm{y}) \in \mathbb{R}^2 - \{(0,0)\}$. H f(0,0) is positive semidefinite.
(c) $Hf(x,y,z)$ is negative definite$\forall(\mathrm{x},\mathrm{y},\mathrm{z}) \in \mathrm{Dom}(f) = \{(\mathrm{x},\mathrm{y},\mathrm{z}) \in \mathbb{R}^3/\mathrm{x} > 0, \mathrm{y} > 0, \mathrm{z} > 0\}$.

Problem 10 Return

(a) negative semidefinite. (b) indefinite.

Problem 11 Return

(a) Positive Semidefinite.

(b) $a = 2$.

(c) Positive Definite.

Answers to Multiple Choice Questions

Return to Contents

1. Which of the following sets is convex? Back
   a. $\{(x, y) \in \mathbb{R}^2 / x^2 + y^2 \leq 1\}$
   b. $\{(x, y) \in \mathbb{R}^2 / x^2 + y^2 = 1\}$
   c. $\{(x, y) \in \mathbb{R}^2 / x^2 + y^2 \geq 1\}$

2. The closed line segment between $(1,1)$ and $(-1,-1)$ can be written as the set Back

   a. $B = \{(x,y) \in \mathbb{R}^2/(x,y) = (2\lambda - 1, 2\lambda - 1), \forall\lambda \in [0,1]\}$
   b. $B = \{(x,y) \in \mathbb{R}^2/(x,y) = (\lambda, 1 - \lambda), \forall\lambda \in [0,1]\}$
   c. $B = \{(x,y) \in \mathbb{R}^2/x = y\}$

3. Given $S \subseteq \mathbb{R}^2$ a convex set, the function $f : S \rightarrow \mathbb{R}$ will be convex if Back

   a. the Hessian matrix $Hf(x, y)$ is negative definite for all (x, y) in S
   b. the sets $\{(x, y) \in S / f(x, y) \leq k\}$ are convex for all k in $\mathbb{R}$
   c. f is a lineal function

4. The set $S = \{(x, y, z) \in \mathbb{R}^3 / x + y^2 + z^2 \leq 1\}$ Back
   a. is convex because the Hessian matrix of the function $f(x, y) = x + y + z^2$ is positive semidefinite
   b. is convex because the function $f(x, y) = x + y^2 + z^2$ is lineal
   c. in not convex

5. Which of the following sets is not convex? Back

   a. $\{(x,y) \in \mathbb{R}^2 / x \leq 1, y \leq 1\}$
   b. $\{(x,y) \in \mathbb{R}^2 / x, y \in [0,1]\}$
   c. $\{(x,y) \in \mathbb{R}^2 / xy \leq 1,\ x, y \geq 0\}$

6. Which of the following Hessian matrices belongs to a concave function? Back

   a. $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$
   b. $\begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}$
   c. $\begin{pmatrix} -2 & 2 \\ 2 & 2 \end{pmatrix}$

7. The function $f(x, y) = \ln x + \ln y$ is concave on the set Back
   a. $S = \{(x, y) \in \mathbb{R}^2 / x, y > 0\}$
   b. $\mathbb{R}^2$
   c. $S = \{(x, y) \in \mathbb{R}^2 / x, y \neq 0\}$

8. Which of the following sets is convex? Back

   a. $A = \{(x, y) \in \mathbb{R}^2 / xy \geq 1, x \geq 0, y \geq 0\}$
   b. $B = \{(x, y) \in \mathbb{R}^2 / xy \geq 1\}$
   c. $C = \{(x, y) \in \mathbb{R}^2 / xy \leq 1, x \geq 0, y \geq 0\}$

1. Which of the following points belongs to the feasible set of the optimization problem Back

$$\begin{aligned} opt. &: \quad x^2\sqrt{y} \\ s.t. &: \quad x+y=3 \text{ ?} \end{aligned}$$

a. $(1,2)$
b. $(-1,2)$
c. $(4,-1)$

2. If the feasible set of an optimization problem is unbounded then Back
   a. no finite optimum point exists
   b. it has an infinite number of feasible points
   c. the existence of a finite optimum point cannot be assured

3. Given $f(x, y) = ax + by$ with $a, b \in \mathbb{R}$ and the set Back

$$S = \{(x, y) \in \mathbb{R}^2 / x + y = 2, x \geq 0, y \geq 0\},$$

a. f has a global maximum point and a global minimum point in S
b. f has a global maximum point in S if a and b are positive
c. there is no maximum or minimum point of f in S

