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Strategic sharing of a costly network.

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Abstract

We study minimum cost spanning tree problems for a given set of users connected to a source. We propose a rule of sharing such that each user may pay her cost for such a tree plus an additional amount to the others users. A reduction of her cost appears as a compensation from the other users. Our first result states the existence of a sharing such that no agent is willing to choose a different tree from the minimum cost tree (mcst) offered by Prim's algorithm. Therefore, the mcst emerges as both a social and individual optimal solution. Given a sharing system, we implement the above solution as a subgame perfect equilibrium of a sequential game where players decide sequentially with whom to connect. Moreover, the proposed solution is at the core of the monotone cooperative game associated with a minimal cost spanning tree problem.

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1 Introduction

Cost-Sharing Network is a challenging issue. (See Harks and Miller, 2011; Babonneau, Nesterov and Vial, 2012). Given a common project developed for several agents, they should pay the total cost to implement the most-liked network. Therefore, two questions arise about how to allocate the cost and how agents agree to implement such network. A large literature aims the first problem. A common approach consists of the design of rules that satisfy axioms representing properties as incentive, fairness, among others. See for instance, Bergantiños and Kar (2010), Bergantiños and Lorenzo-Freire (2008a, 2008b), Bergantiños and Vidal-Puga (2007a, 2007b, 2009), Bogomolnaia and Moulin (2010a, 2010b), among others). The second problem stems on strategic issue. In many network settings, users are motivated by self-interest; hence, it solution should be in tune with incentive-compatible behavior. See Bergantiños and Lorenzo-Freire (2004), Bergantinñs and Vidal-Puga (2007a), Chen, Rougharden and Valiant (2010), etc.

We focus on complete weighted networks with n users and an additional special node named the source. By Prim's algorithm (Prim, 1957), it is known how to connect the users in such a way the total cost of the project "all connected to the source" was minimal. The algorithm provides an optimal social solution in the sense that if an external planner had to choose a tree, she would choose this one involving the lowest cost for the society. Nevertheless, if every user could choose a tree and with whom she would be connected, the minimal tree offered by Prim may not be a practical solution. Users may find profitable to secede from the "social" optimal solution taking her individual best choice. We ask what should be a sharing of the total cost satisfying incentive compatible condition, in other words, what are the conditions such that any user prefers to implement the Prim's minimal cost spanning tree rather than any other tree.

We propose a sharing cost rule based on side payments or compensations. A collection of transfers x_{ij} for each user k is put forward. It means the amount that user i pays to j if she connects with k. Consequently, for all tree where j and k are linked, the user i shares the cost of this link by x_{ij}. Fix a tree where each user is connected directly or indirectly to the source. Taking into account such transfers, the final cost of implementing such a tree for each user is characterized as the sum of three terms. First of all, each user pays the cost of the link she uses, her direct cost. A second term corresponds to the user pays to whom is connected, her immediate predecessor. This second term is understood as the amount paid for being linked to the other users in order to implement the tree. The last term is the quantity she receives from other users to properly connect with her. Therefore, additional amounts are transferred across agents in order to sustain the specific tree. Our first result states the existence of a family of compensations, *cost sharing structures*, such that no agent has incentives to choose another tree than the minimum cost tree offered by Prim. Therefore, the minimal spanning tree emerges as both a social and an individual solution.

The family of cost sharing structures has two desirable features taking into account the non-cooperative and the cooperative perspectives. Following a non- cooperative approach, a finite sequential game is defined. The users of the network are the agents in this extensive game. At each stage, the set of actions for agent i is the set of users and source with whom she connects. In this game, payoff vectors depend on the cost sharing structure. Each terminal node is associated to a path which represents a feasible tree. This describes a finite extensive game with perfect information. The existence of a pure subgame perfect equilibrium is guaranteed for the finitely conditions. Our second result states that the path associated to the minimal cost spanning tree is the unique subgame perfect equilibrium.

Following the approach in Granot and Maschler (1998), or Kar (2002), it is possible to associate to every cost spanning tree problem a cooperative monotone game. A coalition is formed if its members agree to connect each one of them to the source. We allow the members of a coalition to connect themselves using cheaper edges, even they may choose to be connected through players that do not belong to the coalition. However, we require that the formed coalition should be a connected subgraph. Given this cooperative approach, the transferable utility game is generated by considering the minimum cost spanning tree for each coalition. Now, by considering the usual solutions of cooperative games, a solution of the problem of distributing the minimal cost of a spanning tree problem is obtained. Given an optimal cost structure, our last result states that the cost distribution we provide is in the core of the cooperative game.

The rest of the paper is organized as follows. Section 2 presents the model of our network structure together with the extensive game and the cooperative game associated to the network. Several examples drive to the goal of the paper. Sections 3 states the main results by building up an optimal cost structure. Section 4 closes the paper with the strategic and cooperative properties of our solution.

2 Model

2.1 Networks

Let $N = \{1, \dots, n\}$ be a finite set of agents and 0 the *source* they want to be connected. Denote by $N_0 = N \cup \{0\}$. A *graph* over N_0 is a function $p : N \mapsto N_0$ so that i connects $p(i)$. We only consider graphs where any agent is (directly or indirectly) connected to the source; that is, p such that for all i there is some $t \in \mathbb{N}$:

$p \circ \ \ldots \circ p = p^t(i) = 0$ and we name *feasible tree* to a connected graph where there is a unique path from i to the source for all $i \in N$.

