Intertemporal restrictions in structural vector autoregressions*

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Abstract

We propose an innovate identification scheme, which we call intertemporal identification, that focuses on the relative importance of identified shocks at different frequencies to establish a structural interpretation of the responses in VAR models. The method is highly intuitive and can be used to identify structural shocks either independently or in conjunction with other approaches, such as sign restrictions. Thus, or approach is particularly useful when other approaches are questionable or when they generate wide sets of admissible responses. Theoretically and through simulation-based scenarios, we show that the intertemporal identifications tend to significantly reduce the identified sets of sign restrictions and derive the conditions under which one of the restrictions becomes redundant. We illustrate the usefulness of our approach with three empirical examples: (i) identification of technology shocks; (ii) identification of oil-specific demand shocks; and (iii) identification of monetary policy shocks.

Keywords: VAR models; Impulse responses; Identification problem

JEL Classification: C18, C32, E32,

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1. Introduction

Vector Autoregressive (VAR) models are widely used in policy and empirical economic analyses and have gained popularity since the groundbreaking work by Sims (1980). As emphasized in the survey conducted by Ramey (2016), the success of VAR models stems from their ability to offer a straightforward framework for capturing the dynamic interactions among multiple variables over time. This framework enables researchers to examine how each variable responds to shocks in its own value as well as to changes in the values of other variables within the system. Consequently, VAR models provide valuable insights into the interdependencies and reactions within an economic system.

Notwithstanding this remarkable track record, one of the main challenges in building VAR models is the issue of identification. If the variables exhibit feedback effects, then it may not be possible to uniquely estimate the structural parameters of the model based on the observed data. The Structural VAR (SVAR) identification scheme has been the standard approach to deal with this drawback. The idea behind this approach is viewing the residuals of the estimated VAR model as combinations of the underlying structural shocks, which are assumed to be orthogonal to each other and to have a particular economic interpretation. This is allowed by imposing restrictions that set conditions as to how certain variables would behave according to economic theory.

Not intending to exhaustively list all the contributions made by researchers in this rapidly evolving field, numerous approaches have been proposed over the past few decades to impose economically motivated restrictions that identify structural shocks.[1] The early-stage approaches suggested in the literature focused on exact (point) identification restrictions, which limit the relative importance of the identified shock at different time/frequency periods or horizons. The most popular are zero restrictions, which do not allow some variables to react to shocks in the short-run (Sims, 1980), in the long-run (Blanchard and Quah, 1989), or at a specific horizon (Uhlig, 2004). Despite their initial success, these methods have been criticized for being too restrictive and offering unrealistic results, sometimes confronted with economic theory.

To address the challenge of horizon determination in exact identification restrictions, more recent approaches have employed min/max restriction methods. Faust (1998) proposed identifying shocks associated with the maximum forecast-error variance share in specific variables and horizons. Francis et al. (2014) recognized this method as a finite-horizon alternative for identification of SVARs

[1] While acknowledging the significance of statistical methods in identifying restrictions, we intentionally omitted their discussion in the current text because our emphasis centers on economically motivated restrictions. Interested readers seeking a detailed exploration of statistical methods are encouraged to consult, among other valuable references, Herwartz, Lange, and Maxand (2021).

with long-run restrictions and DiCecio and Owyang (2010) proposed a variant of this approach in the frequency domain. In a similar frequency domain context, Lovcha and Perez-Laborda (2020) identified shocks by minimizing the distance between empirical contributions to the fluctuations of system variables at selected frequency ranges and a set of targets derived from theoretical models. However, a limitation of min/max restriction methods lies in their restrictiveness. These methods constrain the contribution of the shock of interest relative to other shocks on the variance of variables at specific time/frequency horizons, demanding that these contributions be either exact or closely approximate predefined arbitrary values.

Overcoming some of the drawbacks of exact identification restrictions, set identification restrictions have emerged as one widely adopted method for identifying structural shocks in VAR models. Among them, sign restrictions impose restrictions on the direction of the responses (Faust, 1998; Canova and Nicolo, 2002; Uhlig, 2005). Since sing restrictions imply weaker restrictions than classical identification schemes, they have achieved extensive success in economic applications. However, pure sign-identified VAR models are typically consistent with a large set of responses that are compatible with the restrictions. In these cases, inferring policy recommendations from the resulting distribution of admissible responses can be misleading as the resulting identified sets are compatible with cases that overestimate the response of some variables and/or underestimate the response of others.

The refinements of the sign restriction approach proposed in the literature often involve complementing sign restrictions with additional economically motivated bound restrictions. These bound restrictions are set-identified constraints that impose narrow bands on transformations of the reduced form parameters, effectively narrowing down the set of admissible responses. For example, Kilian and Murphy (2012) integrated sign restrictions with additional empirically plausible bounds on the magnitude of short-run elasticities and on the impact responses. Volpicella (2022) combined sign restrictions with bounds on the Forecast Error Variance Decomposition (FEVD).

In this paper, we contribute to this growing literature by presenting an alternative approach to identification, which falls within the realm of set identification approaches. In a frequency domain setup, our novel procedure involves identifying the frequency or frequency ranges where the relative contributions of the shock of interest to variables variances are concentrated. This involves imposing inequality restrictions on the FEVD of a selected variable at specific frequencies, an operation that represents a nonlinear transformation of the structural parameters, that can be attributed to a certain shock.[2] Therefore, we term these restrictions intertemporal restrictions because the focus is on

[2] In this sense, the method can be viewed as a form of sign restriction on the nonlinear transformation of structural parameters in an SVAR

determining which shocks contribute more to one set of frequencies than to others, regardless of whether the shock provides the largest contribution under certain conditions for a given set of frequencies.

As in the related literature, the restrictions are typically guided by the results of other empirical analyses or by economic theory. For example, the method could rely on the fact that the contributions of technology shocks tend to be more concentrated in long-run frequencies than in short-run frequencies or that the contributions of transitory market specific shocks are typically concentrated on short-run frequencies. Therefore, the method is very flexible and can be adapted to numerous empirical setups.

In the spirit of the sign restrictions approach, our method also involves a rejection identification method. It implies rotating a base set of orthogonal shocks in a SVAR specification to form a new set of orthogonal shocks from which impulse responses are obtained. Then, a response is retained if the relative contribution of the structural shock to the variance of a target variable at different frequencies agrees with that expected by economic theory, while the response is discarded if it does not. The result of this procedure is a set of identified responses to economically interpretable shocks.

When the rotation matrix is of a Givens form, we develop a theoretical framework that characterizes the system of inequalities of model parameters provided by the intertemporal restrictions which can be solved for the angles that are compatible with the restrictions. This results in an identified set of angles that generate the admissible impulse responses. The analysis of the resulting sets of angles help to derive the conditions under which intertemporal restrictions, in isolation or in conjunction with other identification procedures like sign restrictions, provide reasonable sets of identified responses. We use this setup to enhance our understanding of the advantages, limitations, and potential synergies between sign and intertemporal restrictions in VAR models.

In our comparative analysis, we observe that intertemporal restrictions tend to narrow down the identified set of admissible responses provided by sign restrictions. Notably, sign restrictions alone can be redundant in many cases when combined with intertemporal restrictions. To further illustrate and substantiate these findings, we conduct a simulation study where we have control over the data generating process because we can isolate the effects of each restriction and observe how they interact. Using these controlled scenarios, we also illustrate the usefulness of intertemporal restrictions, especially when they are combined with sign restrictions.

Through three empirical scenarios, we address the applied value of intertemporal restrictions to identify structural responses and evaluate their relative accuracy as compared to sign restrictions.

Our empirical setups have been prolific sources of numerous contributions in the literature of VAR specifications. Specifically, they involve the identification of technology shocks, as in Gali (1999) and Christiano, Eichenbaum and Vigfusson (2003); the identification of oil-specific demand shocks, as in Kilian (2009) and Kilian and Murphy (2012); and the identification of monetary policy shocks, as in Uhlig (2005), Christiano, Eichenbaum, and Evans (2005) and Altig et al. (2011). Our empirical results highlight the importance of imposing intertemporal restrictions, either in isolation or in conjunction with sign restrictions, when such information exists.

The rest of the paper is organized in the following way. Section 2 introduces the intertemporal restrictions. Section 3 examines the comparative accuracy of intertemporal restrictions in relation to sign restrictions and offers the analytical analysis on the effectiveness of the proposed method. Section 4 develops simulations to address the empirical small-sample reliability of VAR models identified with intertemporal and sign restrictions. Section 4 collects three empirical examples that illustrate the usefulness of our approach. Section 5 concludes.

2. Intertemporal restrictions

2.1. Identification in VAR models

Start with a set of N variables, which are collected in the vector y_t, that admits a covariance-stationary reduced-form VAR model

$$F(L)y_t = u_t, \tag{1}$$

with $F(L) = (I - F_1 L - \cdots - F_p L)$, $F(1)$ is of full rank, $E(u_t)=0$, $E(u_t u_t^{'}) = \Sigma$, $E(u_t u_{t-h}^{'}) = 0$ for $h > 0$, and $t = 1,...,T$.[3] The process can be expressed in moving average notation as

$$y_t = C(L)u_t, \tag{2}$$

were $C(L) = F(L)^{-1} = (I + C_1 L + C_2 L^2 + \cdots)$. In this expression, C_h refers to the *h*-period lagged response of y_t to unit impulses to each of the elements of the reduced-form shocks u_t. In the reduced form, parameters and u_t can be estimated from the data.

