Super-majorites, one-dimensional policies, and social surplus *

Daniel Cardona†
Universitat de les Illes Balears and CREB

Clara Ponsatí
Institut d'Anàlisi Econòmica-CSIC and Barcelona GSE

May 3, 2012

Abstract

In the setting of a one-dimensional legislative bargaining game, we characterize qualified majority rules maximizing social surplus - i.e., the sum of individual benefits. The simple majority rule maximizes social surplus when individual utilities are tent-shaped. When the utilities are strictly concave, the surplus maximizing rule is a strict super-majority.

1 Introduction

Democratic polities take decisions by bargaining and voting: Proposals are submitted to the floor until one receives the favorable votes of a (super)majority. Weather the required support is, for instance, a simple majority, a 2/3 majority, or unanimity, is not inconsequential, as the different rules may deliver different outcomes. A natural question is: What rules deliver maximal collective benefits? In this paper, we build on previous results that establish the uniqueness of the equilibrium for the model of bargaining and voting over one-dimensional policies to address this question precisely.

*Financial support from grants ECO2009-08820, ECO2009-06953, and SGR2009-1051 is gratefully acknowledged. Part of this paper was included in an early draft entitled "Bargaining one-dimensional policies and the efficiency of super-majority rules".

†Corresponding author: Departament d'Economia Aplicada, Universitat de les Illes Balears and CREB, Campus UIB, 07122 Palma, Illes Balears, Spain. E-mail: d.cardona@uib.cat.

This paper adds to the literature on bargaining in legislatures initiated by Baron and Ferejohn (1989). More specifically, our paper contributes to the literature addressing the relationship between the voting rules and the bargaining outcomes (e.g., Banks and Duggan (2000, 2006), Cardona and Ponsatí (2007, 2009, 2011), Cho and Duggan (2003), Predtetchinski (2011)). We refer the reader to Cardona and Ponsatí (2011) for a detailed discussion of this literature.

We consider environments where policies must be selected from a continuous one-dimensional set, where individuals have single-peaked and concave utilities, and they are heterogeneous only in the locations of peaks. Bargaining takes place over time. At each period a randomly selected individual makes a proposal which is approved if it receives the vote of a qualified majority; otherwise, a new proposer is selected next period, and so on. Assuming that individual payoffs are discounted utilities, we have proved in Cardona and Ponsatí (2011) that this game has a unique equilibrium.[1] Thanks to the uniqueness of the equilibrium, and its explicit characterization, the welfare performance of the different majority requirements can be assessed. Cardona and Ponsatí (2011) supply sharp results for the special case of symmetric populations: Assuming strict impatience unanimity is the unique (ex-ante) Pareto optimal rule. However, when the population is not symmetric the Pareto criterion is ineffective since any (super)majority may be Pareto optimal.[2] In this paper we take a utilitarian approach and we examine the social performance of the different vote quotas in terms of their delivered social surplus - i.e., the sum of the individual utilities attained in equilibrium.

What qualified majority requirement maximizes the social surplus? To supply an answer free of discount factor considerations, we address this question by examining the (equilibrium) social surplus in the limit as individual impatience vanishes. In this limit, the equilibrium outcome collapses to a unique alternative. In Proposition 1 we supply the equation that determines this policy for each qualified majority. Building on this result, answering our main question is a rather direct comparative statics exercise. Under simple majority rule the median individual determines the outcome, which may be positively or negatively correlated to social surplus depending on the utilities and the distribution of the peaks. Sometimes it is optimal to let the median determine the outcome; but it may be that a super-majority that allocates bargaining strength towards the extremes delivers more surplus. The precise answer depends mostly on the specification of the utilities.

When individual utilities are tent-shaped (that is, when the cost of selecting an alternative different from the peak is linear in its distance to the peak) the simple majority rule delivers the first best policy. The reason is simple: Under tent-shaped utilities social surplus increases as we move towards the median regardless of the distribution of peaks. I.e., the policy that maximizes surplus, the first best policy, always coincides with

[1] Existence of a Stationary Subgame Perfect Equilibrium follows by the results of Banks and Duggan (2000, 2006).

[2] See Examples in Cardona and Ponsatí (2011), p. 72.

the median. Thus, simple majority (and generically no other rule) delivers the first best policy.

When the utilities are strictly concave (the cost of selecting an alternative different from the peak is strictly convex in the distance), the surplus maximizing rule is a strict super-majority. With concavity, the social surplus does not necessarily increase as policies approach the peak of the median. Then, under distributions of peaks satisfying very mild regularity conditions, the optimal rule is a strict super-majority. Although the first best policy might not be attained in general, for the case of quadratic utilities we can prove that a super-majority weaker than unanimity delivers the first best - which is the peak of the average individual.

In order to fit the size of optimal quotas for strictly concave utilities, a detailed specifications of the peaks distribution and explicit computations are necessary. We carry out this computations for quadratic utilities and a rich set of parametrizations and the results suggest that the optimal consensus requirement is always a rather demanding quota rule that lies in the interval $[0.80, 0.95]$.

The remainder of the paper is organized as follows. Section 2 presents the model and describes the unique bargaining outcome. In section 3 we present our results about the surplus-maximizing quota rules. Section 4 describes the simulation exercises that complement our theoretical results. The Appendix contains proofs and simulation results.

2 Bargaining under (super)majority decisions: The unique outcome

The basic set up is that of Cardona and Ponsati (2011). A group of n individuals, must collectively select an alternative within the one dimensional policy space $[0, 1]$. They negotiate over discrete time, $t = 0, 1, 2....$ with a procedure that combines alternating proposals and voting. The negotiations begin at $t = 0$ and proceed as follows: At each $t \geq 0$ an individual is selected at random (all with equal probability) to make a proposal. Then, she chooses an alternative in $[0, 1]$ and all other players, sequentially in any fixed order, reply with acceptance or rejection. A collective decision under qualified majority $q \in [1/2, 1]$ requires the support of a subset $S \in \{T \subset I : |T| \geq nq\}$ of the players. Thus, when at least $nq-1$ responders accept the proposal, it is implemented and the game ends. Otherwise, the game moves to $t + 1$, a new proposer is selected, and so on.

Upon a collective decision that selects alternative $x \in [0, 1]$ at date t, individual i obtains utility $\delta^t u(x, i)$ where $\delta \in (0, 1)$ is the common discount rate. Utilities satisfy, $u(x, i) = v(|i - x|)$ where v is twice differentiable for any $|i - x| > 0$, decreasing and concave, with $v(0) > v(1) \geq 0$.[3] Disagreement yields zero utility to all agents.