Let $\mathbf{C}$ be the cost matrix, where $c_{ij} \in \mathbb{R}_+$ represents the connection cost between agents i, $j \in N_0$. The social network is represented by $(N_0, \mathbf{C})$. Assume, as usual, that $c_{ii} = 0$ and $\mathbf{C}$ is symmetric, *i.e.*, $c_{ij} = c_{ji}$.

Prim (1957) offers an algorithm for solving the problem of connecting all agents to the source such that the total cost of creating the network is minimum. The achieved solution may not be unique. Denote by m the tree with minimal cost obtained through Prim's algorithm and by C_m its cost. That is,

$$C_m = \sum_{i=i}^{n} c_{im(i)} \leq \sum_{i=i}^{n} c_{ip(i)}$$

for all feasible p. Our target is to set proposals on how the total (minimal) cost C_m is distributed amongst agents verifying individual incentive condition, *i.e.*, each agent prefers to implement the minimal spanning tree.

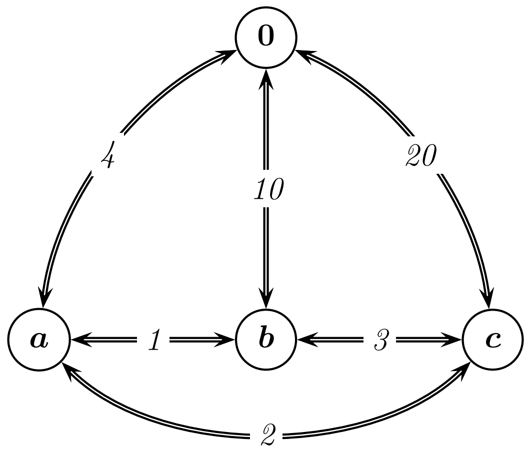
2.2 Cost structure

Before introducing our notion of *cost structure*, we analyse the following example.

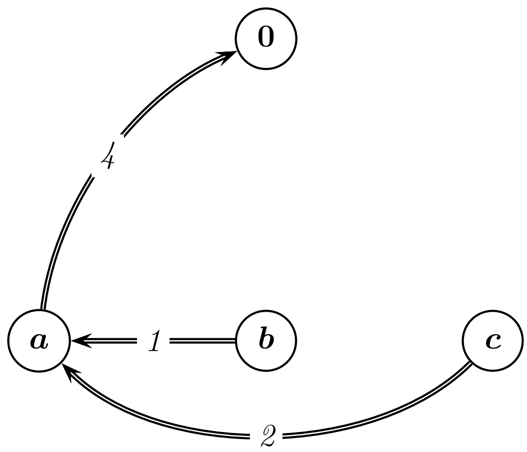
Example 1 *Consider individuals a, b, and c, who want the service of cable network in their respective houses. There is only one cable operator, whom we will call the source, in the locality. It is not necessary for each one to be connected directly to the source. For instance, a could be connected to b and b to the source, thereby providing an indirect connection of a to the source. The costs of connections are represented by a cost matrix* $\mathbf{C}$,

$$\mathbf{C} = \begin{bmatrix} 4 & 0 & 1 & 2 \\ 10 & 1 & 0 & 3 \\ 20 & 2 & 3 & 0 \end{bmatrix}$$

where the first column represents the cost of direct connections to the source. The following figure shows the graphical representation of this problem:



The first objective is to find the minimum cost cable network in which a, b, and c are connected to the source. The minimum cost will be 7 units and the minimum cost spanning tree is:



The remaining question is how the total cost of 7 units is distributed amongst a, b, and c. In order to discuss this problem, if we fix our attention on agent a, her cost to directly connect with the source is 4 units, whereas she can connect to b with a cost of 1 unit. So, she would be willing to pay up to 3 units to agent b, in order that b connects to the source and then a can connect through b. By using this idea we define a set of side payments:

We represent $x_{ij}(k)$ as the amount that player i pays to player j if she connects with k. In other words, player i pays $x_{ij}(k)$ to player j for all tree p such that $p(j) = k$. For each node $k \in N$ we designate the following matrix:

$$X(k) = [x_{ij}(k)]_{n \times n} = \begin{bmatrix} x_{11}(k) & x_{12}(k) & \cdots & x_{1n}(k) \\ x_{21}(k) & x_{22}(k) & \cdots & x_{2n}(k) \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1}(k) & x_{n2}(k) & \cdots & x_{nn}(k) \end{bmatrix}$$

The cost structure is the family of matrices $x = \{X(k)\}_{k \in N_0}$. Notice that, unlike what happens with the cost matrix $\mathbf{C}$, matrices $X(k)$ need not be a symmetric matrix. We are now ready to write down the cost of implementation $C(i;p)$ for an agent $i \in N$, a given feasible tree p, and a cost structure x:

$$C(i;p)_x = c_{ip(i)} + x_{ip(i)}(p^2(i)) - \sum_{\{k:p(k)=i\}} x_{ki}(p(i))$$

where $c_{ip(i)}$ corresponds to the direct cost for agent i to be connected to player $p(i)$. The second term is the amount that player i pays to $p(i)$ her *immediate predecessor* (the agent with whom i connects) to sustain the tree p, given the cost structure. Therefore, for being linked to the other agents to follow the tree p. The last term is the amount received by agent i from the other agents to properly connect with her. Given a tree p and a cost structure $x = \{X(k)\}_{k \in N_0}$, it is immediate to observe that the sum of the individual costs $C(i;p)_x$ is equal to the cost of the tree denoted by C_p.

$$\sum_{i=i}^{n} C(i;p)_x = C_p = \sum_{i=i}^{n} c_{ip(i)}$$

So, the cost structure provides a way of distributing the cost of the tree.