4. Which of the following is the feasible set of the optimization problem Back

$$\begin{aligned} \max. \quad &: \quad 2x + y \\ s.t. \quad &: \quad x + y = 1 \\ &: \quad x^2 + y^2 \leq 5 \text{ ?} \end{aligned}$$

a. $\{(x, y) \in \mathbb{R}^2 / x + y = 1, x \geq 0, y \geq 0\}$
b. $\{(x, y) \in \mathbb{R}^2 / (x, y) = \lambda(5, 0) + (1 - \lambda)(0, 5), \forall \lambda \in [0, 1]\}$
c. $\{(x, y) \in \mathbb{R}^2 / (x, y) = \lambda(2, -1) + (1 - \lambda)(-1, 2), \forall \lambda \in [0, 1]\}$

1. The function $f(x, y) = x^2 + y^2$ Back
   a. has no stationary point
   b. has a stationary point at $(0, 0)$
   c. has a stationary point at $(1, 1)$

2. The function $f(x, y, z) = (x-2)^2 + (y-3)^2 + (z-1)^2$ has, at point $(2, 3, 1)$, Back
   - a. a global maximum point
   - b. a global minimum point
   - c. a saddle point

3. The function $f(x,y) = xy^2(2-x-y)$ has, at point $(0,2)$,

Back

   a. a local maximum point
   b. a local minimum point
   c. a saddle point

4. The function $f(x, y) = x^2y + y^2 + 2y$ has Back
   a. a local maximum point
   b. a local minimum point
   c. a saddle point

5. The function $f(x, y) = \dfrac{\ln(x^3 + 2)}{y^2 + 3}$: Back

   a. has a stationary point at $(1, 0)$
   b. has a stationary point at $(0, 0)$
   c. has no stationary points

6. If the determinant of the Hessian matrix of $f(x, y)$ on a stationary point is negative, then Back

   a. the stationary point is a saddle point
   b. the stationary point is a local minimum point
   c. the stationary point is a local maximum

7. If (a, b) is a stationary point of the function $f(x, y)$ such that

$$\frac{\partial^2 f(a,b)}{\partial x^2} = -2 \text{ and } |Hf(a,b)| = 3$$

then Back

a. (a, b) is a local maximum point
b. (a, b) is a local minimum point
c. (a, b) is a saddle point

8. If $(2,1)$ is a stationary point of the function $f(x,y)$ such that

$$\frac{\partial^2 f(2,1)}{\partial x^2} = 3 \text{ and } |Hf(2,1)| = 1$$

then Back

a. $(2,1)$ is a local maximum point
b. $(2,1)$ is a local minimum point
c. $(2,1)$ is a saddle point

9. The Hessian matrix of function $f(x, y, z)$ is

$$Hf(x,y,z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

If the function had a stationary point, this would be Back

a. a local maximum point
b. a global maximum point
c. a global minimum point

10. Let $B(x, y)$ be the profit function of a firm which produces two output goods in quantities x and y. If (a, b) is a stationary point of function $B(x, y)$, for it to be a global maximum point it must occur that Back

   a. the profit function is concave for all (x, y) in $\mathbb{R}^2$
   b. the profit function is convex for all (x, y) in $\mathbb{R}^2$
   c. the profit function is concave in a neighborhood of the point (a, b)

11. The Hessian matrix of function $f(x,y)$ is given by Back

$$Hf(x,y) = \begin{pmatrix} x^2+2 & -1 \\ -1 & 1 \end{pmatrix}.$$

If $f(x,y)$ had a stationary point then this point would be

a. a global maximum point
b. a global minimum point
c. a local minimum point that couldn't be global

12. If $(2,1)$ is a stationary point of the function $f(x,y)$, which of the following conditions assures that $(2,1)$ is a global maximum point of the function? Back

   a. $Hf(2,1)$ is negative definite
   b. $Hf(x,y)$ is negative definite fort all (x,y) in $\mathbb{R}^2$
   c. $Hf(2,1)$ is positive definite

1. The Lagrange function associated with the problem Back

$$Opt.f(x, y, z) \text{ subject to } g(x, y, z) = c \text{ is}$$

- a. $\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - b)$
- b. $\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda(g(x, y) + b)$
- c. $\mathcal{L}(x, y, \lambda) = g(x, y) - \lambda(f(x, y) - b)$

2. The maximum production of a firm is 500 units of a certain good and the shadow price of the available resource is 3. What would be the effect on the maximum production level if the resource were increased by one unit? Back
   a. The maximum production level would not be affected
   b. The maximum production level would reduce by 3 units
   c. The maximum production level would increase by 3 units