To derive the impulse responses to structural-form errors, the focus is finding the linear combinations of the reduced-form shocks, which are determined by the invertible matrix A providing orthogonal transformations

[3] We omitted the intercepts to simplify the exposition.

$$\varepsilon_t = A^{-1}u_t, \tag{3}$$

that arise from the structural form of the VAR

$$A(L)y_t = \varepsilon_t, \tag{4}$$

with $A(L) = (A^{-1} - A_1L - \cdots - A_pL)$. To ensure that the structural shocks are mutually uncorrelated, it is standard to look for A satisfying $\Sigma = AA^{'}$, which implies $E(\varepsilon_t\varepsilon_t^{'}) = A^{-1}\Sigma A^{-1'} = I$.

The moving average representation of the structural shocks is

$$y_t = C(L)A\varepsilon_t, \tag{5}$$

and the responses to unit impulses of the structural-form shocks are $R_h = C_hA$, with h=1,2,... As $C_0 = I$, then $R_0 = A$ and the matrix A of structural parameters defines the contemporaneous responses of the variables to unit structural shocks. Unfortunately, the appropriate set of weights on u_t given by A^{-1} cannot be derived uniquely from $\Sigma = AA^{'}$ because this system involves $(N^2 + N)/2$ more unknowns than linearly dependent equations.

Since the seminal proposal of Sims (1980), one of the most popular ways to approximate the matrix A is via the Cholesky decomposition of the reduced-form covariance matrix, $\Sigma = HH^{'}$. In this case, the Cholesky combination of the reduced-form shocks is

$$\eta_t = H^{-1}u_t. \tag{6}$$

The moving average representation of these shocks is $y_t = C(L)H\eta_t$, which implies that the contemporaneous responses to unit shocks are given by the lower triangular matrix H. This decomposition imposes an arbitrary recursive structure on the VAR as variables ordered first do not respond to contemporaneous shocks of variables ordered later. Unless the dynamics of the time series yield this restrictive contemporaneous response, it is quite improbable that the Cholesky shocks correspond to the actual structural shocks of interest.

2.2. Rotating a base set of shocks

To alleviate this restrictive assumption and following the lines suggested by the literature on sign restrictions, we propose rotating an initial set of orthogonal shocks, sometimes called base set of shocks, to form a new set of orthogonal shocks from which impulse responses are obtained. To ensure that the intertemporal restrictions apply, we employ a rejection method, which is achieved by discarding the rotations generating structural matrix A that does not satisfy the restrictions in the way we describe below.

To start with, we follow Fry and Pagan (2011) and rely on Cholesky decompositions to obtain base shocks. They are just a rescaled version of the reduced-form shocks, which are contemporaneously uncorrelated.[4] Then, we propose a large set of M candidates of structural shocks as linear combinations of these base shocks

$$\varepsilon_t^m = P_m^{-1}\eta_t, \tag{7}$$

where P_m is an orthonormal rotation square matrix satisfying $P_m^{-1}P_m^{-1'} = P_m^{-1'}P_m^{-1} = I$ and $m = 1,...,M$. Generated in this way, the candidates of structural shocks are mutually uncorrelated as $E(\varepsilon_t^m \varepsilon_t^{m'}) = P_m^{-1}P_m^{-1'} = I$. The moving average representation of the candidates of structural shocks is

$$y_t = C(L)HP_m\varepsilon_t^m. \tag{8}$$

In this way, the candidate of impulse responses are

$$R_h^m = C_h HP_m, \tag{9}$$

and the contemporaneous responses are $R_0^m = HP_m$, with $m = 1,...,M$.

There are many ways to find orthonormal square matrices that provide combinations of the base shocks. The two most popular methods used to obtain P_m focus on Givens (Canova and de Nicolo, 2002) and Householder (Rubio-Ramirez, Waggoner, and Zha, 2010) transformations. According to the results suggested by Fry and Pagan (2011) the two methods are equivalent, although the latter are computationally faster when the number of variables included in the system grows. In line with Fisher and Huh (2019b), as the theoretical part of this paper focuses in small-size VARs, we will follow the former approach.

2.3. Setting intertemporal restrictions

According to our rejection method to find identified impulse responses, we require finding only the set of rotations of the base shocks that generate the structural matrix A that satisfies pre-specified intertemporal restrictions. To define these restrictions, we follow Stiassny (1996), who focus on the cross spectral density matrix of the VAR model. The density matrix of the process at frequency ω is

$$f_y(\omega) = \frac{1}{2\pi} C\left(e^{i\omega}\right)\Sigma C\left(e^{-i\omega}\right)', \tag{10}$$

[4] Another way of obtaining base shocks is the eigenvalue-eigenvector decomposition $\Sigma = QDQ' = VV'$, where Q is a matrix of eigenvectors, D is a diagonal matrix with eigenvalues on the main diagonal and $V = PD^{1/2}$. For instance, this strategy was pursued by Fisher and Huh (2019).

where i denotes the imaginary unit ($i=\sqrt{-1}$), $C\left(e^{-i\omega}\right)$ is the complex conjugate of $C\left(e^{i\omega}\right)$, and the frequency is defined for $\omega_j = 2\pi j/T$, $j=0,...,T/2$.

The n-th element of the main diagonal of the spectral density matrix contains the univariate spectral density of y_n at frequency ω

$$f_{y_n}(\omega)=\frac{1}{2\pi}s_n{}'C\left(e^{i\omega}\right)\Sigma C\left(e^{-i\omega}\right)' s_n, \tag{11}$$

where s_n is a column vector with the n-th element equals to one and the other elements equal to zero, $\omega\in[0,\pi]$, and $n=1,...,N$. Intuitively, this expression measures how much variance of y_n is concentrated at each frequency and is typically used to assess the importance of the different frequencies to the overall variance.

Using the relation $\Sigma=R_0^m R_0^{m'}=HP_m P_m' H'$, we define the causality spectrum of a shock in say variable y_k on variable y_n at frequency ω as

$$f_{y_k\to y_n}(\omega)=\frac{1}{2\pi}s_n{}'C\left(e^{i\omega}\right)\left[R_0^m\right]_k\left[R_0^m\right]_k' C\left(e^{-i\omega}\right)' s_n, \tag{12}$$

where $\left[R_0^m\right]_k$ refers to the k-th column of $R_0^m=HP_m$. It can be shown that the spectral density of y_n at frequency ω can be decomposed into their N different sources, originating from shocks in the time-series collected in the vector y_t

$$f_{y_n}(\omega)=\sum_{k=1}^{N}f_{y_k\to y_n}(\omega). \tag{13}$$

To determine the share contribution of the innovations in the k-th time series on the n-th time series, we define the causality spectrum share as

$$w_{nk}^{\omega}=\frac{f_{y_k\to y_n}(\omega)}{f_{y_n}(\omega)}, \tag{14}$$

where $\sum_{k=1}^{N}w_{nk}^{\omega}=1$. This represents the portion of the spectrum of y_n at frequency ω that can be attributed to shocks in y_k.

This expression can be integrated over sets of different frequencies of interest to assess the shares of structural shocks in fluctuations at a range of frequencies. For example, let $\Omega=[\omega_1,\omega_2]$ be the selected frequency range. Then, the share of the k-th disturbance in the fluctuations of the n-th variable at the range of frequencies Ω is given by

$$w_{nk}^{\Omega} = \frac{\int_{\Omega} f_{y_k \to y_n}(\omega) d\omega}{\int_{\Omega} f_{y_n}(\omega) d\omega}, \tag{15}$$

where the definite integrals can be approximated by summations for the frequencies $\omega_j = 2\pi j/T$ for $j = 1, 2, ...T/2$ belonging to the selected range.

The causality spectrum share can be used to impose the intertemporal restrictions established by economic theory on the impulse responses. In particular, economic theory often provides guidance to assert that some shocks are expected to have larger effects on a target variable at determined frequencies than at others. For example, fundamental shocks are expected to have greater effects in long-run frequencies while transitory market specific shocks are typically concentrated on the short-run frequencies.

These intertemporal restrictions are easy to translate into meaningful inequality constraints on the causality spectrum share. Without loss of generality, we use as an example an economic theory indicating that the magnitude of the influence of the *k*-th disturbance in the fluctuations of the *n*-th variable is different at frequencies Ω_1 than at frequencies Ω_2, which implies a specific sign for $w_{nk}^{\Omega_1} - w_{nk}^{\Omega_2}$.[5] In the spirit of the literature of sign restrictions, the way we proceed to recover the structural matrix A from intertemporal restrictions is a rejection method that consists of the following steps:

1. Run an unrestricted VAR to get consistent estimates of Σ and C_h, with $h = 1, 2, \ldots$. Then, perform a Cholesky decomposition of Σ to obtain H.
2. Use Givens or Householder transformations to obtain a random orthogonal square matrix P_m and the candidate structural matrix $A_m = HP_m$, compute $w_{nk}^{\Omega_1}$ and $w_{nk}^{\Omega_2}$.
3. If the postulated intertemporal restriction holds (correct sign of $w_{nk}^{\Omega_1} - w_{nk}^{\Omega_2}$), then save the candidate structural matrix A_m. If not, discard the draw.
4. Repeat 2 and 3 for $m = 2, 3, ..., M$ and store the set of admissible structural matrices.