Example 2 *A possible cost structure for the problem in Example 1 is given by:*

$$X(0) = \begin{bmatrix} 0 & 3 & 2 \\ 9 & 0 & 7 \\ 18 & 17 & 0 \end{bmatrix}; X(a) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 7 \\ 0 & 17 & 0 \end{bmatrix}; X(b) = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 18 & 0 & 0 \end{bmatrix}; X(c) = \begin{bmatrix} 0 & 3 & 0 \\ 9 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that each agent i is paying to any other agent j the difference between her cost to connect directly with the source, and the cost to connect with her. Now consider the tree m defined by:

$$m(a) = 0;\ m(b) = a;\ m(c) = a.$$

Then, the associated cost for each agent is:

$$\begin{aligned} C(a;m)_x &= c_{a0} - (x_{ba}(0) + x_{ca}(0)) = 4 - 9 - 18 = -23 \\ C(b;m)_x &= c_{ba} + x_{ba}(0) = 1 + 9 = 10 \\ C(c;m)_x &= c_{ca} + x_{ca}(0) = 2 + 18 = 20. \end{aligned}$$

Notice that the second term $x_{am(a)}(m^2(a)) = x_{a0}(m(0))$ in $C(a;m)$ has zero value since a connects to 0 and $m(0) = \emptyset$. For $C(b;m)$ and $C(c;m)$, the third term has zero value since for the tree m there is not k such that $m(k) = b$ and $m(k) = c$.

One may think that agents b deviates from this cost distribution since she prefers the feasible tree $p$such that $p(a) = 0$; $p(b) = 0$ and $p(c) = b$. The new cost for each agent is:

$$\begin{aligned} C(a;m)_x &= c_{a0} - (x_{ba}(0) + x_{ca}(0)) = 4 \\ C(b;m)_x &= c_{b0} - x_{cb}(0) = 10 - 17 = -7 \\ C(c;m)_x &= c_{cb} + x_{cb}(0) = 3 + 17 = 20. \end{aligned}$$

Then, the agent b has incentive to deviate from m to another tree with total cost 17. A question arises: is it possible to guarantee the prevalence of the minimum spanning tree for a cost structure?

2.2.1 Conditions on $C(i,p)_x$

We now explore some conditions on the matrices $X(k)$ and the corresponding costs $C(i;p)_x$.

First, the cost structure $C(i;p)_x$ must fulfil some "logical" properties with respect to the problem and to the cost function:

- Each agent will pay, at least, his minimal link connexion:

$$C(i;p)_x \geq min\{c_{ik} : k = 0, 1, \ldots, n; k \neq i\}$$

- Each agent will pay, at most, their direct cost to the source:

$$C(i;p)_x \leq c_{i0}$$

- No agent i pays other agent j for j connecting to her:

$$x_{ij}(i) = 0$$

Note that the cost distribution provided in Example 2 does not fulfil this conditions. On the other hand, we will ask the cost structure supports the minimum cost spanning tree, as we will define later. Moreover, under these conditions, matrices $X(k)$ have null the diagonal elements; and elements in row and column k are also null, for $k \neq 0$. In the next example we show a cost structure satisfying these conditions.

Example 3 *Let us consider the following cost structure for Example 1:*

$$X^*(0) = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}; X^*(a) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; X^*(b) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}; X^*(c) = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then, we have

$$C(a;m)_{x^*} = 1; C(b;m)_{x^*} = 3; C(c;m)_{x^*} = 3.$$

which could be a more reasonable distribution of the cost amongst the agents.

2.3 The spanning tree extensive game.

Our objective now is to define a family of sequential games parametrized by the cost distribution problem. For a finite set $\{1, 2, \ldots, n\}$ consider Π_n the set of permutations over $\{1, 2, \ldots, n\}$. The element $\pi = (\pi(1), \ldots, \pi(n)) \in \Pi_n$ determines the order of choices among players. We denote by

$$\Gamma_\pi = (N, K, S_i, Z, u_i : \prod_{i=1}^{N} A_i \to \mathbb{R})$$

the sequential game, where:

- $N = \{1, 2, \ldots, n\}$ is the set of players.
- K is the set of not terminal nodes, and any node k in K is an information set. We denote by $i(k)$ the agent playing at node k. Z is the set of terminal nodes.
- The set of actions for player i, denoted by S_i, is the set of agents (included the source) with whom player i may connect. Note that when some connections are already done, the set of feasible players to connect in order to implement a feasible tree that may not be the remainder of players. The set of actions at stage t will depend on the sub-graph (a feasible tree) already implemented; in order words, on the links already done. Namely, it depends on the permutation π.