The different locations of the peaks are the only source of heterogeneity within the population. Each $i \in I$ denotes both a generic individual and the location of her peak, so that all the information regarding heterogeneity within the population is embedded in the cumulative distribution function of peaks, denoted by F. Since we are interested in set-ups where n is large, it will be convenient to describe a population with a continuous cumulative distribution function of peaks F, with a positive density f on $(0, 1)$. A population is *symmetric* if $f(i) = f(1-i)$ for every $i \in [0, 1]$. The *median policy* x^m is the alternative that coincides with the peak of the median individual; i.e., x^m satisfies $F(x^m) = 1/2$. The *mean policy* x^e is the alternative that coincides with the expectation of individual peaks; i.e., $x^e = \int_0^1 xf(x)\,dx$.

A *stationary subgame perfect equilibrium* (SSPE) is a profile of stationary strategies that are mutually best responses at each subgame. Cardona and Ponsatí (2011) characterize and prove the uniqueness of an SSPE for each specification (n, q, F, v, δ). An SSPE is associated to a unique profile of expected utilities; agents exploit the impatience of others and propose the alternative which is closest to her peak and it is approved by a q majority. Thus the unique SSPE is fully characterized by a unique approval set; i.e., the set of alternatives that in equilibrium receive the acceptance of a q majority. The bounds of the approval set depend on the range of proposals that are individually acceptable to the *boundary players,* individuals $l(q)$ and $r(q)$ that constitute a (tight) q majority with all the individuals on their right and on their left, respectively; i.e., $F(l(q)) = 1 - q$ and $F(r(q)) = q$. When the agents' impatience vanishes, the advantage of the proposer also vanishes and the approval set collapses to a singleton. This asymptotic equilibrium outcome $x(q)$ is the unique alternative at which the (marginal) deviations induced by the proposers with peak $i \leq x(q)$ weighted by their mass $F(x(q))$, are exactly compensated by the (marginal) deviations induced by proposers with peak $i > x(q)$, which occur with the complementary probability. Proposition 1 supplies the equation that characterizes this unique asymptotic outcome.

Proposition 1 UNIQUE BARGAINING OUTCOME. *Consider a sequence of environments* (q, F, v, δ_k)*, where* $\delta_k \to 1$. *In the limit, as* $\delta_k \to 1$*, the SSPE approval set converges to a singleton* $x(q)$*, where* $x(q)$ *is the unique solution to*

$$K_F(x, q) \equiv F(x)\,\frac{u_x^+(x, l(q))}{u(x, l(q))} + [1 - F(x)]\,\frac{u_x^-(x, r(q))}{u(x, r(q))} = 0. \tag{1}$$

Proof. See Cardona and Ponsatí (2009) or Predtetchinski (2011). ■

[3] Note that the right and left derivatives with respect to x, $u_x^+(x, i)$ and $u_x^-(x, i)$, are always well defined, and that they coincide when $i \neq x$.

A unique equilibrium outcome $x(q)$ yields a unique payoff $u(x(q),i)$ for each $i \in I$, which in turn induce collective benefits.[4] Equipped with Eq. (1), we are ready to address the comparative statics for $x(q)$ and its induced individual and collective benefits with respect to q. We turn to this exercise next.

3 (Super)majority rules and social surplus maximization

The social surplus associated to policy x is the sum of individual utilities delivered by x:

$$S(x) = \int_0^1 u(x,i)f(i)\,di.$$

Hence, we can define the social surplus associated to each q, via $x(q)$, as

$$W(q) = \int_0^1 u(x(q),i)f(i)\,di.$$

A *first best policy* x^{fb} is an alternative that maximizes $S(x)$. When the maximization of social surplus is the welfare-maximizing criterion, the best conceivable performance for $W(q)$ would be delivered by a *first best rule* - i.e., a quota rule q^{fb} such that $x(q^{fb}) = x^{fb}$.

It is trivial to check that in the special case of symmetric populations, Eq. (1) yields $x^* = x^m = 1/2$ for all q. Moreover, it is immediate that symmetry implies that the median policy x^m (which coincides with mean policy) is also the first best policy. Thus, in this special case any q is (trivially) a first best rule. However, this result is not robust to perturbations of F away from symmetry. For an asymmetric F the bargaining outcome varies with q, and thus the value of q must be fine tuned to attain the first best policy. The following simple examples illustrate this point.

Example 1 *Consider a population distributed according to a triangular density function*

$$f_T(x;d) = \begin{cases} 2\frac{x}{\theta} & 0 \leq x \leq \theta \\ 2\frac{(1-x)}{1-\theta} & \theta < x \leq 1 \end{cases}$$

and tent-shaped utilities $u(x,i) = 1 - |x-i|$. Figure 1 displays equilibrium outcomes $x(q)$ and the first best policy for parameters $\theta \in \{0.7, 0.9\}$.

[4] A property of the function $K_F(x,q)$, which is used to derive the uniqueness result, is that it is strictly decreasing in x (see Lemma 4.3. in Predtetchinski, 2010).

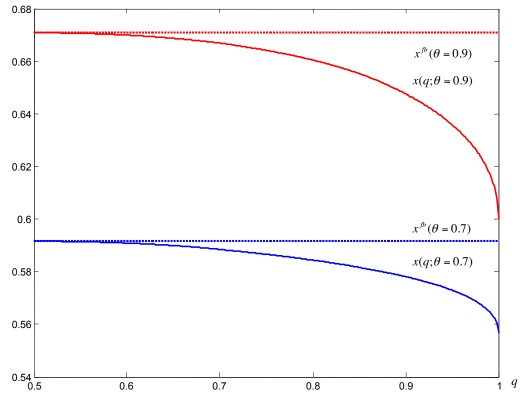

Figure 1: Tent-shaped preferences

Example 2 *Consider a population distributed according to a triangular density function and quadratic utilities* $u(x, i) = 1 - (x - i)^2$*. Figure 2 displays equilibrium outcomes* $x(q)$ *and the first best policy for parameters* $\theta \in \{0.7, 0.9\}$*.*

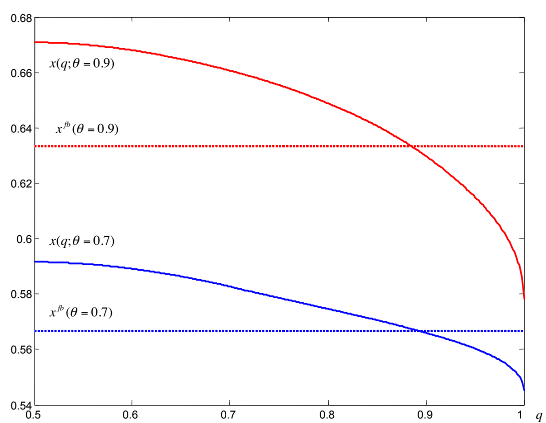

Figure 2: Quadratic preferences

These examples show three things: First, there exists one (and only one) q that delivers the first best policy. Second, when preferences are tent shaped, then the simple majority delivers the first best policy. Third, under quadratic preferences the first best rule is a strict super-majority, which is smaller than unanimity. In particular, the first best rule is around 0.9. In the remainder of this section, we show that these observations apply in general under very mild regularity properties of the population.