In summary, economic theory guides the expected intertemporal restrictions, which are imposed as meaningful restrictions on the causality spectrums share at different frequencies. The set of admissible impulse responses can be calculated based on the admissible set of structural matrices. The resulting impulse responses align with economic reasoning and policy implications, discarding those that do not agree with the intertemporal restrictions.

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[5] For example, if the influence of the *k*-th disturbance in the fluctuations of the *n*-th variable is greater at frequencies Ω_1 than at frequencies Ω_2, then $w_{nk}^{\Omega_1} > w_{nk}^{\Omega_2}$ and $w_{nk}^{\Omega_1} - w_{nk}^{\Omega_2} > 0$.

It is unlikely that the result of this method will be only one rotation given by P_m satisfying the intertemporal restrictions. Thus, in practice, there will be many R_h^m satisfying the restrictions, each of one referring to a different model. This results in an impulse-response graph with set of estimated responses. Although there are some initiatives in the literature of sign restrictions to summarize the multiple responses result (Fry and Pagan, 2011), we will focus the analysis on the whole set of responses that are consistent with the intertemporal restrictions.[6]

2.4. The identified set of intertemporal restrictions

The identified set refers to the compilation of generated structural matrices A that are compatible with the imposed restrictions and, therefore, are not discarded by the rejection method. To provide a theoretical characterization of the set of responses induced by intertemporal restrictions, we will start with a simple bivariate VAR(1) model. In this case, the structural framework is the following:

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \left[I - \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} L \right]^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}, \tag{16}$$

where the structural matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies $\Sigma = AA'$.

Let $H = \begin{pmatrix} \sigma_{11} & 0 \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$ be a lower-triangular Cholesky decomposition of Σ, with the diagonal elements satisfying $\sigma_{11} > 0$ and $\sigma_{22} > 0$. Let us consider a Givens rotation as the weighting to generate $m = 1,...,M$ combinations of the base shocks

$$P_m = \begin{pmatrix} \cos\rho_m & -\sin\rho_m \\ \sin\rho_m & \cos\rho_m \end{pmatrix}, \tag{17}$$

for $\rho_m \in [-\pi/2, \pi/2,]$, where m indexes the different values of the angle used to rotate the base set of shocks. In this way, the candidate structural matrices are

$$A_m = \begin{pmatrix} \sigma_{11} cos\rho_m & -\sigma_{11} sin\rho_m \\ \sigma_{21} cos\rho_m + \sigma_{22} sin\rho_m & -\sigma_{21} sin\rho_m + \sigma_{22} cos\rho_m \end{pmatrix},$$

$$R_0^m = \begin{pmatrix} \sigma_{11}\cos\rho_m & -\sigma_{11}\sin\rho_m \\ \sigma_{21}\cos\rho_m + \sigma_{22}\sin\rho_m & -\sigma_{21}\sin\rho_m + \sigma_{22}\cos\rho_m \end{pmatrix}, \tag{18}$$

[6] Despite this comment, for the sake of facilitating comparisons, we will occasionally reference the median value among the range of admissible responses for each h-period lagged response. Furthermore, we will present these median values in certain figures to provide visual clarity.

which represent the structural contemporaneous responses $R_0 = A$. To facilitate notation, we will denote ρ as a particular realization of the angles $\{\rho_1,...,\rho_M\}$, so that we will skip the sub-index m

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Let us assume there is an economic theory providing the intertemporal restriction that the importance of the second shock for the variances of the first and second variables in the longest frequency ($\omega = 0$) is different than that in the shortest frequency ($\omega = \pi$). The identified set of impulse responses must satisfy the corresponding restrictions in the sign of the difference of causality spectrum shares $w_{12}^{0} - w_{12}^{\pi}$ and $w_{22}^{0} - w_{22}^{\pi}$, which imply two intertemporal restrictions that we call IR1 and IR2, respectively. In this framework, the identified set can alternatively be viewed as the set of angles ρ generating structural matrices that are compatible with the restrictions, which is denoted by IS_{IR}^{ρ}.

Appendix A shows that IR1 and IR2 will lead to the following four zeros that satisfy $w_{12}^{0} = w_{12}^{\pi}$ and $w_{22}^{0} = w_{22}^{\pi}$ in the interval $[-\pi/2, \pi/2]$:

$$R_1 = \arctan\left[\frac{\sigma_{22}}{\sigma_{21} + \sigma_{11}\dfrac{\alpha - F_{22}}{F_{12}}}\right] \text{ and } R_3 = \arctan\left[\frac{\sigma_{22}}{\sigma_{21} + \sigma_{11}\dfrac{\beta - F_{22}}{F_{12}}}\right] \qquad (19)$$

for IR1, and

$$R_2 = \arctan\left[\frac{\sigma_{22}}{\sigma_{21} + \sigma_{11}\dfrac{F_{21}}{\gamma - F_{11}}}\right] \text{ and } R_4 = \arctan\left[\frac{\sigma_{22}}{\sigma_{21} + \sigma_{11}\dfrac{F_{21}}{\delta - F_{11}}}\right] \qquad (20)$$

for IR2,

where

$$\alpha = \frac{K_{110} - K_{11\pi}}{K_{110} + K_{11\pi}},\ \beta = \frac{K_{110} + K_{11\pi}}{K_{110} - K_{11\pi}},\ \gamma = \frac{K_{220} - K_{22\pi}}{K_{220} + K_{22\pi}}, \text{ and } \delta = \frac{K_{220} + K_{22\pi}}{K_{220} - K_{22\pi}},$$

with

$$K_{110}^2 = \frac{1}{\left[(1-F_{11})(1-F_{22}) - F_{12}F_{21}\right]^2}\frac{1}{S_{11}(0)},\ K_{11\pi}^2 = \frac{1}{\left[(1+F_{11})(1+F_{22}) - F_{12i}F_{21}\right]^2}\frac{1}{S_{11}(\pi)},$$

$$K_{220}^2 = \frac{1}{\left[(1-F_{11})(1-F_{22}) - F_{12}F_{21}\right]^2}\frac{1}{S_{22}(0)}, \text{ and } K_{22\pi}^2 = \frac{1}{\left[(1+F_{11})(1+F_{22}) - F_{12}F_{21}\right]^2}\frac{1}{S_{22}(\pi)}.$$

Note that these zeros do not depend on the entries of the structural matrix A and remain constant for whatever realization of the rotation matrix P. In Appendix B, we derive the following characterization of these zeros.

Proposition 1. *The length of the intervals for ρ in each of the two intertemporal restrictions is $\pi/2$, i.e. $|R_3 - R_1| = |R_4 - R_2| = \pi/2$.*

This implies that $|(-\pi/2, R_1) \cup (R_3, \pi/2)| = |(-\pi/2, R_2) \cup (R_4, \pi/2)| = \pi/2$. In addition, the zeros that satisfy the restrictions cannot be zero because $\sigma_{22} > 0$ and the parameters in F and the angle ρ are finite.

Corollary 1.1. *The identified set that satisfies the two intertemporal restrictions IR1 and IR2, IS_{IR}^{ρ}, is smaller than $\pi/2$.*

The identified set is the collection of angles that lay on the intersection of the two intervals corresponding to the intertemporal restrictions IR1 and IR2, each of which of the length $\pi/2$ on $[-\pi/2, \pi/2]$. Thus, if these two intervals do not coincide, the length of this intersection must smaller than $\pi/2$.[7]

Corollary 1.2. *The zeros for IR1 (R_1 and R_3) and IR2 (R_2 and R_4) are always of opposite sign.*

This follows from the fact that the length the intervals for ρ in each of the two intertemporal restrictions is $\pi/2$ on $[-\pi/2, \pi/2]$ and that none of the zeros can be zero.

To provide a graphical intuition of this characterization, Figure 1 shows in blue arrows the four identified sets of angles for which both intertemporal restrictions hold, which are denoted by IS_{IR}^{ρ}. In each graph, points R_1 and R_3 refer to the zeros of IR1 while R_2 and R_4 refer to the zeros in IR2. The upper-left and lower-right panels show identified sets that refer to rotations of base shocks that admit positive and negative angles, which lay in the intervals $(R_1, R_3) \cap (R_2, R_4) = (R_2, R_3)$ and $\left\{\left(-\frac{\pi}{2}, R_1\right) \cup \left(R_3, \frac{\pi}{2}\right)\right\} \cap \left\{\left(-\frac{\pi}{2}, R_2\right) \cup \left(R_4, \frac{\pi}{2}\right)\right\} = \left\{\left(-\frac{\pi}{2}, R_1\right) \cup \left(R_4, \frac{\pi}{2}\right)\right\}$, respectively.