- Let's start with the first stage with $k = 1$. The set of actions is

$$S_{i(1)} = \{0, 1, \ldots, i(1) - 1, i(1) + 1, \ldots, n\}.$$

  That is, player $i(1)$ selects the player (other agent, or the source) with whom she connects.

- For k such that $i(k) \neq \pi(n)$

$$S_{i(k)} = (N \cup \{0\}) - \{i(k)\} - \{j \in N : \exists k' < k \text{ such that } j = i(k') \text{ and } s_j = i(k)\}$$

  That is, player $i(k)$ selects, among the available agents or the source, the one she wants to connect.

- For $i(k) = \pi(n)$ we have two cases:

  * If there is some agent $j \neq \pi(n)$ such that her action was to connect to the source $s_j = 0$, then

$$S_{i(k)} = N \cup \{0\} - \{i(k)\} - \{j \in N : \exists k' < k \text{ such that } j = i(k') \text{ and } s_j = i(k)\}$$

  * In other case (no agent is already connected to the source)

$$S_{i(k)} = \{0\}$$

- At the set of terminal nodes Z, payoffs $\{u_i\}_{i \in \mathbf{N}}$ are realized. Notice that the cardinality of Z is the number of different feasible trees. Then, for all $z \in Z$ there exists a unique path, h_z associated to one tree denoted by p_z.

- The payoff function of player i, $u_i(s_1, s_2, \ldots, s_n)$ represents the amount player i must pay for implementing the tree. We can assign to the path $(s_1, s_2, \ldots, s_n)$ the corresponding path h_z where the payoff is realized. Then,

$$\begin{aligned} u_i(s_1, s_2, \ldots, s_n) &= u_i(h_z) = u_i(p_z)\mathbf{C}(i; p_z)_x \\ &= c_{ip_z(i)} + x_{ip_z(i)}(p_z(p_z(i))) - \sum_{\{k : p_z(k) = i\}} x_{ki}(p_z(i)) \end{aligned}$$

This extensive game has $n+1$ stages. At stage 0 there are $n!$ realizations of Π_n. At first stage, the initial node of the following subgame, player $i(1)$ has n actions corresponding by linking with any other agent and the source. Let $f_1(n) = 1$ the number of initial notes. At stage 2, the action set of player $i(2)$ depends on that player $i(1)$ has already chosen. The number of nodes at stage k is computed by the following recursive[1] formulae:

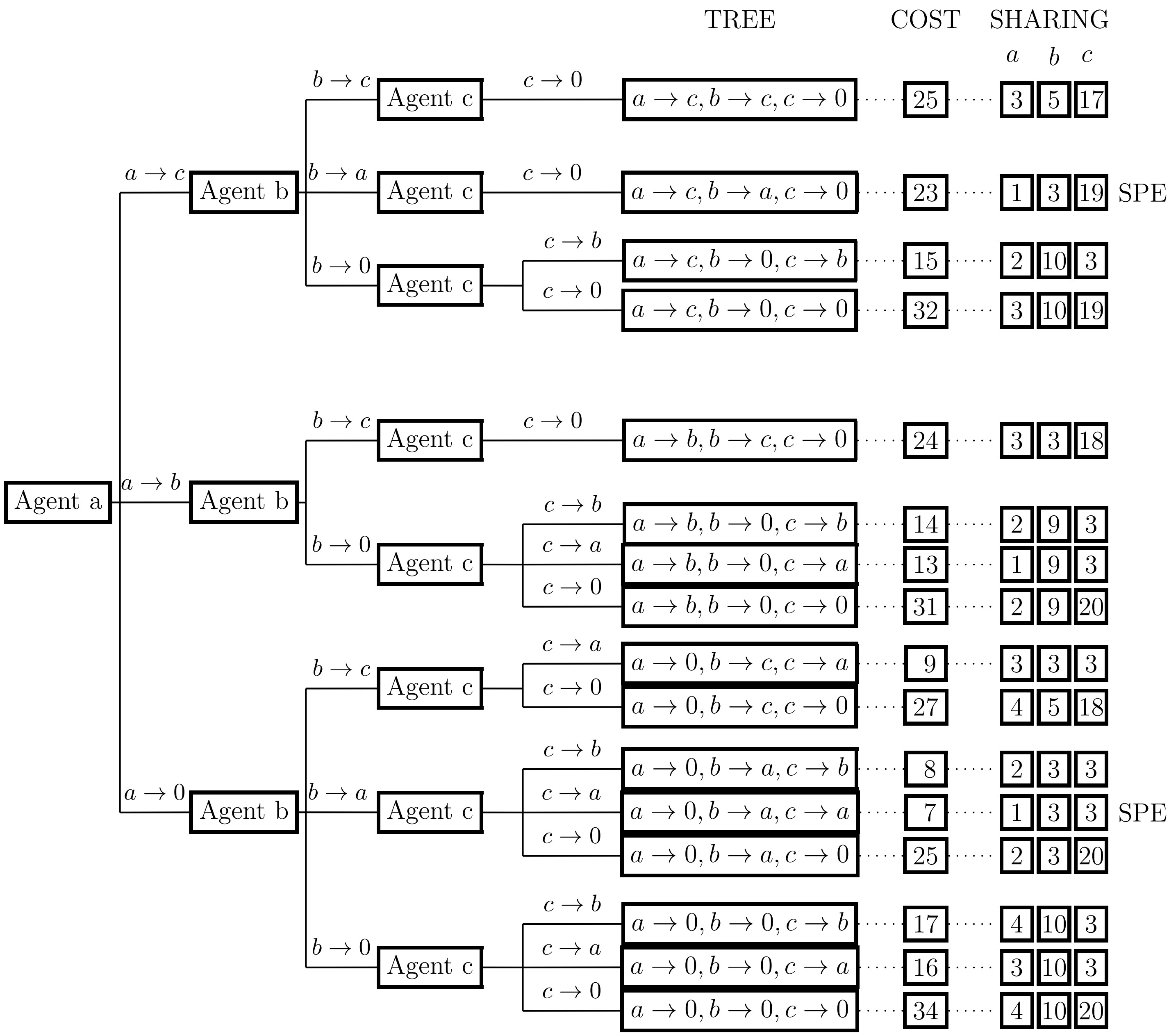
$$f_k(n) = nf_{k-1}(n) - \sum_{s=1}^{k} D^s[f_{k-1}(n)] \; for \; 2 \leq k \leq n$$