Population Regularity Properties A population F with density f satisfies

1. *Asymmetry Regularity (AR)* if (a) there exists $\varepsilon > 0$ such that $f(x^m - y) \neq f(x^m + y)$ for all $y \in (0, \varepsilon)$, and (b) $f(x^m + y) - f(x^m - y)$ changes sign at most once for $y > 0$.
2. *Mean-Median Regularity (MMR)* if either (a) $x^m < x^e \leq 1 - F(x^e)$ or (b) $x^m > x^e \geq 1 - F(x^e)$.

AR (b) is a weakening of the condition provided by Groeneveld and Meeden (1979) that guarantees that a distribution with a unique mode M satisfies the well know mean-median-mode inequality, that either $x^e \leq x^m \leq M$ or $x^e \geq x^m \geq M$, with strict inequalities whenever $x^m \neq M$.[5] AR and MMR are independent mild conditions that hold for most common specifications of bounded distributions.[6] We will rely on AR or MMR to show that the features displayed in the examples are general.

We first show that under tent-shaped utilities the simple majority is optimal.

Proposition 2 Optimal simple majority. *Consider an asymmetric population F. If $u(x, i) = a - b\,|x - i|$ then $q^{fb} = 1/2$. Moreover, if F satisfies AR, then (i) no other quota delivers the first best if $x^m \neq 1/2$ and (ii) both, simple majority and unanimity, uniquely deliver the first best policy whenever $x^m = 1/2$.*

Proof. Since x^{fb} maximizes

$$S(x) = a - b \int_0^1 |x - i|\, f(i)\, di.$$

The first order condition that characterizes x^{fb} is $F(x) = 1 - F(x)$, and therefore $x^{fb} = x^m$. Since $x(1/2) = x^m$ for all F, $q = 1/2$ is a first best rule. If $x^m = 1/2$, then $x(1) = 1/2$ as well, and the unanimity rule also delivers the first best. See Lemma 2 in Appendix A for the proof that if F satisfies AR then no other rule delivers x^m. ■

[5]Specifically, their condition demands AR (b) and that $f(M + y) - f(M - y)$ does not change sign. Van Zwet (1979) shows that a weaker sufficient condition guaranteeing $M \leq x^m \leq x^e$ is $F(x^m + x) + F(x^m - x) \leq 1$.

[6]Both conditions hold in Triangular, Two-block, Beta and Standard Two-Sided Power distributions, and for most of Kumaraswamy distributions.

The optimality of the simple majority does not extend beyond the tent-shaped specification. Specifically, when preferences are quadratic MMR assures that the rule maximizing social surplus is a strict super-majority weaker that unanimity which, in addition, is the first best rule. Furthermore, for general strictly concave utilities, AR assures that the optimal rule is a strict super-majority.

Proposition 3 Optimal super-majorities. *Consider an asymmetric population F.*

1. *If $u(x,i) = a - b\,(x-i)^2$, $a \geq b$ and F satisfies MMR, then $x(q) = x^{fb} = x^e$ for some $q \in (1/2, 1)$.*

2. *If utilities are strictly concave and F satisfies AR, then the rule that maximizes social surplus is a strict super-majority. Moreover, if $x^m = 1/2$ this majority is smaller than unanimity.*

Outline of the Proof.

1. For quadratic utilities, $u(x,i) = a - b(x-i)^2$, $a \geq b$, the first order condition for social surplus maximization is $\int_0^1 (x-i) f(i)\, di = 0$, which implies that $x^{fb} = \int_0^1 i f(i)\, di = x^e$. The strict concavity of u combined with MMR assure the existence of a unique $q \in (1/2, 1)$ such that $x(q) = x^e$.

2. Given F that satisfies AR, an auxiliary distribution $\widehat{F}$ with symmetric density $\widehat{f}$ can be constructed such that (a) $\widehat{F}$ has median x^m, and (b) f and $\widehat{f}$ cross only once. For population $\widehat{F}$ the first best policy is x^m, and $q = 1/2$ is the first best rule. The argument then relies on the strict concavity of u and the first order stochastic dominance relation between F and $\widehat{F}$.

See Appendix A for the detailed arguments.

4 Simulations

Proposition 2 provides a precise characterization of the optimal rule for tent-shaped utilities. Likewise, for strictly concave preferences, Proposition 3 establishes that the optimal rule is a strict super-majority weaker than unanimity. However, it gives no hint about its precise size. To this end, we carried out numerical simulations computing the optimal super-majority rule for quadratic utilities $u(x,i) = 1 - (x-i)^2$ and extensive parametrizations of 4 natural specifications of F: Two-block,Triangular, Beta and Kuramaswamy distributions. The results of these computations are in Appendix B.

Although optimal rules depend (continuously) on the concavity of the preferences, the size of the optimal super-majority grows quickly when increasing this concavity. Figure 3 displays this dependence in an example.

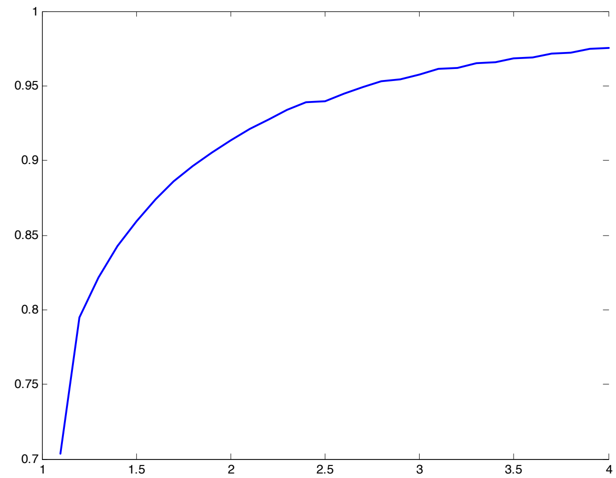

Figure 3: First best rule as a function of k for populations distributed according to a beta $(2,7)$, and utilities given by $u = 1 - |x-i|^k$.