3. Given the optimization problem Back

$$\min f(x,y) \text{ subject to } 3x-6y=9. \tag{P}$$

If $(x,y,\lambda)=(1,-1,3)$ is a stationary point of the associated Lagrange function, it can be assured that $(1,-1)$ is a global minimum of problem (P) when the function $f(x,y)$ is

a. convex
b. concave
c. neither convex nor concave

4. Given the following optimization problem Back

$$\min f(x,y) \text{ subject to } x^2 + y = 5. \qquad \text{(P)}$$

Let $(x, y, \lambda) = (1, 4, 3)$ be is a stationary point of the associated Lagrange function $\mathcal{L}(x, y, \lambda)$. Then, if the Hessian matrix of function $\mathcal{L}(x, y, 3)$ is positive semidefinite then $(1, 4)$ is a

- a) is a global maximum point of problem (P)
- b) is a global minimum point of problem (P)
- c) It can't be assured that it is a global extreme point for problem (P)

5. The Hessian matrix of the Lagrange function $\mathcal{L}(x, y, z, \lambda^*)$ is given by Back

$$H(\mathcal{L}(x, y, z, \lambda^*)) = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & -4 \end{pmatrix},$$

then, a stationary point is a

a. global maximum
b. global minimum
c. neither of the above

6. Given the optimization problem

$$Opt. f(x,y,z) \text{ subject to } g(x,y,z) = c$$

and given $(1,2,3,4)$, a stationary point of the Lagrange function ($\lambda = 4$ is the Lagrange multiplier) if the Hessian matrix of $\mathcal{L}(x,y,z,4)$ is

$$H\mathcal{L}(x,y,z,4) = \begin{pmatrix} x^2+1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4y^2+3 \end{pmatrix}.$$

then Back

a. the problem has no solution
b. point $(1,2,3)$ is a global minimum point
c. $(1,2,3)$ is a global maximum point

7. Given the optimization problem

$$Opt. f(x,y,z) \text{ subject to } g(x,y,z) = c,$$

it is known that the Hessian matrix of the Lagrange function when $\lambda = 4$ is given by

$$H\mathcal{L}(x,y,z,4) = \begin{pmatrix} -x^2-7 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

Then, if $(1,2,3,4)$ is a stationary point of the Lagrange function, Back

a. the problem has no solution
b. the problem has a global maximum point
c. the problem has a global minimum point

8. Given the optimization problem

$$Opt. f(x,y) \text{ subject to } g(x,y) = c,$$

it is known that the Hessian matrix of the Lagrange function when $\lambda = 4$ is given by

$$H\mathcal{L}(x,y,4) = \begin{pmatrix} -x^2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Then, if $(1,2,4)$ is a stationary point of the Lagrange function, Back

a. the minimum value of the objective function is 4
b. $(1,2)$ is a global minimum point
c. $(1,2)$ is a global maximum point

9. In the maximization of profits with a linear constraint on costs $x+y+z=89$, the Lagrange multiplier is $-0,2$. Is it worth increasing the level of cost? Back
   a. No. The maximum profit would decrease
   b. Yes, since the maximum profit would increase
   c. Yes, because we would continue with positive profits

1. The matrix $A = \begin{pmatrix} 4 & 1 & -3 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \end{pmatrix}$ is Back
   a. a lower triangular matrix
   b. an upper triangular matrix
   c. a diagonal matrix

2. Which of the following is the true definition of a symmetric matrix? Back
   a. A square matrix A is said to be symmetric if $A = -A$
   b. A square matrix A is said to be symmetric if $A = -A^t$
   c. A square matrix A is said to be symmetric if $A = A^t$

3. Given A and B matrices of order $m \times n$ and $n \times p$, $(AB)^t$ equals to: Back
   a. $B^t A^t$
   b. $A^t B^t$
   c. AB

4. Let A be a matrix such that $A^2 = A$ then, if $B = A - I$, then: Back
   a. $B^2 = B$
   b. $B^2 = I$
   c. $B^2 = -B$

5. Let A and B be square matrices of order 3. If $|A| = 3|$ and $|B| = -1$ then: Back
   a. $|2A \, . \, 4B| = (-4)2^3$
   b. $|2A \, . \, 4B| = (-3)2^3$
   c. $|2A \, . \, 4B| = (-3)2^9$

6. Which of the following properties is $\boxed{\text{NOT}}$ always true? Back
   a. $\left|A^2\right| = |A|^2$
   b. $|A+B| = |A| + |B|$
   c. $|A^t B| = |A|\,|B|$

7. Given the 3 by 3 matrices A, B, C such that $|A| = 2$, $|B| = 4$ and $|C| = 3$, compute $\left|\frac{1}{|A|}B^tC^{-1}\right|$: Back

   a. $\frac{2}{3}$
   b. $\frac{1}{6}$
   c. 6

8. Let A and B be matrices of the same order, which of the following properties is always true? Back
    a. $(A-B)(A+B)=A^2-B^2$
    b. $(A-B)^2=A^2-2AB+B^2$
    c. $A(A+B)=A^2+AB$