The upper-right panel alludes to rotations that admits only negative angles, as it refers to angles that are located in the intersection $(R_1, R_3) \cap \left\{\left(-\frac{\pi}{2}, R_2\right) \cup \left(R_4, \frac{\pi}{2}\right)\right\} = (R_1, R_2)$. Finally, the lower-left panel stands for rotations that admits only positive angles as it refers to solutions that lay on the interval $\left\{\left(-\frac{\pi}{2}, R_1\right) \cup \left(R_3, \frac{\pi}{2}\right)\right\} \cap (R_2, R_4) = (R_3, R_4)$.

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[7] The length of IS_{IR}^{ρ} would be $\pi/2$ if and only if $1 - F_{22} = F_{21}$ and $1 - F_{11} = F_{12}$, which cannot happen because the matrix $(I - F)$ has been assumed of full rank.

3. Connecting intertemporal and sign restrictions

3.1. Sign restrictions

There are deep connections between intertemporal restrictions and sign restrictions. Although both approaches follow a rejection method described Section 2.3, the main difference between these two approaches is the restrictions used to obtain identified sets. In this case, sign restrictions discards the rotations of the base shocks whose signs of the generated impulse responses, R_h^m, do not agree with the pre-specified signs of the same elements of the structural responses, R_h.

To facilitate comparisons with intertemporal restrictions, we restrict the analysis to the VAR(1) with two variables suggested in Section 2.4. In addition, it is quite common in the literature on sign restrictions imposing restrictions on the signs of impact responses, $R_0 = A$, as for example, in Faust (1998) or Canova and Nicolo (2002). To start with, we impose a first set of sign restrictions that consist of normalizing the impact responses of the variables to their own shocks to be positive. This implies that $\sigma_{11}\cos\rho \geq 0$ and $-\sigma_{21}\sin\rho + \sigma_{22}\cos\rho \geq 0$.

In addition, we restrict the rotation of the base shocks to the set of angles that generate a predetermined sign in the response of y_1 to a unit shock ε_2. Thus, we will consider only angles that generate rotations of the base shocks with sign of $-\sigma_{11}\sin\rho$ that agrees with the sign of the structural response, which is determined by the parameter b in A. In practice, this restriction implies that the identified set supported by the sign restrictions, denoted by IS_{SR}^{ρ}, only admits angles ρ that have opposite sign to that of b.

Depending on the signs of b and σ_{21}, the sign restrictions imply four different possible identified sets for ρ. Appendix C shows that these identified sets are $\left[-\pi/2, 0\right]$ when $b \geq 0$ and $\sigma_{21} \geq 0$; $\left[\arctan(\sigma_{22}/\sigma_{21}), 0\right]$ when $b \geq 0$ and $\sigma_{21} < 0$; $\left[0, \arctan(\sigma_{22}/\sigma_{21})\right]$ when $b < 0$ and $\sigma_{21} \geq 0$; and $\left[0, \pi/2\right]$ when $b < 0$ and $\sigma_{21} < 0$. Thus, the length of the resulting identified set of the sign restrictions is $\pi/2$ when the sign of b coincides with the sign of the reduced-form parameter σ_{21}, while it is $\left|\arctan(\sigma_{22}/\sigma_{21})\right| < \pi/2$ when their signs do not coincide.

To facilitate interpretation, Figure 1 shows the identified sets that are compatible with all the combinations of b and σ_{21}, which are represented with red arrows. In each graph, the main diagonal panels represent the cases where b and σ_{21} have the same signs. In these cases, the size of the identified sets is $\pi/2$. The off main diagonal panels of each graph refer to the cases where b and σ_{21} have the different signs and the identified sets are delimited by the location of $\arctan(\sigma_{22}/\sigma_{21})$.

3.2. Combining sign and intertemporal restrictions

The success of sign restrictions in empirical studies is often attributed to their simplicity and intuitive appeal, making them easy to impose. However, it is important to note that purely sign-identified VAR models often yield a wide range of admissible responses, especially when the signs restrictions are not very informative. For example, when b and σ_{21} have the same signs, the identified set can encompass a broad set of responses. This can lead to potential challenges when inferring policy implications from the set of admissible responses, as some cases may overestimate the response of certain variables while underestimating the response of others.

Indeed, one approach to refine the set of admissible responses in VAR models is to combine sign restrictions with additional restrictions. Researchers have employed various types of additional restrictions to further narrow down the set of feasible responses. To name a few, Canova and De Nicolo (2002) imposed additional structure in the form of sign restrictions on dynamic cross-correlations; Uhlig (2005) and Baumeister and Benati (2013); Kilian and Murphy (2012) combined sign restrictions with plausible bounds on the magnitude of the short-run responses; and Fisher and Huh (2019) combined long-run neutrality restrictions together with signs and contemporaneous zero restrictions.

Building on the existing literature, we propose in this section to combine sign and intertemporal restrictions. Specifically, we aim to compare the joint set of admissible responses, which are compatible with both type of restrictions, with each of the identified sets obtained separately. The objective is to determine when both approaches can be beneficial in obtaining narrower sets of admissible responses, when one of the restrictions may not contribute to obtain the joint identify set, and when the two types of restrictions are incompatible.

For this purpose, let $IS^{\rho}_{IR+SR} = IS^{\rho}_{IR} \cap IS^{\rho}_{SR}$ be the identified set of angels ρ that is compatible with both intertemporal and sign restrictions. We say that the identified set of sign restrictions is redundant if $IS^{\rho}_{IR+SR} = IS^{\rho}_{IR}$, which happens when $IS^{\rho}_{IR} \subset IS^{\rho}_{SR}$. In the same way, we say that the identified set of intertemporal restrictions is redundant if $IS^{\rho}_{IR+SR} = IS^{\rho}_{SR}$, which happens when $IS^{\rho}_{SR} \subset IS^{\rho}_{IR}$. In the cases where no type of restrictions is redundant, combining intertemporal and sign restrictions can be beneficial in narrowing down the set of admissible structural matrices and responses to a more specific range. In addition, we say that the two identified sets are not compatible when $IS^{\rho}_{IR+SR} = \varnothing$.

Using Figure 1 to characterize the identified sets, we can identify some combinations for which intertemporal and sign restrictions are not compatible. For example, when $IS^{\rho}_{IR} = [R_1, R_2]$ as in the upper-right graph while the sign restrictions imply the assumption that $b < 0$; when $IS^{\rho}_{IR} = [R_3, R_4]$ as in the bottom-left graph while the sign restrictions imply the assumption that $b > 0$; or when $IS^{\rho}_{IR} = \{[-\pi/2, R_1] \cup [R_4, \pi/2]\}$ as in the bottom-right graph while the sign restrictions imply the assumptions that $b > 0$ and $\sigma_{21} < 0$ and $\arctan(\sigma_{22}/\sigma_{21}) \in [R_2, 0]$.

When intertemporal and sign restrictions are compatible, the combination of both restrictions produces a joint identification set that is strictly smaller than the identification sets that uses only intertemporal restrictions or only sign restrictions, unless in the cases that one of the two types of restrictions is redundant. In this case, the joint set will coincide with the narrower of the separate sets.

With the help of Figure 1, we find that the sign restrictions are redundant in two different scenarios. First, if b and σ_{21} have the same signs, then sign restrictions are redundant when $b > 0$ and $\sigma_{21} \geq 0$ and $IS^{\rho}_{IR} \in [R_1, R_2]$ as in first quadrant of the upper-right graph, or when $b < 0$ and $\sigma_{21} < 0$ and $IS^{\rho}_{IR} \in [R_3, R_4]$ as in the fourth quadrant of the bottom-left graph. Second, if b and σ_{21} have opposite signs, then sign restrictions are redundant when $b \geq 0$, $\sigma_{21} < 0$ and $\arctan(\sigma_{22}/\sigma_{21}) \in [-\pi/2, R_1]$ and $IS^{\rho}_{IR} \in [R_1, R_2]$ as in third quadrant of the upper-right graph, or when $b < 0$, $\sigma_{21} \geq 0$ and $\arctan(\sigma_{22}/\sigma_{21}) \in [R_4, \pi/2]$ and $IS^{\rho}_{IR} \in [R_3, R_4]$ as in the second quadrant of the bottom-left graph.

Interestingly, Figure 1 shows that intertemporal restrictions cannot be redundant if b and σ_{21} have the opposite sign. When these parameters are of the same sign, there are only two situations for which intertemporal restrictions are redundant, both occurring when $IS^{\rho}_{IR} \in [R_2, R_3]$, as in the upper-left graph. The first situation refers to $b > 0$, $\sigma_{21} < 0$ and $\arctan(\sigma_{22}/\sigma_{21}) \in [R_2, 0]$, which appears in the third quadrant, while the second situation refers to $b < 0$, $\sigma_{21} \geq 0$ and $\arctan(\sigma_{22}/\sigma_{21}) \in [0, R_3]$, which depicted in the second quadrant.[8]

[8] The authors derived an on-line appendix describing the situations for which sign and intertemporal restrictions are redundant as a function of the reduced-form and structural form parameters. This is available from the websites of the authors.