Moreover, our extensive computations confirm the features of Example 2: Under quadratic preferences, first best rules are large. As shown in the Appendix B, with slight differences between populations, the first best rule ranges from 80% to 95%. Surplus maximization requires a super-majority in order to avoid that extreme players (those who suffer high "transportation" costs) are completely excluded from the bargaining. On the other hand, since the mass of extreme agents is relatively low, their influence must be limited. Thus, the optimal rule is lower than unanimity. It is worth to note that we did not find a clear monotonicity relationship between the mass of extreme players, measured in terms of skewedness of the distribution, and the optimal rule.[7] This is due to two effects that appear when increasing the skewedness of the population. First, the first best policy moves away from the increased tail. Second, the boundary players also move away from the increased tail, so does the bargaining outcome. Hence, the total effect on the optimal rule due to an increase in the heterogeneity is unclear, as it depends on the relative

[7]The skewness is a measure of the asymmetry of the probability distribution of a real-valued random variable. Although there are many measures, the skewness of a random variable x is usually taken as the third standardized moment, defined as $E\left[\left(\frac{x-x^e}{\sigma}\right)^3\right]$, where σ is the standard deviation.

size of each effect. In particular, when changes in the bargaining outcome exceed (are smaller than) the variation in the optimal policy, then the optimal rule must be increased (reduced) accordingly, in order to weaken (strengthen) the bargaining power of the agents in the largest tail of the distribution.

Our computations examine first the optimal rules under a simple one-paramenter (the median) specification where densities are constant over *two-blocks* with the same mass of players each. For this specification increasing the skewedness - equivalent to decreasing the median - of the population, decreases the optimal super-majority. Starting from the optimal rule in any given population, the change in the optimal policy due to an increase in the skewedness always exceeds the corresponding change in the bargaining outcome. Hence, the super-majority required must be weaker in order to decrease the bargaining power of agents in the largest tail. However, these considerations are nuanced in the other specifications for which we have computed optimal rules. Under *Triangular distributions*, for example, the difference between the change of the optimal policy and the change in the bargaining outcome due to an increase in the skewedness of the population depends on the initial distribution. Thus, the optimal quota is not monotone in the skewedness.

Things are even more complex, when considering of *Beta distributions*. These distributions are characterized by two parameters $\alpha, \beta > 0$, and offer a very flexible family of specifications. Roughly speaking, the absolute value of the difference between the parameters is positively related to the skewedness of the density function. Now, when $\alpha < \beta$, the skewedness might be increased either (i) by decreasing α while maintaining β fixed or (ii) by fixing α and increasing β. Moreover, our computations show that the optimal super-majority rule decreases in case (i) and it increases in case (ii). Thus, while large super-majorities are obtained in all cases, they are not monotone in the skewedness. As an illustration, consider the beta distribution with parameters $(\alpha, \beta) = (3, 6)$. The optimal rule is 0.9146. If the skewedness is increased by reducing α to 2 then the optimal rule decreases to 0.9134. If, instead, the skewedness increases by changing β to 7, the optimal rules goes up to 0.9163.[8]

Finally, we carry out computations for *Kumaraswamy distributions*, which (unlike the previous specifications) do not necessarily satisfy neither AR nor MMR. For example (see Table 4 in the Appendix), the distribution $F_K(x; 3, 6)$ is such that $x^e < x^m < M < 1/2$, which implies that (i) $f(x^m+y)-f(x^m-y)$ changes sign twice, and (ii) $1-F(x^e) > 1/2 > x^m > x^e$. Despite the fact that the sufficient conditions of Proposition 3 do not apply, the computed optimal rules are very similar to those obtained with the beta distributions.

[8] In the light of this example, one could think that the important variable in detemining the size of the optimal rule might be the sum of a and b, which is related to the variance of the distribution. However, it can be easily checked that this is not the case.

5 Final remarks

In the context of a multilateral one-dimensional bargaining game, we analyzed the performance of alternative (super)majority rules in achieving outcomes that maximize collective surplus. We showed that simple majority is generically the unique optimal rule only when the preferences of the agents are tent-shaped. Moreover, this is independent of how the population is distributed. However, under natural specifications of the population and strictly concave preferences, the surplus-maximizing quota rule is shown to be a strict super-majority. This super-majority is smaller than unanimity, and it allows to achieve the first best under the additional hypothesis of quadratic preferences. Although in general our results do not determine how strong the surplus-maximizing majority requirement should be, numerical simulations show that when preferences are quadratic the optimal rule is always higher than $q = 0.8$ and smaller that $q = 0.95$.

We adopted an utilitarian approach to determine the social performance of the different qualified majorities. An alternative welfare approach would be to consider a Rawlsian social welfare specification. In our setting, this would mean that, regardless of the distribution of peaks, the socially optimal policy is $1/2$, which, not surprisingly, would be obtained in symmetric populations. We cannot characterize the Rawlsian optimal rule in general. However, in asymmetric populations characterized by either a concave or a convex distribution function of peaks, it can be show that a Rawlsian first best rule does not exist, since there is no q delivering $x(q) = 1/2$. (See Proposition 4 in the Appendix A).

References

[1] Banks, J.S. and Duggan, J. (2000) A Bargaining Model of Collective Choice, *American Political Science Review*, Vol. 94 pp. 73-88.

[2] Banks, J.S. and Duggan, J. (2006) A General Bargaining Model of Legislative Policy-making, *Quarterly Journal of Political Science*, Vol. 1 pp. 49-85.

[3] Baron, D. and Ferejohn, J. (1989) Bargaining in Legislatures, *American Political Science Review*, Vol. 83 pp. 1181-1206.

[4] Cardona, D. and Ponsati, C. (2007) Bargaining one-dimensional social choices, *Journal of Economic Theory*, Vol. 137 pp.- 627-651.

[5] Cardona, D. and Ponsati, C. (2009) Bargaining one-dimensional policies and the efficiency of super-majority rules, *UFAE and IAE Working Papers 762.08.*

[6] Cardona, D. and Ponsati, C. (2011) Uniqueness of stationary equilibria in bargaining one-dimensional policies under (super) majority rules, *Games and Economic Behavior,* Vol 73 (1), pp. 65-75.

[7] Cho, S. and Duggan, J. (2003) Uniqueness of stationary equilibria in a one-dimensional model of bargaining, *Journal of Economic Theory*, Vol. 113 pp. 118-130.

[8] Cho, S. and Duggan, J. (2009) Bargaining Foundations of the Median Voter Theorem, *Journal of Economic Theory*, Vol. 144 pp. 851-868.

[9] Groeneveld, R.A. and Meeden, G. (1977). The mode, median and mean inequality, *American Statistician* Vol. 33, 120-121.