9. Which of the following properties is $\boxed{\text{NOT}}$ true? Back
   a. $|A^2| = |A|^2$
   b. $|-A| = |A|$
   c. $|A^t| = |A|$

10. Let A and B be symmetric matrices, then which of the following is also a symmetric matrix: Back
    a. BA
    b. $A+B$
    c. AB

11. Let A be an n by n real matrix. Then, if $k \in \mathbb{R}$ one has that

Back

a. $|kA| = k\,|A|$
b. $|kA| = |k|\,|A|$, being $|k|$ the absolute value of the real number k
c. $|kA| = k^n\,|A|$

12. Given the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} B = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} yC = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$$

the solution to the matrix equation
$3X + 2A = 6B - 4A + 3C$ is Back

a. $X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
b. $X = \begin{pmatrix} -4 & 1 \\ -4 & 0 \end{pmatrix}$
c. $X = \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}$

13. Let A, B be matrices of order n. Then: Back
   a. $(AB)^2 = A^2B^2$
   b. $(AB)^2 = B^2A^2$
   c. $(AB)^2 = A(BA)B$

14. Given the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$, the adjoint element a_{12} is:

Back

a. -1
b. 1
c. -2

1. Which of the following is a quadratic form? Back
   a. $Q(x, y, z) = x^2 + 3z^2 + 6xy + 2z$
   b. $Q(x, y, z) = 2xy^2 + 3z^2 + 6xy$
   c. $Q(x, y, z) = 3xy + 3xz + 6yz$

2. Let $Q(x, y, z)$ be a quadratic form such that $Q(1, 1, 0) = 2$ and $Q(5, 0, 0) = 0$, then

Back

   a. $Q(x, y, z)$ could be indefinite
   b. $Q(x, y, z)$ is positive definite
   c. $Q(x, y, z)$ could be negative semidefinite

3. The quadratic form $Q(x, y, z) = (x - y)^2 + 3z^2$ is

Back

a. positive definite
b. positive semidefinite
c. indefinite

4. The quadratic form $Q(x, y, z) = -y^2 - 2z^2$ is

Back

- a. negative definite
- b. negative semidefinite
- c. indefinite

5. The quadratic form $Q(x, y, z) = x^2 + 2xz + 2y^2 - z^2$ is

Back

a. positive definite
b. positive semidefinite
c. indefinite

6. A 2 by 2 matrix has a negative determinant, then the matrix is

Back

   a. negative definite
   b. negative semidefinite
   c. indefinite

7. The matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ is

Back

   a. indefinite
   b. positive semidefinite
   c. positive definite

8. The leading principal minors of a 4 by 4 matrix are $|\mathrm{A}_1| = -1$, $|\mathrm{A}_2| = 1$, $|\mathrm{A}_3| = 2$, and $|\mathrm{A}_4| = |A| = 0$.Then,

Back

   a. the matrix is negative semidefinite
   b. its definiteness cannot be determined with this information
   c. the matrix is indefinite

9. The leading principal minors of a 4 by 4 matrix are $|\mathrm{A}_1| = -1$, $|\mathrm{A}_2| = 1$, $|\mathrm{A}_3| = -2$ and $|\mathrm{A}_4| = |A| = 0$. Then,

Back

   a. the matrix is negative semidefinite
   b. its definiteness cannot be determined from this information
   c. the matrix is indefinite

10. The leading principal minors of a 4 by 4 matrix are $|A_1| = -1, |A_2| = 1$, $|A_3| = -2$, and $|A_4| = |A| = 1$.Then, the matrix is

Back

   a. negative definite
   b. indefinite
   c. positive definite and negative definite

11. The quadratic form in three variables $Q(x, y, z)$, subject to $x + 2y - z = 0$, is positive semidefinite. Then, the unconstrained quadratic form is:

Back

a. positive semidefinite or indefinite
b. positive semidefinite
c. positive definite or positive semidefinite

12. If $Q(x, y, z)$ is a negative semidefinite quadratic form such that $Q(-1, 1, 1) = 0$, then $Q(x, y, z)$ subject to the constraint $x + 2y - z = 0$

Back

a. is negative semidefinite
b. cannot be classified with this information
c. is negative definite or negative semidefinite

13. The quadratic form $\mathrm{Q}(\mathrm{x}, \mathrm{y}, \mathrm{z}) = (\mathrm{x} - y)^2 + 3\mathrm{z}^2$ subject to the constraint $x = 0$ is:

◂ Back

a. indefinite
b. positive semidefinite
c. positive definite

14. The quadratic form $Q(x, y, z) = (x - y)^2 + 3z^2$ subject to the constraint $z = 0$ is:

Back

a. indefinite
b. positive semidefinite
c. positive definite