where D^s denotes the s-derivative and n is the number of agents.

Given a permutation π a subgame of the extensive game is constructed and payoff depends on the cost structure. Next figure illustrate the extensive game $\Gamma_{\pi(i)=i}$ with the cost structure in the previous example 3.

[1]For instance:

- $f_1(n) = 1$
- $f_2(n) = n$
- $f_3(n) = nf_2(n) - D\ f_2(n) - D^2\ f_2(n) = n^2 - 1$



EXTENSIVE GAME FOR EXAMPLE 3

The next example illustrate the above consideration:

Example 4 *Let now construct* $\Gamma_{\pi(i)=i}$*:*

- *At the first stage agent* a *decides the agent (or the source) she wants to connect* s_a*.*
- *Now, agent* b *decides where to connect by taking in account* s_a*. Suppose that* $s_a = b$*. This implies* $s_b \neq a$*.*
- *Finally, agent* c *connects to the source if no one is previously connected to it, or she connects to some available agent otherwise.*

Then, the cost structure determines the payoffs. For instance, if a *and* b *select her minimum connection, we have:*

$$s_a = b; s_b = c; s_c = 0$$

This path corresponds with the fifth terminal node. For the cost structure in Example 3, the cost of this tree p *is 24 units distributed in the following way:*

$$C(a;p)_{x^*} = 3; C(b;p)_{x^*} = 3; C(c;p)_{x^*} = 18.$$

Note that this tree does not coincide with the minimum cost spanning tree m*. Moreover, the distribution of the tree cost among agents depends on the cost structure.*

This kind of sequential game is studied by Bergantiños and Lorenzo-Freire (2005). They prove the existence of a permutation such that the minimum spanning tree is assigned as a subgame perfect equilibrium. The payoff profile is exactly the direct cost for each user. In our set up, the minimum spanning tree emerges as strategic solution independent of the permutation.

Given the sequential structure of Γ_π, we want to characterize the subgame perfect equilibria (SPE henceforth). The equilibrium strategies should specify optimal behavior from any information node up to the end of the game. That is, any agent's strategy should prescribe what is optimal from that node onwards given her opponents' strategies. Notice that the final payoffs do depend on the values of x^* and therefore on the cost structure of the network.

2.4 Minimum cost spanning tree cooperative game.

Following the approach in Granot and Maschler (1998), or Kar (2002), it is possible to associate to every cost spanning tree problem a cooperative monotone game in a natural way. We say that a coalition $S \subseteq N$ is formed if its members agree to connect each one of them to the source. Obviously, the members of S will choose to connect themselves using least expensive edges. They may even choose to be connected through players not in S. Thus, we allow free riders[2], where members of a coalition S are not allowed to use vertices occupied by members of $N - S$. This is called the *monotone minimum cost spanning tree cooperative game.* However, we require that the set of vertices and agents T used by a formed coalition S, $S \subseteq T$ should be a connected subgraph.

Following this cooperative approach, the transferable utility game is generated by considering the minimum cost spanning tree for each coalition $S \subseteq N$, *i.e.*, the characteristic function of this cooperative game is defined as follows:

- Given a subset of agents $S \subseteq N$ and a tree p, the cost for agents in S for this tree is:
$$C(S,p) = \min_{S \subseteq T} \{ \sum_{i \in T} c_{ip(i)} \}$$
- Then, the characteristic function is
$$C(S) = \min_{p} \{ C(S,p) \}$$

Now, by considering the usual solutions of cooperative games, a solution of the problem of distributing the minimal cost of a spanning tree problem is obtained. For instance, Kar (2002) provides a solution which is based in the Shapley value of this cooperative game. Moreover he gives an axiomatic characterization of his solution. Granot and Huberman (1984) and Granot and Mashler (1998) analyse the core and nucleolus.

Example 5 *With the cost matrix in Example 1, the characteristic function is:*
$v(\{a\}) = 4; v(\{b\}) = 5; v(\{c\}) = 6;$
$v(\{a,b\}) = 5; v(\{a,c\}) = 6, v(\{b,c\}) = 7;$
$v(\{a,b,c\}) = 7.$

[2]This is in contrast to other models (see, e.g., Bird (1976), Granot and Huberman (1981)).