[10] Predtetchinski, A. (2011) One–dimensional Bargaining, *Games and Economic Behavior*, Vol. 72 (2) pp. 526–543.

[11] Van Zwet, W.R. (1979) Mean, median, mode, II. *Statistica Neerlandica* 33, 1-5.

A Proofs

Induced Symmetric Densities

As a tool to examine the equilibrium outcome of asymmetric populations we will rely on the comparison with an auxiliary symmetric distribution constructed as follows. Given a positive density f on $(0,1)$ we define its *Induced Symmetric Density* $\widehat{f}$ as follows

1. If f is such that $x^m \leq 1/2$ then $\widehat{f}(x) = f(x)$ for $x \in [0, x^m]$, $\widehat{f}(x) = f(2x^m - x)$ for $x \in [x^m, 2x^m]$, and $\widehat{f}(x) = 0$ for $x \in [2x^m, 1]$.

2. If f is such that $x^m > 1/2$ then $\widehat{f}(x) = f(x)$ for $x \in [x^m, 1]$, $\widehat{f}(x) = f(2x^m - x)$ for $x \in [1 - 2x^m, x^m]$, and $\widehat{f}(x) = 0$ for $x \in [0, 1 - 2x^m]$.

Note that condition AR (b) is equivalent to requiring that f and $\widehat{f}$ cross only once. Consider for instance $x^m \leq 1/2$ so that $\widehat{f}(x) = f(x)$ for all $x \in [0, x^m]$, $\widehat{f}(x^m + y) = f(x^m - y)$ for $y \in (0, x^m]$ and $\widehat{f}(x) = 0$ for $x \in [2x^m, 1]$. Thus, f and $\widehat{f}$ can cross only at $x \in (x^m, 2x^m]$. Since $f(x^m + y) - \widehat{f}(x^m + y) = f(x^m + y) - f(x^m - y)$ for any $y \in (0, x^m]$, then AR implies that f and $\widehat{f}$ cross only once.

The function $\widehat{f}$ is constructed either by transferring mass from one of the tails of the distribution towards the median or vice-versa. As an illustration, in Figure 4, $\widehat{f}$ is constructed from f by replacing agents with higher peaks with agents with lower peaks (from A to B). Note that in this example, $\widehat{F}(x) \geq F(x)$ for all $x \in [0, 1]$.

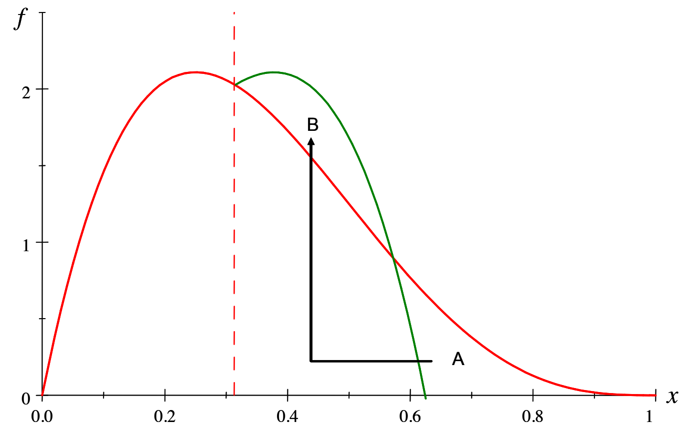

Figure 4.

A direct implication of AR (as f and $\widehat{f}$ cross only once) is a *first order stochastic dominance* relationship (denoted by $\succ_{(1)}$) which, in turn, determines the relationships between the boundary players stated in the following Lemma:

Lemma 1 *If F satisfies AR then*

1. *If $x^m < 1/2$ then $f \succ_{(1)} \widehat{f}$, $l(q) = \widehat{l}(q)$ and $r(q) > \widehat{r}(q)$ for all $q \in (1/2, 1)$.*
2. *If $x^m > 1/2$ then $\widehat{f} \succ_{(1)} f$, $l(q) < \widehat{l}(q)$ and $r(q) = \widehat{r}(q)$ for all $q \in (1/2, 1)$.*
3. *If $x^m = 1/2$ then either (i) $f \succ_{(1)} \widehat{f}$, $l(q) = \widehat{l}(q)$ and $r(q) > \widehat{r}(q)$ for all $q \in (1/2, 1)$, or (ii) $\widehat{f} \succ_{(1)} f$, $l(q) = \widehat{l}(q)$ and $r(q) < \widehat{r}(q)$ for all $q \in (1/2, 1)$.*

   where $\widehat{l}(q)$ and $\widehat{r}(q)$ denote the boundary players under the induced symmetric distribution.

Proof. Consider $x^m < 1/2$. As $\widehat{f}(x) = 0 <$ for all $x \in (2x^m, 1)$, AR implies that generically there exists z such that $\widehat{f}(x) > f(x)$ for all $x \in (x^m, z)$ and $\widehat{f}(x) < f(x)$ for all $x \in (z, 1)$, while $\widehat{f}(x) = f(x)$ for any $x \in [0, x^m]$. Hence, $F(x) = \widehat{F}(x)$ for all $x \leq x^m$, which implies $l(q) = \widehat{l}(q)$, and $\widehat{F}(x) > F(x)$ for all $x \in (x^m, 1)$, implying $r(q) > \widehat{r}(q)$ for all $q \in (1/2, 1)$. This proves (1). A similar argument will prove case (3). In case that $x^m = 1/2$, by AR there exists z such that either (i) $\widehat{f}(x) > f(x)$ for all $x \in (x^m, z)$ and $\widehat{f}(x) < f(x)$ for all $x \in (z, 1)$, or (ii) $\widehat{f}(x) < f(x)$ for all $x \in (x^m, z)$ and $\widehat{f}(x) > f(x)$ for all $x \in (z, 1)$. In case (i), the previous argument applies; and in case (ii), $F(x) = \widehat{F}(x)$ for all $x \leq x^m$ and $\widehat{F}(x) < F(x)$ for all $x \in (x^m, 1)$ so $\widehat{l}(q) = l(q)$ and $\widehat{r}(q) > r(q)$. ■

Note that AR implies that $x^e \neq x^m$, as the mean and the median of the induced symmetric distribution $\widehat{f}$ coincide and there is a first order stochastic dominance relationship between f and $\widehat{f}$.