4. Simulations

4.1. VAR with *p*=1 and *N*=2

To illustrate the theoretical analysis developed in Sections 2 and 3, we conduct simulations using a VAR(1) model with two variables. We begin by defining the matrices F and Σ that characterize the reduced form of the model. Using these matrices, we calculate the set of moving average representation matrices, C_h, with $h=1,...,20$. We also obtain the lower triangular matrix H by performing the Cholesky decomposition of Σ. For the intertemporal restrictions, we also find the zeros R_1, R_2, R_3 and R_4 .[9] Additionally, we specify the structural matrix A, which is utilized to compute the true structural responses $R_h = C_h A$.

Next, we proceed to identify the partition of the interval $\left[-\pi/2, \pi/2\right]$ as determined by the intertemporal restrictions, IS_{IR}^{ρ}. This partition is characterized by the restrictions on the sign of the difference of the causality spectrums $w_{12}^{0} - w_{12}^{\pi}$ and $w_{22}^{0} - w_{22}^{\pi}$. Additionally we determine the identified set of sign restrictions, IS_{SR}^{ρ}, characterized by $d \geq 0$ and a sign restriction on b . Lastly, we determine the identified set resulting from the combination of both restrictions, IS_{IR+SR}^{ρ}. To achieve this, we select the angles ρ_m, with $m=1,...,M$, that are consistent with these three identified sets. These angles are utilized to generate the matrices P_m and the corresponding admissible responses based on the respective identified sets, $R_h^m = C_h H P_m$. The resulting responses are displayed in Figure 2 and 3.[10]

To facilitate comparisons, the graphs show the responses of the first variable to shocks in the second variable. In these graphs, the responses of the data generating process are represented by black lines. The blue dashed lines depict the upper and lower limits of the responses provided by IS_{IR}^{ρ}, with the blue solid lines representing the median values. Similarly, the red dashed lines depict the upper and lower limits of the responses provided by IS_{SR}^{ρ}, with the red solid lines representing the median

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[9] Recall that the zeros R_1, R_2, R_3 and R_4 do not depend on the angle ρ used to rotate the base shocks.

[10] It is worth pointing out that our simulation exercise does not require generating data or estimating model parameters. This avoids unnecessary uncertainty.

values. Finally, the green dashed lines display the upper and lower limits of the responses provided by IS^{ρ}_{IR+SR}, with the green solid lines representing the median values.[11]

With the aim of exploring a wide range of possibilities, we generate four data generating processes, DGP1, DGP2, DGP3 and DGP4 characterized by a specific combination of the matrices F and Σ. For each DGP we choose the sign of the structural parameter b that defines the sign of impulse response of the first variable to the second shock. Given the sign of b and the sign of σ_{21}, we can recover the identified set for the sign and intertemporal restrictions for each DGP.

DGP1: $F = \begin{bmatrix} 0.7 & 0.2 \\ 0.1 & 0.4 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$, $b > 0$ and $\sigma_{21} > 0$ => $IS^{\rho}_{SR} = [-\pi/2, 0]$,

DGP2: $F = \begin{bmatrix} 0.7 & 0.2 \\ 0.1 & 0.4 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$, $b < 0$ and $\sigma_{21} > 0$ => $IS^{\rho}_{SR} = [0, 1.0472]$,

DGP3: $F = \begin{bmatrix} 0.9 & -0.2 \\ 0.4 & 0.5 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$, $b > 0$ and $\sigma_{21} < 0$ => $IS^{\rho}_{SR} = [-1.0472, 0]$,

DGP4: $F = \begin{bmatrix} 0.5 & 0.2 \\ 0.4 & 0.5 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$, $b > 0$ and $\sigma_{21} < 0$=> $IS^{\rho}_{SR} = [-1.0472, 0]$.

Figure 2 presents examples of DGP where either intertemporal restrictions or sign restrictions are redundant, while Figure 3 depicts the responses where both intertemporal restrictions and sign restrictions are not redundant when employed jointly in the identification process and produce narrower identification set than when employed by separately. To facilitate comparisons, in Figure 3, the left-hand-side graphs exhibit the limits of the responses determined by intertemporal restrictions (blue lines) and sign restrictions (red lines). Meanwhile, the right-hand-side graphs replace the responses determined by intertemporal restrictions with those that conform to the joint restrictions (green lines).

In DGP1, the structural parameters b and σ_{21} are of opposite sign (or ρ and σ_{21} are of the same sign) that is situation when the intertemporal restrictions cannot be redundant given that the identified set for the sign restrictions is wide: $IS^{\rho}_{SR} = [-\pi/2, 0]$. This situation is represented in the first quadrant in all graph in Figure 1. The zeros of the intertemporal restrictions depend on the parameters of the reduced-form model only and take values: $R_1 = -1.5157$, $R_2 = -0.6187$, $R_3 = 0.0551$, and $R_4 = 0.9521$. Let us consider the situation represented in the first quadrant of the upper-right graph in Figure 1 when the sign restrictions are redundant. If $\rho = -1$ in the DGP1, the

[11] Recall that using median responses to characterize the responses can be misleading. They are included in the graphs only to facilitate comparisons.

corresponding structural matrix takes form $A = HP(\rho) = \begin{bmatrix} 0.5403 & 0.8415 \\ -0.4586 & 0.8887 \end{bmatrix}$. The intertemporal restrictions in this case are given by $w_{12}^{0} \geq w_{12}^{\pi}$ and $w_{22}^{0} \geq w_{22}^{\pi}$; and the identified set is determined by $IS_{IR}^{\rho} = [R_1, R_2] = [-1.5157, -0.6187]$. The recovered impulse responses with sign and intertemporal restrictions are presented in the upper-left graph of Figure 2. In this case, there exists a large range of impulse responses that satisfy the contemporaneous sign restrictions. However, when intertemporal restrictions are imposed in conjunction with sign restrictions, the sign restrictions become redundant due to the overlap of the identified sets, implying $IS_{IR+SR}^{\rho} = IS_{IR}^{\rho}$. Therefore, in this scenario, the intertemporal restrictions prove valuable in significantly narrowing down the set of admissible responses generated by the sign restrictions.

The top graph of Figure 3 represents the case of non-redundancy of the sign and intertemporal restrictions for DGP1, corresponding to the first quadrant of the upper-left graph of Figure 1. If in DGP1 the angle $\rho = -0.3$, then and the structural matrix takes form $A = HP(\rho) = \begin{bmatrix} 0.9553 & 0.2955 \\ 0.2217 & 0.9751 \end{bmatrix}$. The corresponding intertemporal restrictions $w_{12}^{0} \geq w_{12}^{\pi}$ and $w_{22}^{0} < w_{22}^{\pi}$ result in the identified set $IS_{IR}^{\rho} = [R_2, R_3] = [-0.6187, 0.0551]$. The joint set of the sign and intertemporal restriction is narrower than the individual set: $IS_{IR+SR}^{\rho} = [-0.6187, 0]$. In this case the application of both restrictions is appropriate.

The reduced-form model in the DGR2 is the same as in the DGP1, but now we consider the case when the parameter b and σ_{21} are of the same sign (or ρ and σ_{21} are of the opposite signs). In this case the identified set of the sign restrictions is narrower than in DGP1: $IS_{\rho}^{SR}(b < 0, \sigma_{21} \geq 0) = \left[0, \arctan\left(\frac{\sigma_{22}}{\sigma_{21}}\right)\right] = [0, 1.0472]$. The possible identified sets for this DGP are depicted in the second quadrant in all graphs in Figure 1. Given that the zeros of the intertemporal restrictions depend on the reduced-form parameters only, they take the same values as in DGP1: $R_1 = -1.5157$, $R_2 = -0.6187$, $R_3 = 0.0551$, and $R_4 = 0.9521$. As before, let us illustrate the case where the sign restrictions are redundant as in the second quadrant in the down-left graph of Figure 1, when the limit of the identified set for the sign restrictions is marked with arrow 3. In the upper-right graph of Figure 2, we plot the responses of DGP2 with $\rho = 0.3$ and the corresponding structural matrix $A = HP(\rho) = \begin{bmatrix} 0.9553 & -0.2955 \\ 0.7336 & 0.6796 \end{bmatrix}$, identified with intertemporal restrictions given by $w_{12}^{0} \geq w_{12}^{\pi}$ and $w_{22}^{0} \geq w_{22}^{\pi}$. The resulting identified set for intertemporal restrictions is $IS_{IR}^{\rho} = [R_3, R_4] = [0.0551, 0.9521]$. In this case, the joint identified set IS_{IR+SR}^{ρ} coincides with IS_{IR}^{ρ}, leading to redundant sign restrictions. However, if in DGP2, $\rho \notin [R_3, R_4]$ then the sign restriction is not redundant. For example, if $\rho = 1$ and the corresponding structural matrix $A = HP(\rho) = \begin{bmatrix} 0.9610 & -0.5713 \\ 0.7287 & 0.4679 \end{bmatrix}$, the intertemporal restrictions $w_{12}^{0} < w_{12}^{\pi}$ and $w_{22}^{0} \geq w_{22}^{\pi}$ produces the identified set $IS_{IR}^{\rho} = \left[-\frac{\pi}{2}, R_1\right] \cup \left[R_4, \frac{\pi}{2}\right] = \left[-\frac{\pi}{2}, -1.5157\right] \cup \left[0.9521, \frac{\pi}{2}\right]$ as in the second quadrant of the down-right graph in Figure 1, where the limit of the set identified with the sign restrictions is marked with the arrow 3. In this case, the joint identified set $IS_{IR+SR}^{\rho} = [0.9521, 1.0472]$ is significantly more restrictive than the separate identified sets, leading to a substantial reduction in the set of admissible responses. This results are presented in the middle row of Figure 3.