3 Optimal cost structure

This section is devoted to study the existence of an optimal cost structure for a given network. What is the meaning of *"optimal"* cost structure? We wish that payments $x_{ij}(k)$ fulfil the individual incentive compatibility: no agent can found an alternative tree p in which her cost is lower than in the minimum cost tree m:

$$C(i;p)_x \geq C(i;m)_x \quad \forall p,\ \forall i.$$

Next proposition provides a way to construct an optimal cost structure x. We proceed generating the matrices $X(k)$ for $k \in N_0$ by steps. Fixing the tree m, let r be the maximal number of links needed to connect any node to the source. At step $t \in \{1, \ldots, r\}$, we distinguish the set of agents connected to the source with t links, named N_t. At step t, for any $i \in N_t$, we obtain the column i at each $X(k)$. Namely, the quatities $x_{ij}(k)$ is twofold: on the one hand, they guarantee the implementation of m, and, on the other hand, they preclude any individual deviation of implementing another feasible tree than m. So payments from N_t to $N_{t'}$ for $t' \in \{t+1, \ldots, r\}$, is merely an incentive to deter the deviation from the social goal, although these payments are be not effective in m.

Proposition 1 *There exist a cost structure x such that $C(i;p)_x \geq C(i;m)_x$ $\forall p$ and for all player i.*

Proof.
In order to prove this existence of a cost structure satisfying $C(i;p)_x \geq C(i;m)_x$, we are going to define, given a social network $(N_0, \mathbf{C})$, a cost structure x of (possible) payments satisfying our requirements. We proceed by steps from 1 to r:

step 1

Consider the set of agents $N_1 = \{\alpha \in N : m(\alpha) = 0\}$; that is, those agents that in the minimum cost tree connect directly to the source. Now we have two possibilities: all agents go directly to the source, or there is some agent that connects indirectly to the source.

case 1 $N_1 = N$

In this case, $m(\alpha) = 0, \forall \alpha \in N$. So, no agent connects to any other agent and payments will be not effective Then, we define

$$x_{ij}(k) = 0, \forall i, j \in N, k \in N \cup \{0\}$$

Let us suppose that there is some agent i and some tree p such that

$$C(i;p)_x = c_{ip(i)} < C(i;m)_x = c_{im(i)}$$

Consider the tree q defined by:

$$q(j) = \begin{cases} m(j) = 0 & j \neq i \\ p(i) \end{cases}$$

Then, we obtain

$$C_q = C_m + \left(c_{ip(i)} - c_{im(i)}\right) < C_m$$

contradicting that m is the minimum cost tree. The proof concludes.

case 2 $N_1 \neq N$

Let β an agent that does not connect directly to the source in the minimal cost tree m, $\beta \in N - N_1$; then we define

$$x_{\alpha\beta}(k) = \max\left\{c_{\alpha 0} - c_{\alpha\beta}, 0\right\}, \forall \alpha \in N_1$$

What are we doing? By defining these payments, agents that connect to the source within m will prefer (or will be indifferent) this connection to any other provided by an alternative tree. That is, we are given incentives to the agents to chose the socially efficient tree m. Effectively, for any $\alpha \in N_1$

$$C(\alpha;p)_x = c_{\alpha p(\alpha)} + x_{\alpha p(\alpha)}\left(p^2(\alpha)\right) - \sum_{i:p(i)=\alpha} x_{i\alpha}(p(\alpha))$$

(note that the third term is equal to zero, since no agent pays anything to the ones that connect directly to the source)

$$= c_{\alpha p(\alpha)} + x_{\alpha p(\alpha)}\left(p^2(\alpha)\right)\big) = \begin{cases} c_{\alpha p(\alpha)} & \text{if } c_{\alpha p(\alpha)} \geq c_{\alpha 0} \\ c_{\alpha p(\alpha)} + (c_{\alpha 0} - c_{\alpha\beta}) = c_{\alpha 0} & \text{otherwise} \end{cases}$$

and

$$C(\alpha;m)_x = c_{\alpha 0} + x_{\alpha 0}\left(m^2(\alpha)\right) - \sum_{i:m(i)=\alpha} x_{i\alpha}(0) = c_{\alpha 0}$$

so,

$$C(\alpha;m)_x \leq C(\alpha;p)_x \qquad \forall \alpha \in N_1$$

At this point we have proved that agents in N_1, which are the only ones that have non-zero payments, attach their optimal choice in the minimum cost tree m. In the next steps we follow a similar argument in order to define the cost structure for the remaining agents so they also attach their optimal choice at m.

step 2

Now, the costs of the links involving elements outside N_1 have been modified, since these agents can receive some quantities from the agents in N_1. Let us name

$$c_{ij}^2 = c_{ij} - \sum_{r=1}^{n} x_{ri}(j)$$

This is the amount she pays for connecting agent j, once we have discounted the amount she receives from the agents in N_1 for join j. Consider now the set of agents $N_2 = \{\alpha \in N - N_1 : m^2(\alpha) = 0\}$ (the agents that arrive to the source with exactly two links). If any remaining agent is in this set, $N = N_1 \cup N_2$ and we only need to define the payments between these agents. In other case, we will need to continue with the other agents.