Lemma 2 *Under tent shaped utilities, if F satisfies AR, only $q = 1/2$ or $q = 1$ can be first best rules.*

Proof. We must show that under AR no $q \in (1/2, 1)$ delivers x^m. To see that, note that, by Lemma 1, either (i) $f \succ_{(1)} \widehat{f}$ and $l(q) = \widehat{l}(q)$ and $r(q) > \widehat{r}(q)$ or (ii) $\widehat{f} \succ_{(1)} f$ with $l(q) < \widehat{l}(q)$ and $r(q) = \widehat{r}(q)$ for any $q \in (1/2, 1)$. Moreover, it is easily checked that $\left| \frac{u_x(x,i)}{u(x,i)} \right|$ increases in $|x - i|$.

Hence, when $f \succ_{(1)} \widehat{f}$ we obtain that

$$K_F(x^m, q) > K_{\widehat{F}}(x^m, q) = 0 \text{ for any } q \in (1/2, 1),$$

where the last equality comes from the fact that $\widehat{f}$ is symmetric. This implies that there is no $q \in (1/2, 1)$ such that $x(q) = x^m = x^{fb}$.

Similarly when $\widehat{f} \succ_{(1)} f$ we obtain $K_F(x^m, q) < K_{\widehat{F}}(x^m, q) = 0$ for all $q \in (1/2, 1)$.

To see that $x(1) \neq x^m$ when $x^m \neq 1/2$ just note that in this case

$$\left| \frac{u_x(x^m, 0)}{u(x^m, 0)} \right| \neq \left| \frac{u_x(x^m, 1)}{u(x^m, 1)} \right|,$$

so that $K_F(x^m, 1) \neq 0$. ■

Proof of Proposition 3 .

Part 1. For quadratic utilities, $u(x, i) = a - b(x - i)^2$, $a \geq b$, the first order condition for social surplus maximization is $\int_0^1 (x - i) f(i)\, di = 0$, which implies that $x^{fb} = \int_0^1 i f(i)\, di = x^e$.

Strict concavity of u implies that the function

$$B(x, q) \equiv F(x)\, u_x(x, l(q)) + [1 - F(x)]\, u_x(x, r(q))$$

is continuous in x and q and decreasing in x.

Denote by $\widetilde{x}$ the (unique) solution to $B(x, 1) = 0$. By MMR either $x^m < x^e \leq 1 - F(x^e)$ or $x^m > x^e \geq 1 - F(x^e)$. Assume, w.l.o.g., the first. Note first that it implies $x^m < x^e < 1/2$. Thus, $B(x^m, 1) > 0$. Moreover, $x(1)$ solves

$$K_F(x, 1) = F(x)\frac{u_x(x, 0)}{u(x, 0)} + [1 - F(x)]\frac{u_x(x, 1)}{u(x, 1)} = 0,$$

implying that $x(1) \in (x^m, 1/2)$ and therefore $u(x(1), 0) > u(x(1), 1)$. Thus,

$$\begin{aligned} B(x(1), 1) &= F(x(1))\, u_x(x(1), 0) + [1 - F(x(1))]\, u_x(x(1), 1) \\ &< K_F(x(1), 1) = 0, \end{aligned}$$

which implies that $\widetilde{x} \in (x^m, x(1))$.

Moreover, $B(x^e) = -F(x^e)2bx^e + [1 - F(x^e)]\, 2b(1 - x^e) = 2b[1 - x^e - F(x^e)] \geq 0$ so that $x^m < x^e \leq \widetilde{x} < x(1)$. Therefore, by continuity of $x(q)$, $x(q) = x^e$ for some $q \in (1/2, 1)$.

A similar argument applies if $x^m \geq x^e \geq 1 - F(x^e)$, in which case it must be that $x^e > 1/2$.

Part 2. We prove the result for setups where $x^m \leq 1/2$. The symmetric argument applies when $x^m > 1/2$.

Assume first $x^m < 1/2$. By Lemma 1 $f \succ_{(1)} \widehat{f}$. Moreover, since utilities are strictly concave (i.e., $\partial u_x(x^m, i)/\partial i > 0$ for all $i > x^m$), we have that

$$\begin{aligned} S'(x^m) &= \int_0^1 u_x(x^m, i)\, f(i) di = \int_0^{x^m} u_x(x^m, i)\, f(i) di + \int_{x^m}^1 u_x(x^m, i)\, f(i) di \\ &> \int_0^{x^m} u_x(x^m, i)\, \widehat{f}(i) di + \int_{x^m}^1 u_x(x^m, i)\, \widehat{f}(i) di = 0, \end{aligned}$$

because x^m is the first best allocation under $\widehat{f}$. Hence, social surplus is increasing at x^m, implying $x^{fb} > x^m$. Moreover, as $K_F(x, q)$ decreases in x and

$$K_F(x^m, 1) = \frac{1}{2}\left[\frac{u_x(x^m, 0)}{u(x^m, 0)} + \frac{u_x(x^m, 1)}{u(x^m, 1)}\right] > 0,$$

it must be that $x(1) > x^m$. Since $x(q)$ is continuous in q, social surplus is maximal at some $q \in (1/2, 1]$.

If $x^m = 1/2$ there are two possibilities: either $f \succ_{(1)} \widehat{f}$ or $\widehat{f} \succ_{(1)} f$. Assume, w.l.o.g., the first case. As previously, dominance implies $x^{fb} > x^m$. Moreover, $l(q) = \widehat{l}(q)$ and $r(q) > \widehat{r}(q)$. Thus,

$$K_F(x^m, q) > K_{\widehat{F}}(x^m, q) = 0 \text{ for any } q \in (1/2, 1)$$

which, as $K_F(x^m, q)$ is a decreasing function of x, implies that $x(q) > x^m$. Moreover, as $x(1) = x^m$, the rule that maximizes social surplus must be stronger than simple majority and weaker than unanimity.

Proposition 4 *If F is concave (resp. convex) then there is no q such that $x(q) = 1/2$.*

Proof. We prove the statement for F concave. The argument for F convex is analogous.