The responses corresponding to DGP3 depicted in the lower-left graph of Figure 2 for the case where the sign restrictions are redundant and in the last row of Figure 3 where the application of both restrictions is preferable. Given that in the DGP3, the parameters b and σ_{21} are of the same sign (or ρ and σ_{21} are of the opposite signs), the identified set of the sign restrictions is $IS_{\rho}^{SR}(b > 0, \sigma_{21} < 0) = \left[\arctan\left(\frac{\sigma_{22}}{\sigma_{21}}\right), 0\right] = [1.0472, 0]$. The zero of intertemporal restrictions are $R_1 = -0.3206$, $R_2 = -0.0790$, $R_3 = 1.2502$, and $R_4 = 1.4918$. The former case stands for DGP3 with $\rho = -0.3$ and the corresponding structural matrix $A = HP(\rho) = \begin{bmatrix} 0.9553 & 0.2955 \\ -0.7336 & 0.6796 \end{bmatrix}$. For this scenario, the intertemporal restrictions are specified as $w_{12}^{0} < w_{12}^{\pi}$ and $w_{22}^{0} < w_{22}^{\pi}$, resulting in the identified set $IS_{IR}^{\rho} = [R_3, R_4] = [-0.3206, -0.0790]$. This example is represented in the third quadrant of the upper-right graph in Figure 1, where the limit of the set identified with sign restrictions is marked with the arrow 3. In this case, the sign restrictions are redundant since $IS_{IR+SR}^{\rho} = IS_{IR}^{\rho}$. In the latter case we choose $\rho = -1$ and the corresponding structural matrix $A = HP(\rho) = \begin{bmatrix} 0.5403 & 0.8415 \\ -0.9989 & 0.0472 \end{bmatrix}$. The intertemporal restrictions $w_{12}^{0} < w_{12}^{\pi}$ and $w_{22}^{0} \geq w_{22}^{\pi}$ lead to the intertemporal and joint identified set of $IS_{IR}^{\rho} = \left[-\frac{\pi}{2}, R_1\right] \cup \left[R_4, \frac{\pi}{2}\right] = \left[-\frac{\pi}{2}, -0.3206\right] \cup \left[1.4918, \frac{\pi}{2}\right]$ and $IS_{IR+SR}^{\rho} = [-1.0472, -0.3206]$ respectively. This case is represented in the third quadrant of the down-right graph in Figure 1, where the limit of the set identified with the sign restrictions is marked with the arrow 3. The combination of intertemporal and sign restrictions substantially reduces the set of admissible responses, indicating the effectiveness of using both sets of restrictions together.

In n the lower-right graph of Figure 2, we observe the responses of DGP4 with $\rho = -0.3$, the corresponding structural matrix is $A = HP(\rho) = \begin{bmatrix} 0.9553 & 0.2955 \\ -0.7336 & 0.6796 \end{bmatrix}$. The identified set of the sign restriction is $IS_{\rho}^{SR}(b > 0, \sigma_{21} < 0) = \left[\arctan\left(\frac{\sigma_{22}}{\sigma_{21}}\right), 0\right] = [1.0472, 0]$. The zero of intertemporal restrictions are $R_1 = -1.4204$, $R_2 = -1.3752$, $R_3 = 0.1504$, and $R_4 = 0.1956$. The intertemporal restrictions are given by $w_{12}^{0} \geq w_{12}^{\pi}$ and $w_{22}^{0} \geq w_{22}^{\pi}$, result in the identified set of $IS_{IR}^{\rho} = \left[-1.3752, 0.1504\right]$. This situation is represented in the third quadrant of the upper-left graph in Figure 1, where the limit of the set identified with sign restrictions is marked with the arrow 1. In this scenario, we can observe one of the few cases where the intertemporal restrictions are indeed redundant, as $IS_{IR+SR}^{\rho} = IS_{SR}^{\rho}$.

4.2. VAR with *p*>1 and *N*>2

To derive theoretical results within tractable models, we have restricted the theoretical analysis and the simulations to the case of a VAR model containing one lag. However, relaxing these restrictions would have little qualitative impacts on the main results presented in the paper. Specifically, allowing for a higher number of lags up to *p* in the VAR model would result in minor adjustments to the calculation of w_{nk}^{Ω} from

$$C(L) = \left(I - \sum_{j=1}^{p} F_j e^{i\omega j} \right). \tag{19}$$

In addition, the conditions for sign restrictions remain unaffected regardless of the number of lags considered in the VAR model.

With regards to relaxing the dimension of the VAR to include up to *N* variables, the resulting models become mathematically challenging, making it difficult to derive analytical conclusions. To address this issue, we performed extensive simulations that show how intertemporal restrictions, especially when they were combined with sign restrictions, significantly narrow down the set of admissible responses that are provided by sign restrictions alone. In many cases, the sign restrictions become redundant due to the effectiveness of intertemporal restrictions in constraining the responses.[12]

5. Empirical examples

In this section, we illustrate the application of intertemporal identification to provide identified sets for impulse responses used separately or in conjunction with sign restrictions. It is important to note that our analysis is not intended to be exhaustive, does not aim to include the latest developments in the literature on the topic, and does not incorporate policy recommendations.

We focus on three well-known empirical models used in the literature. The first example pertains to the identification of a technology shock in a bi-variate VAR, while the second and third

[12] These results are available from the authors upon request.

examples pertain to the identification of monetary policy shocks and oil market specific demand shocks in a VAR with three variables. All examples are based on 100,000 draws of the orthogonal matrix.

5.1 Identification of the technology shock

The identification of technology shocks in a SVAR has garnered significant attention in the macroeconomic literature, aiming to discern the effects of such shocks. Building upon the influential proposal by Gali (1999), our analysis focuses solely on its simplest specification. This specification relies on a SVAR with two variables: hours worked, assumed to be stationary as in Christiano, Eichenbaum, and Vigfusson (2003), and (log) labor productivity, assumed to be integrated of order one. Consistent with the approach taken by Lovcha and Perez-Laborda (2021), we estimate the bivariate model using U.S. postwar quarterly data from 1948:1 to 2009:4, sourced from the Federal Reserve Bank of St. Louis database.

Our measure of labor input in per-capita terms was obtained by taking the logarithm of the non-farming business sector hours of all persons (HOANBS) and subtracting the logarithm of the civilian non-institutional population over the age of 16 years (CNP16OV). On the other hand, our measure of output productivity is represented by the logarithm of nonfarm business sector labor productivity (output per hour) for all workers (OPHNFB). To ensure comparability, all the time series, except for the population, were seasonally adjusted. Based on this dataset, we estimated the reduced form of the VAR model with four lags.

Next, we proceed to identify the model and calculate the impulse responses to technology shocks applying both intertemporal and sign restrictions. Figure 4 displays the impulse responses of hours worked to technology shocks in three distinct scenarios. In this figure, the responses identified with the two intertemporal restrictions are depicted as shaded blue areas, while the set of identified responses that agrees with the two sign restrictions is delimited by the two black lines.

To facilitate comparisons, we conduct a sensitivity analysis in three distinct scenarios. In the first scenario, the intertemporal restrictions are employed to determine the relative contribution of a technology shock to the variances of productivity and hours worked at frequencies zero (long run) and π (short-run). This implies imposing restrictions on the signs of $w_{12}^{0} - w_{12}^{\pi}$ and $w_{22}^{0} - w_{22}^{\pi}$. Specifically, we assume that a technology shock has a larger effect on the variance of productivity and hours worked in the long run compared to the short run, leading to $w_{22}^{0} - w_{22}^{\pi} \geq 0$, and that

$w_{12}^{0} - w_{12}^{\pi} \geq 0$.[13] In terms of the sign restrictions, we constrain the contemporaneous reaction of productivity to a technology shock to be positive, $d \geq 0$. Additionally, we set the impact response of hours worked to a technology shock as positive, $b \geq 0$, in line with the standard predictions of calibrated Real Business Cycle (RBC) models.

The left graph of Figure 4 displays the responses of hours worked to technology shock obtained under the scenario described above. In this case, both the intertemporal restrictions and sign restrictions lead to positive responses of labor to the technology shock. However, the impact of the technology shock is more pronounced when intertemporal restrictions are imposed. An interesting observation is that the set of the responses that satisfy the intertemporal restrictions is a subset of the set of responses that align with the sign restrictions. This implies that the sign restrictions are redundant, and their exclusion from the analysis would not have any effect on the results.

The second scenario differs from the first scenario in letting the impact response of hours worked to technology shocks to be negative, $b < 0$. This is consistent with the results provided by Keynesian models. In this case, the responses displayed in the middle graph of Figure 4 show that the intertemporal and sign restrictions are not compatible because the set of responses that satisfy both restrictions jointly empty. Unlike sign restrictions, intertemporal restrictions do not permit negative responses of hours worked to technology shocks. Similar to the first scenario, the application of intertemporal restrictions significantly reduces the set of admissible responses.