case 1 $N_2 = N - N_1$

In this case, $m^2(\alpha) = 0, \forall \alpha \in N$. Then, we modify our vector x of payments in the following way:

$$x_{ij}(0) = \max\{c_{im(i)} - c_{ij}^2, 0\} \qquad \forall i, j \in N_1, i \neq j$$

Note that these changes do not affect to the payments from agents in N_1, so only null coefficients are changed. Moreover, only payments are made from i to j for j joining the source, in such a way that the agents do not wish to deviate from tree m due the amounts they are now receiving. We continue naming x to the new cost structure. Observe that the cost function remains invariant for all agents in N_1 and then these agents still are in an optimal situation at tree m, that is

$$C(\alpha; m)_x \leq C(\alpha; p)_x \qquad \forall \alpha \in N_1$$

In order to compute the new cost of a tree p, for agents outside N_1, we need to distinguish if through p they arrive to the source in one or two steps (like in m) or if they need more links.

sub-case a $p^2(\alpha) = 0$

Then,[3]

$$\begin{aligned} C(\alpha;p)_x &= c_{\alpha p(\alpha)} + x_{\alpha p(\alpha)}\left(p^2(\alpha)\right) - \sum_{i:p(i)=\alpha} x_{i\alpha}(p(\alpha)) = \\ &= c_{\alpha p(\alpha)} + x_{\alpha p(\alpha)}\,(0) - \sum_{i:p(i)=\alpha} x_{i\alpha}(p(\alpha)) \geq \\ &\geq c_{\alpha p(\alpha)} + x_{\alpha p(\alpha)}\,(0) - \sum_{i=1}^{n} x_{i\alpha}(p(\alpha)) = c^2_{\alpha p(\alpha)} + x_{\alpha p(\alpha)}\,(0) \geq \\ &\geq c_{\alpha m(\alpha)} = C(\alpha;m)_x \end{aligned}$$

sub-case b $p^2(\alpha) \neq 0$

Let us suppose that there is some agent i and some tree p such that

$$C(i;p)_x < C(i;m)_x, p^2(i) \neq 0.$$

Consider the tree q defined by:

$$q(j) = \begin{cases} m(j) = 0 & j \neq i \\ p(i) & \end{cases}$$

Then, it is easy to check that for all $j \neq i$ we have

$$\begin{aligned} C(j;q)_x &= C(j;m)_x; \quad \text{and} \\ C(i;q)_x &= C(i;p)_x \end{aligned}$$

So,

$$C_q = \sum_{i=1}^{n} C(i;q)_x < \sum_{i=1}^{n} C(i;m)_x = C_m$$

contradicting that m is the minimum cost tree. The proof concludes.

case 2 $N_2 \neq N - N_1$

Let β an agent that does not connect *via* m to the source in one or two links, $\beta \in N - (N_1 \cup N_2)$; then if we modify the payments by considering

[3] The argument for the last inequality is analogous to the one used in step 1, case 2.

$$x_{\alpha\beta}(k) = \max\{c_{\alpha m(\alpha)} - c^2_{\alpha\beta}, 0\} \qquad \forall \alpha \in N_2, k \in (N - (N_1 \cup N_2)) \cup \{0\}, k \neq \alpha, \beta$$

we obtain

$$\begin{aligned} C(\alpha;p)_x &= c_{\alpha p(\alpha)} + x_{\alpha p(\alpha)}\left(p^2(\alpha)\right) - \sum_{i:p(i)=\alpha} x_{i\alpha}(p(\alpha)) = \\ &= c_{\alpha p(\alpha)} + x_{\alpha p(\alpha)}\left(p^2(\alpha)\right) - \sum_{i:p(i)=\alpha} x_{i\alpha}(p(\alpha)) \geq \\ &\geq c_{\alpha p(\alpha)} + x_{\alpha p(\alpha)}\left(p^2(\alpha)\right) - \sum_{i=1}^{n} x_{i\alpha}(p(\alpha)) = \\ &= c^2_{\alpha p(\alpha)} + x_{\alpha p(\alpha)}\left(p^2(\alpha)\right) \geq c_{\alpha m(\alpha)} = C(\alpha;m)_x \end{aligned}$$

so,

$$C(\alpha;m)_x \leq C(\alpha;p)_x \qquad \forall \alpha \in N_2$$

Therefore we have that the agents in $N_1 \cup N_2$ attach her optimal tree at the minimum cost tree m. By repeating an analogous argument as in [step 2] we conclude the proof. ∎

3.1 Optimal cost structure is not unique

We can observe that the cost structure in Example 3 is optimal. In the next example we define an alternative cost structure which is also optimal.

Example 6 *Consider the cost structure defined by:*

$$\hat{X}(0) = \begin{bmatrix} 0 & 5 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}; \hat{X}(a) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \hat{X}(b) = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}; \hat{X}(c) = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then, we have

$$C(a;m)_{\hat{x}} = 4; C(b;m)_{\hat{x}} = 1; C(c;m)_{\hat{x}} = 2.$$

The main problem is not about the existence of several optimal cost structures, but they propose different distributions of the cost among the agents. An interesting property of optimal cost structures is that of convexity.