We first show that $-l'(q) < r'(q)$. By definition $l(q) = F^{-1}(1-q)$ and $r(q) = F^{-1}(q)$. If F is concave, then $H = F^{-1}$ is convex. Thus, since $q \in [0.5, 1]$ we obtain

$$-l'(q) = H'(1-q) \leq H'(q) = r'(q).$$

Assume there is q such that $x(q) = 1/2$. I.e.,

$$F(1/2)\frac{u_x^+(1/2, l(q))}{u(1/2, l(q))} + (1 - F(1/2))\frac{u_x^-(1/2, r(q_1))}{u(1/2, r(q_1))} = 0.$$

That is,

$$\frac{-\dfrac{u_x^+(1/2, l(q))}{u(1/2, l(q))}}{\dfrac{u_x^-(1/2, r(q))}{u(1/2, r(q))}} = \frac{1 - F(1/2)}{F(1/2)}.$$

Since F is concave we have that $F(1/2) > 1/2$. Moreover, since $\left|\dfrac{u_x^-(x, i)}{u(x, i)}\right|$ increases in $|x - i|$, it must be that

$$r(q) - 0.5 > 0.5 - l(q) \Longleftrightarrow r(q) + l(q) > 1.$$

However, F^{-1} is a convex function with $F^{-1}(0) = 0$, and $F^{-1}(1) = 1$. Thus, $r(q) = F^{-1}(q) < q$ and $l(q) = F^{-1}(1-q) < 1-q$, contradicting $r(q) + l(q) > 1$. ■

B Simulations

Recall that under quadratic utilities, the first best policy is the mean of the peaks x^e. Thus, the optimal rule is q such that Eq. (1) yields $x(q) = x^e$. Our numerical evaluations require the use of MatLab for the Beta and Kuramaswamy distributions. In this case, intervals of size $\Delta = 1/10000$ are used.

B.1 Two block distributions

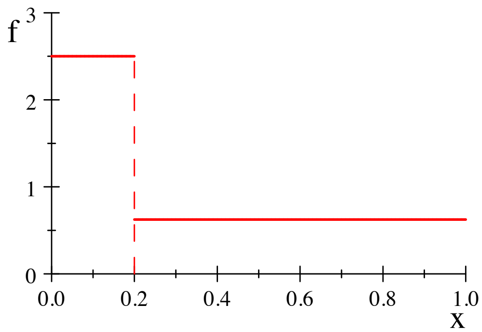

Figure 5: $f_b\left(x;0.2\right)$

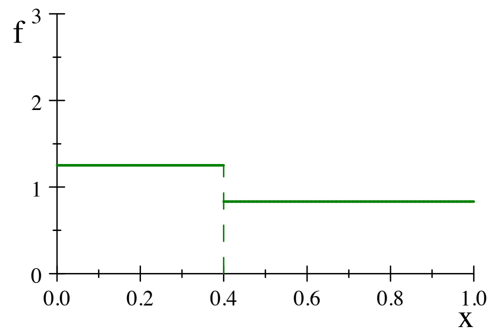

Figure 6: $f_b\left(x;0.4\right)$

The *two-block* density and cumulative distribution functions, defined over $[0,1]$, are given by

$$f_b(x;\theta)=\begin{cases}\frac{1}{2\theta} & 0\le x\le\theta\\ \frac{1}{2(1-\theta)} & \theta<x\le 1\end{cases}\quad\text{and } F_b(x;\theta)=\begin{cases}\frac{x}{2\theta} & 0\le x\le\theta\\ 1-\frac{1-x}{2(1-\theta)} & \theta<x\le 1\end{cases}$$

where $x^m=\theta\in[0,1]$, $x^e=(1+2\theta)/4$, $l\left(q\right)=2\theta(1-q)$ and $r\left(q\right)=\theta+\left(q-\frac{1}{2}\right)2(1-\theta)$. The first best rules, for different values of $\theta\in(0,1/2)$, are presented in Table 1.

$\theta=x^m$	$x^e=x^{fb}$	q^{fb}	l	r
0.1	0.3	0.82332	0.035336	0.68198
0.2	0.35	0.83383	0.066468	0.73413
0.3	0.4	0.84769	0.091386	0.78677
0.35	0.425	0.85640	0.100 52	0.813 32
0.4	0.45	0.8667	0.106 64	0.840 04
0.45	0.475	0.87905	0.108 86	0.866 96
0.49	0.495	0.89081	0.107 01	0.888 63

Table 1.

B.2 Triangular distributions

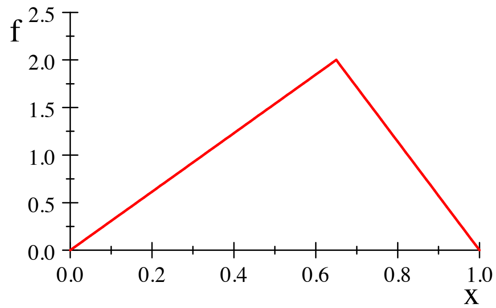

Figure 7: $f_T(x; 0.65)$

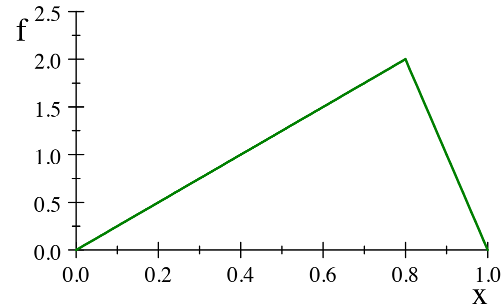

Figure 8: $f_T(x; 0.8)$

The density and cumulative distribution functions of the peaks are given by

$$f_T(x;d) = \begin{cases} 2\frac{x}{\theta} & 0 \le x \le \theta \\ 2\frac{(1-x)}{1-\theta} & \theta < x \le 1. \end{cases} \quad \text{and } F(x;\theta) = \begin{cases} \frac{x^2}{\theta} & 0 \le x \le \theta \\ 1 - \frac{(1-x)^2}{1-\theta} & \theta < x \le 1 \end{cases}$$

We restrict, w.l.o.g., to the case where $\theta \in (1/2, 1]$. In these cases, $x^m = (\theta/2)^{1/2}$ and $x^e = (1+\theta)/3$. Note that $x^e = (1+\theta)/3 < x^m = (\theta/2)^{1/2} < \theta$. Moreover, $l(q) = \sqrt{(1-q)\theta}$ for all q, $r(q) = \sqrt{q\theta}$ if $q \le \theta$ and $r(q) = 1 - \sqrt{(1-q)(1-\theta)}$ if $q \ge \theta$.

The first best rules, for different values of $\theta \in (0, 1/2)$, are presented in Table 2.

θ	x^m	$x^e = x^{fb}$	q^{fb}	l	r
0.6	0.547 72	0.533 33	0.917 65	0.222 28	0.818 51
0.7	0.591 61	0.566 67	0.893 25	0.273 36	0.821 04
0.8	0.632 46	0.6	0.882 53	0.306 56	0.846 72
0.9	0.670 82	0.633 33	0.885 33	0.320 97	0.892 63
0.95	0.689 2	0.65	0.88758	0.326 82	0.893 76

Table 2.