Finally, the third scenario differs from the second scenario in letting technology shocks to have a greater contribution to the variance of hours worked in the short run compared to the long run, indicated by $w_{12}^{0} - w_{12}^{\pi} < 0$. The estimated sets of responses are displayed in the right graph of Figure 4, illustrating the advantage of combining intertemporal and sign restrictions. Specifically, the range of the responses one gets by applying the contemporaneous sign restrictions independently is much wider than that of the model combining sign restrictions and intertemporal restrictions, which is represented by the shaded area appearing in the bottom of the graph. This shows that the inclusion of intertemporal restrictions narrows down the range of admissible responses significantly, providing more precise and informative results.

Additionally, this empirical example exemplifies the valuable role of intertemporal restrictions in distinguishing between different theoretical proposals based on empirical evidence. The second scenario shows that any theoretical model that implies a negative response of labor to a positive technology shock together with a higher impact of this shock to the long-run labor variance than to the short-run is not compatible with data.

[13] This assumption is less restrictive than restricting technology as the only drive of productivity in the long run.

5.2. Identification of oil market-specific demand shocks

As a second example, we direct our attention on the structural VAR model of the global crude oil market proposed by Kilian (2009). Our objective is to show the usefulness of applying intertemporal restrictions in estimating the dynamic effects of an oil-market-specific demand shock on three key variables: the global crude oil production, the index of real economic activity (representing the global business cycle) constructed by Kilian (2009), and the real price of oil. The dataset consists of monthly observations, spanning from 1973:02 to 2008:09.[14] Within this time frame, we employ a reduced-form VAR model with a lag length of 2, selected based on the minimum BIC information criterion.[15]

Kilian and Murphy (2012) proposed a VAR identification based on sign restrictions that imposing that a positive oil-market-specific demand shock on impact tends to have the following effects: raising oil production, lowering real activity, and stimulating the real price of oil. Mathematically, if we denote the impact reactions of global oil production, real activity, and the real price of oil as a_{13}, a_{23}, and a_{33}, respectively, the sign restrictions imply $a_{13} \geq 0$, $a_{23} < 0$, and $a_{33} \geq 0$. The first column of graphs in Figure 5 displays the set of responses that are admissible with the sign restrictions, which is delimited by the two dashed lines plotted in each graph.

Consistent with the findings of Kilian and Murphy (2012), the Figure 5 shows that sign restrictions yield a wide range of response functions, exhibiting varying amplitudes and shapes. Consequently, this diversity of admissible response functions poses challenges in determining the relative importance of an oil-market-specific demand shock for the evolution of global crude oil production, economic activity, and the real price of oil.

With respect to the intertemporal restrictions and considering that demand shocks specific to the global crude oil market are associated with market specific fluctuations, we posit that they represent shocks for the global oil supply and economic activity associated with mostly to the short-run movements. Accordingly, we assume that the contribution of the oil-market specific shock at frequency π is higher than at the frequency zero for these two variables, which implies $w_{13}^{0} - w_{13}^{\pi} < 0$, and $w_{23}^{0} - w_{23}^{\pi} < 0$.

With the aim of providing a varied set of exemplifications, we are more agnostic about the contribution of the oil-market specific demand shock to the real price of oil and do not wish to rule out the possibility that the shock could be either more important in the short term than in the long term or the in other way around. Thus, we consider two scenarios of intertemporal restrictions. In line

[14] For a full discussion of the data sources and construction of the data see Kilian (2009).

[15] For *p*=24, as in Kilian (2009) and Kilian and Murphy (2012), the estimated spectrum in a VAR is not smooth and it is difficult to judge about the relative importance of a shock at different frequencies or ranges.

with Killian (2009), we evaluate a first scenario in which the price shock reflects an increase in precautionary demand for crude oil that causes immediate large changes in the real price of crude oil. Accordingly, we assume that $w_{33}^{0} - w_{33}^{\pi} < 0$.

The first column of graphs in Figure 5 displays with shaded blue areas the impulse response functions generated by the admissible models after imposing the intertemporal restrictions. The sets of responses that agree with intertemporal restrictions in conjunction with sign restrictions appear as shaded green areas in the second column of graphs in Figure 5. One notable result is that the intertemporal identifying assumptions significantly reduce the number of admissible responses in the index of economic activity when they are imposed in conjunction with sign restrictions. Consistent with Kilian and Murphy (2012), the intertemporal restrictions are able to successfully eliminate all the economically implausible responses that do not satisfy $-1.5 < a_{23} < 0$.

In addition, we consider a second scenario where the oil-market specific demand shock is assumed to be more important in the long-run than in the short-run for the real prices, as indicated by the intertemporal restriction $w_{33}^{0} - w_{33}^{\pi} > 0$. The set of admissible responses under this assumption, displayed in the third and fourth columns of graphs in Figure 5, not only confirms the robustness of the results but also yields a narrower range of admissible responses of oil production and prices, especially when both restrictions are imposed at the same time. In this scenario, it is noteworthy that the set of implied impact responses of economic activity remains above the limit of -1.5. However, it is worth noting that the supply elasticity, as defined in Kilian and Murphy (2012) as a_{13}/a_{33}, will exceed significantly the threshold of 0.0258, which is the value used by those authors to impose an additional restriction.

5.3 Identification of monetary policy shocks

This final example serves to illustrate the identification of impulse responses using intertemporal restrictions across a range of frequencies, rather than focusing on specific frequencies. Specifically, we examine the effects of monetary policy on output, a topic that has been extensively explored in the economic literature. Notably, the survey conducted by Boivin, Kiley and Mishkin (2010) documents the substantial body of scholarly work dedicated to analyzing monetary policy transmission mechanism.

In line with the previous examples, we confine ourselves to examining the most basic specification. In particular, we examine output and inflation responses to shocks in the equation of interest rates for the U.S. The data is quarterly, collected from St. Louis FRED, and runs from the first quarter of 1980 to the last quarter of 2018. The GDP growth is computed as log-difference of real gross domestic product (GDPC1), inflation is computed as log-difference of GDP deflator (GDPDEF), and the treatment of fed fund interest rates (FEDFUNDS) is left agnostically open by considering the time series in level and linearly detrended.[16] Based on the Schwarz criteria, we opted for a VAR model with a lag order of 4.

As a basis of comparison, we rely on Uhlig (2005) to set the sign restrictions. We impose the condition that a contractionary monetary supply shock raises interest rates and lowers output and inflation on impact. If a_{i3} represents the immediate impact of a monetary shock on GDP (i=1), inflation (i=2) and the federal funds rate (i=3), the sign restrictions imply $a_{13} < 0$, $a_{23} < 0$, and $a_{33} \geq 0$. Figure 6 plots as dashed black lines the maximum and the minimum of the impulse responses that satisfy this pure-sign-restriction approach after a contractionary monetary policy shock, where the first two columns are obtained by using interest rate and the last two columns refer to the case of detrended interest rate. Regardless of the responses that we consider, the figure shows that multiple specifications can satisfy the restrictions, resulting in uninformative ranges of admissible responses.

Regarding intertemporal identification, we examine the relative contribution of a monetary policy shock, as implied by various credible parameterizations of the theoretical models developed by Christiano et al. (2005) and Altig et al. (2004). For this purpose, we adopt a methodology similar to that proposed by Dedola and Neri (2007) and Canova and Paustian (2011) to establish robust sign restrictions. However, in this particular study, our objective is to identify frequency ranges where the relative contribution of the monetary policy shock remains consistent despite variations in theoretical model parameterization. The procedure is as follows:

1. Determine credible ranges for all model parameters, ensuring they are sufficiently wide to encompass all theoretically plausible values.
2. Randomly draw values for each model parameter from independent uniform distributions within their respective credible intervals, generating a credible parameterization of the model.
3. Calculate the contributions of the monetary policy shock to the variances of the variables at different frequencies or frequency ranges using this parameterization.
4. Repeat steps 2 and 3 until the desired number of credible parameterizations is obtained.
5. Identify the frequencies or frequency ranges where the relative contribution of the monetary policy shock remains robust across all parameterizations. Combine these frequencies or ranges with the findings on the relative importance of the monetary policy shock for intertemporal identification.

[16] The evolution of the fed fun rate has a significant downward sloping trend in our sample period.

In line with this procedure, we postulate that a monetary policy shock has a short-run effect in output and interest rate while it has a long-run effect on inflation. We define the long-run monetary policy shock as one that contributes more to the variance at frequencies with period longer than 30 years, than to frequencies shorter than 1 year. Translated to intertemporal restrictions, these effects imply $w_{13}^{long} - w_{13}^{short} < 0$, $w_{23}^{long} - w_{23}^{short} > 0$, and $w_{33}^{long} - w_{33}^{short} < 0$, where *long* and *short* refer to period above 30 years and below one year, respectively.

Figure 6 displays as blue areas the set of responses that are admissible with intertemporal restrictions and as green areas those that satisfy both intertemporal and sign restrictions. Notably, intertemporal restrictions, used in isolation or combined with sign restrictions, tend to narrow down significantly the identified sets of impulse responses to a monetary policy shock. The results remain robust even when detrending the interest rate. Notably, the resulting sets of admissible responses align with the findings of Christiano, Eichenbaum, and Evans (2005) and Altig et al. (2011) regarding the response patterns of output, prices, and interest rates to a monetary policy shock. Specifically, we find a hump-shaped response of output, a short-run response of interest rates, and substantial inertia in the response of prices.