Proposition 2 *The set of optimal cost structures for a problem* $(N_0, \mathbf{C})$ *is a convex set.*

Proof. Let us consider two optimal cost structures x^1, x^2 and $\lambda \in [0, 1]$. Consider the cost structure $x = \lambda x^1 + (1 - \lambda)x^2$. It is immediate to observe that x satisfies the required conditions and that

$$C(i; m)_x = \lambda C(i; m)_{x^1} + (1 - \lambda)C(i; m)_{x^2}$$

so x is also an optimal cost structure. ■

Moreover, it is immediate to see that this set is also closed and bounded. More precisely, it is a convex polyhedron.

4 Strategic and cooperative properties

Now, we are going to analyse the behavior of the optimal cost structure from strategic and cooperative points of view.

4.1 Strategic game: implementing a SPE

Proposition 3 *Given an optimal cost structure* x *for a problem* $(N_0, \mathbf{C})$*, the minimal cost tree* m *is a subgame perfect equilibrium of the strategic game* Γ_π*.*

Proof. Let X be an optimal cost structure. Let Γ be the extensive game associated to x and $\pi \in \Pi$ any order permutation over the set of player $\{1, \ldots, n\}$ which generates the branches at stage 0. Notice that Γ is a finite extensive game with perfect information. Therefore, there exist pure subgame perfect equilibrium probably not unique. To obtain the proposition is enough to prove that the path in the extensive game associated to the minimal cost tree m is the best response at any node for the associated subgame where each player implement the minimal spanning tree. At any node of the path related to the minimal spanning tree, denoted by h_m, every agent plays the best action given the best strategy of the players onward. By definition of m and x, we know that $C(i; m)_x \leq C(i; p)_x$ for all $p \neq m$. In particular for any tree p that is not m in the subgame where m is involved. Therefore, m is the best response for all player at any node of h_m and the result holds. ■

The strategic game Γ_π may have several SPE different from the one given by the *mcst* m, as we show in the following example. But, if we consider that agents may appear in any possible order π, then m is the only SPE that remains.

Example 7 *Given the cost structure in Example 3, we know that m is a SPE that provides the following cost distribution*

(SPE 1): $[1, 3, 3]$

But, by considering the permutation $\pi(i) = i$, there is another SPE provided by tree p, defined by:

$$p(a) = c; p(b) = a; p(c) = 0$$

The cost of this tree is 23 units which are distributed in the following way

(SPE 2): $[1, 3, 19]$

This equilibrium cannot be regarded as a good solution under the social point of view. Nevertheless, it emerges as a subgame perfect equilibrium. This is so, since the player 3 only has the choice to connect to the source. This mandatory action jointly to the circunstance that the rest of the players in the corresponding subgame have as best response the above tree, entail the prevalence of this unsatisfactory equilibrium.

Proposition 4 *Given an optimal cost structure x for a problem $(N_0, \mathbf{C})$, the minimal cost tree m is the only subgame perfect equilibrium of the strategic game Γ_π, for all permutation π.*

Proof. Let us suppose that the agents chose their strategies in the order $\{1, 2, \dots, n\}$ (that is, we consider the permutation $\pi(i) = i$). It has been already shown that m is a SPE. If there is another tree p that also corresponds to a SPE, then since p does not minimize there is an agent (say agent 2) such that:

$$C(2; m)_x < C(2; p)_x$$

If we now consider a permutation such that $\pi(1) = 2$ (the first agent who decides is agent 2), then clearly tree p will not be chosen by this agent, so p is not a SPE. ■

4.2 Cooperative game: included in the core

Proposition 5 *Given an optimal cost structure x for a problem $(N_0, \mathbf{C})$, the cost distribution $\{C(i; m)_x\}$ for $i \in N$ is in the core of the cooperative game (N, v).*

Proof. The Core of the cooperative games is

$$Co(v) = \left\{ z \in R_+^n : \sum_{k \in S} z_k \leqq v(S) \quad \sum_{k \in N} z_k = v(N) \right\}$$

To prove that the distributions of the minimum cost among the agents provided by the optimal cost structure x belongs to the core let us suppose that there is a subset $S \subseteq N$ such that

$$v(S) < \sum_{i \in S} C(i;m)_x$$

We consider two cases:

case 1

The minimum cost tree connecting each agents in S to the source only involves agents in S. We denote by p_S the network providing $v(S)$. Then,

$$v(S) = \sum_{i,j \in S : p_s(i)=j} c_{ij}$$

Consider a network p_Q, $Q = N - S$ that connects the elements outside S with the source not involving elements in Q . Then,

$$p = p_s \cup p_Q$$

is a network connecting each agent to the source. Note that in this network the agents in S and outside S are no connected, so the cost function only depends on the elements in each subset. So

$$v(S) = \sum_{i,j \in S : p_s(i)=j} c_{ij} = \sum_{i \in S} C(i;p)_x < \sum_{i \in S} C(i;m)_x$$

and then there is some agent $k \in S$ such that $C(k;p)_x < C(k;m)_x$, which contradicts that X fulfils conditions (number lo que sea).

case 2

If the minimum tree for agents in S involves agents outside this set, then $v(S) = v(T)$ $S \subset T$. Since coalition T contains more agents than S, we have

$$v(T) = v(S) < \sum_{i \in S} C(i;p)_x < \sum_{i \in T} C(i;m)_x$$

and we can apply case 1 to the set T to find a contradiction. ■

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