B.3 Beta distributions

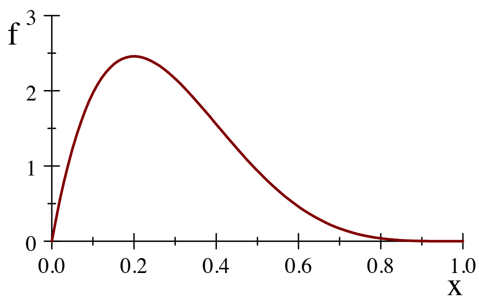

Figure 9: $f_B(x;2,5)$

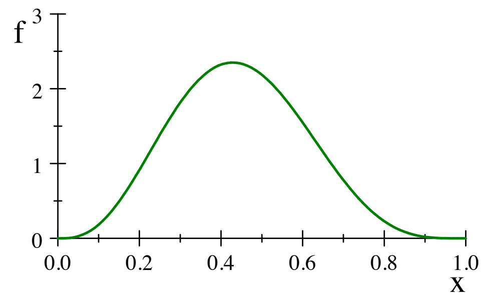

Figure 10: $f_B(x;4,5)$

The Beta density and cumulative distribution functions, defined over $[0,1]$, are parametrized by two positive parameters, α and b, and are given by[9]

$$f_B(x;\alpha,\beta)=\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \text{ and } F_B(x;\alpha,\beta)=\frac{1}{B(\alpha,\beta)}\int_0^x z^{\alpha-1}(1-z)^{\beta-1}\,dz$$

with

$$B(\alpha,\beta)=\int_0^1 z^{\alpha-1}(1-z)^{\beta-1}dz.$$

While the median has no explicit form, for any α,β the mean is given by $x^e=\alpha/(\alpha+\beta)$, and the mode is $(\alpha-1)/(\alpha+\beta-2)$.

The first best rules, for different values of α,β, are presented in Table 3.

[9] $a>1$ and $b>1$ guarantee the single-peakedness of the density function.

α	β	x^m	$x^e = x^{fb}$	q^{fb}	l	r
1	3	0.2063	0.25	0.8989	0.0349	0.5341
2	3	0.3857	0.4	0.9028	0.1403	0.6829
2.5	3	0.4487	0.4545	0.9048	0.1922	0.7266
1	4	0.1591	0.2	0.9053	0.0246	0.4453
2	4	0.3138	0.3333	0.9075	0.1074	0.5930
3	4	0.4214	0.4286	0.9100	0.1926	0.6773
3.5	4	0.4636	0.4667	0.9111	0.2315	0.7074
1	5	0.1294	0.1667	0.9093	0.0188	0.3812
2	5	0.2644	0.2857	0.9109	0.0867	0.5232
3	5	0.3641	0.375	0.9126	0.1606	0.6094
4	5	0.4401	0.4444	0.9142	0.2281	0.6688
4.5	5	0.4718	0.4737	0.9149	0.2589	0.6921
1	6	0.1091	0.1429	0.912	0.0152	0.3331
2	6	0.2285	0.25	0.9134	0.0726	0.4676
3	6	0.3205	0.3333	0.9146	0.1376	0.5533
5	6	0.4517	0.4545	0.9169	0.2541	0.6605
5.5	6	0.4769	0.4783	0.9174	0.2795	0.6796
1	7	0.0943	0.125	0.914	0.0128	0.2957
3	7	0.2862	0.3	0.9163	0.1203	0.5064
4	7	0.3551	0.3636	0.9171	0.1763	0.5677
6	7	0.4595	0.4615	0.9188	0.2742	0.6529
6.5	7	0.4805	0.4815	0.9192	0.2956	0.6692

Table 3.

B.4 Kuramaswamy distributions

The Kumaraswamy density and cumulative distribution functions are defined by

$f_K(x;\alpha,\beta) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}$ and $F_K(x;\alpha,\beta) = 1-(1-x^{\alpha})^{\beta}$, respectively, $\alpha,\beta > 0$.

This distribution resembles to the Beta distribution. In particular, if $x_{\alpha,\beta}$ is a Kumaraswamy distributed random variable with parameters α and β, and $y_{1,\beta}$ denotes a Beta distributed random variable with parameters 1 and β, then one has that $x_{\alpha,\beta} = y_{1,\beta}^{1/\alpha}$. As a comparison, in Figure 11, displays population distributed according to $f_K(x;2,8)$

and a population distributed according to $f_B\left(x; 2.7, 6\right)$.

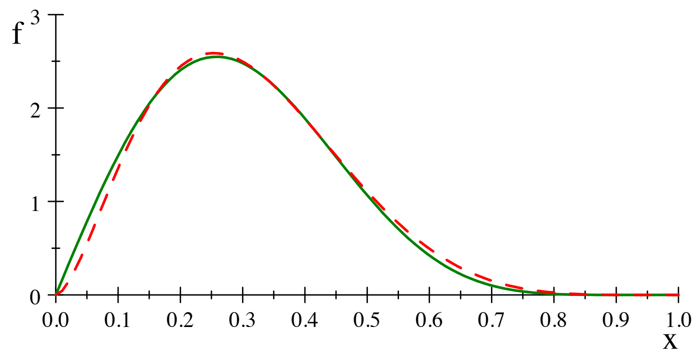

Figure 11: $f_K\left(x; 2, 8\right)$ (solid) and $f_B\left(x; 2.7, 6\right)$ (dash).

The first best rules, for different values of α, β, are presented in Table 4.

α	β	M	x^m	$x^e = x^{fb}$	q^{fb}	l	r
2	5	0.3333	0.3598	0.3694	0.9132	0.1341	0.6219
2	6	0.3015	0.3303	0.341	0.9146	0.1215	0.5800
2	7	0.2773	0.307	0.3183	0.9158	0.1117	0.5457
2	8	0.2582	0.2881	0.2995	0.9168	0.1039	0.5168
3	5	0.5228	0.5059	0.5007	0.9026	0.2727	0.7194
3	6	0.4900	0.4778	0.4743	0.8973	0.2616	0.6810
3	7	0.4641	0.4551	0.4528	0.8871	0.2570	0.6445
3	8	0.4430	0.4362	0.4347	0.8669	0.2605	0.6063
4	5	0.6304	0.5998	0.5884	0.9127	0.3668	0.7882
4	6	0.6100	0.5747	0.5648	0.91282	0.3505	0.7603
4	7	0.5773	0.5541	0.5454	0.91285	0.3373	0.7365
4	8	0.5577	0.5367	0.5288	0.9128	0.3263	0.7160
5	5	0.6988	0.6644	0.6504	0.9164	0.4442	0.8289
5	6	0.6729	0.6420	0.6294	0.91677	0.4281	0.8056
5	7	0.6518	0.6236	0.6119	0.9170	0.4149	0.7856
5	8	0.6342	0.6079	0.5970	0.9172	0.4039	0.7682

Table 4.