6. Conclusion

We contribute to the literature on structural vector autoregression models, by proposing a novel identification procedure for identifying structural shocks. This procedure involves imposing restrictions on the relative importance of shocks across different frequency horizons. Our approach follows a similar procedure to identifying VAR models as that of sign restrictions, as it also explores the responses generated by rotating a set of base shocks derived from the model estimates.

However, our procedure differs from sign restrictions in terms of the type of restrictions used to select the set of admissible responses. While sign restrictions discard rotations that produce responses deviating from expected signs, our intertemporal restrictions eliminate rotations that generate responses inconsistent with the expected relative contribution of some structural shocks to the variances of target variables. Specifically, we focus on the relative contribution of structural shocks at long-run frequencies compared to short-run frequencies.

This method allows for the generation of economically meaningful responses, such as those identifying fundamental shocks that tend to be concentrated in long-run frequencies, or transitory market specific shocks whose effects are primarily observed in short-run frequencies. These assumptions are less restrictive than the standard variance-based medium-term restrictions or targeting contributions of shocks at different frequency horizons because they do not aim to specify a specific share of the shock of interest in the variables' variances.

We contribute to a theoretical framework that characterizes the system of inequalities for model parameters derived from intertemporal restrictions when the rotation matrix takes a Givens form. We solve these inequalities to determine the angles that are compatible with the restrictions, resulting in an identified set of angles that generate the admissible impulse responses. This framework enables us to establish the conditions under which intertemporal restrictions, either alone or in conjunction with other identification procedures like sign restrictions, yield meaningful results.

Our findings characterize the situations for which intertemporal restrictions effectively narrow down the identified set of admissible responses provided by sign restrictions, highlighting the redundancy of the latter in many cases. With the aim of illustrating these results under controlled conditions, we conduct a simulation study based on a carefully designed data generating process.

Finally, we show the applicability of the methodology developed in this article through a threefold empirical application. These applications showcase the broad range of problems in which VAR models can be employed to identify structural shocks. Specifically, we focus on economic issues that have garnered significant attention in the literature, including the identification of technology shocks, oil-specific demand shocks, and monetary policy shocks.

To establish a basis of comparison, we highlight that the pure-sign-restriction approach often results in a wide range of admissible impulse responses, which diminishes the usefulness of the sign-restriction results for policymaking purposes. In an interesting contrast, we observe that the inclusion of intertemporal restrictions, either in isolation or in combination with sign restrictions, significantly reduces the number of accepted responses. This leads to much more informative sets of admissible responses.

Appendix A. Derivation of zeros for the intertemporal restrictions

Let us consider the first intertemporal restriction in the sign of $w_{12}^{0}-w_{12}^{\pi}$. To derive the zeros, let us consider the difference between these two causality spectra

$$\begin{aligned} w_{12}^{0}-w_{12}^{\pi} &= \frac{1}{\left[(1-F_{11})(1-F_{22})-F_{12}F_{21}\right]^2}\frac{\left[-(1-F_{22})\sigma_{11}\sin\rho+F_{12}\left(-\sigma_{21}\sin\rho+\sigma_{22}\cos\rho\right)\right]^2}{S_{11}(0)}- \\ &-\frac{1}{\left[(1+F_{11})(1+F_{22})-F_{12}F_{21}\right]^2}\frac{\left[-(1+F_{22})\sigma_{11}\sin\rho-F_{12}\left(-\sigma_{21}\sin\rho+\sigma_{22}\cos\rho\right)\right]^2}{S_{11}(\pi)} \end{aligned} \tag{A1}$$

Denote $K_{110}^2=\frac{1}{\left[(1-F_{11})(1-F_{22})-F_{12}F_{21}\right]^2}\frac{1}{S_{11}(0)}$ and $K_{11\pi}^2=\frac{1}{\left[(1+F_{11})(1+F_{22})-F_{12i}F_{21}\right]^2}\frac{1}{S_{11}(\pi)}$, and rewrite (A.1) with the new notation:

$$\begin{aligned} w_{12}^{0}-w_{12}^{\pi} &= K_{110}^2\left[-(1-F_{22})\sigma_{11}\sin\rho+F_{12}\left(-\sigma_{21}\sin\rho+\sigma_{22}\cos\rho\right)\right]^2- \\ &-K_{11\pi}^2\left[-(1+F_{22})\sigma_{11}\sin\rho-F_{12}\left(-\sigma_{21}\sin\rho+\sigma_{22}\cos\rho\right)\right]^2 \end{aligned} \tag{A2}$$

Apply $a_1^2-a_2^2=(a_1+a_2)(a_1-a_2)$

$$\begin{aligned} & w_{12}^{0}-w_{12}^{\pi}= \\ &=\left\{K_{110}\left[-(1-F_{22})\sigma_{11}\sin\rho+F_{12}\left(-\sigma_{21}\sin\rho+\sigma_{22}\cos\rho\right)\right]+K_{11\pi}\left[-(1+F_{22})\sigma_{11}\sin\rho-F_{12}\left(-\sigma_{21}\sin\rho+\sigma_{22}\cos\rho\right)\right]\right\}\times \\ &\times\left\{K_{110}\left[-(1-F_{22})\sigma_{11}\sin\rho+F_{12}\left(-\sigma_{21}\sin\rho+\sigma_{22}\cos\rho\right)\right]-K_{11\pi}\left[-(1+F_{22})\sigma_{11}\sin\rho-F_{12}\left(-\sigma_{21}\sin\rho+\sigma_{22}\cos\rho\right)\right]\right\} \end{aligned}$$

The zeros $w_{12}^{0}-w_{12}^{\pi}=0$ will appear when

$$K_{110}\left[-(1-F_{22})\sigma_{11}\sin\rho+F_{12}\left(-\sigma_{21}\sin\rho+\sigma_{22}\cos\rho\right)\right]+K_{11\pi}\left[-(1+F_{22})\sigma_{11}\sin\rho-F_{12}\left(-\sigma_{21}\sin\rho+\sigma_{22}\cos\rho\right)\right]=0$$

and/or

$$K_{110}\left[-(1-F_{22})\sigma_{11}\sin\rho+F_{12}\left(-\sigma_{21}\sin\rho+\sigma_{22}\cos\rho\right)\right]-K_{11\pi}\left[-(1+F_{22})\sigma_{11}\sin\rho-F_{12}\left(-\sigma_{21}\sin\rho+\sigma_{22}\cos\rho\right)\right]=0$$

Simplify and rearrange the first expression:

$$\sin\rho\left[K_{110}F_{22}\sigma_{11}-K_{110}\sigma_{11}+K_{11\pi}\sigma_{11}+K_{11\pi}F_{22}\sigma_{11}-K_{110}F_{12}\sigma_{21}-K_{11\pi}F_{12}\sigma_{21}\right]=-\cos\rho\left(K_{110}+K_{11\pi}\right)F_{12}\sigma_{22}$$

$$E_{12,1}=\frac{\sin\rho}{\cos\rho}=\tan\rho=\frac{-\left(K_{110}+K_{11\pi}\right)F_{12}\sigma_{22}}{\left(K_{110}+K_{11\pi}\right)F_{22}\sigma_{11}-\left(K_{110}+K_{11\pi}\right)F_{12}\sigma_{21}+\left(K_{11\pi}-K_{110}\right)\sigma_{11}} \tag{A.3}$$

$$R_1=\rho_{12,1}=\arctan\left(E_{12,1}\right)=\arctan\left[\frac{-\left(K_{110}+K_{11\pi}\right)F_{12}\sigma_{22}}{\left(K_{110}+K_{11\pi}\right)F_{22}\sigma_{11}-\left(K_{110}+K_{11\pi}\right)F_{12}\sigma_{21}+\left(K_{11\pi}-K_{110}\right)\sigma_{11}}\right]$$

Simplify and rearrange the second expression:

$$\sin\rho\left[K_{110}F_{22}\sigma_{11}-K_{110}\sigma_{11}-K_{11\pi}\sigma_{11}-K_{11\pi}F_{22}\sigma_{11}-K_{110}F_{12}\sigma_{21}+K_{11\pi}F_{12}\sigma_{21}\right]=\cos\rho\left(K_{11\pi}-K_{110}\right)F_{12}\sigma_{22}$$

$$E_{12,2} = \frac{\sin\rho}{\cos\rho} = \tan\rho = \frac{\left(K_{11\pi} - K_{110}\right)F_{12}\sigma_{22}}{-\left(K_{11\pi} - K_{110}\right)F_{22}\sigma_{11} + \left(K_{11\pi} - K_{110}\right)F_{12}\sigma_{21} - \left(K_{110} + K_{11\pi}\right)\sigma_{11}} \quad \text{(A4)}$$

$$R_3 = \rho_{12,2} = \arctan\left(E_{12,2}\right) = \arctan\left[\frac{\left(K_{11\pi} - K_{110}\right)F_{12}\sigma_{22}}{-\left(K_{11\pi} - K_{110}\right)F_{22}\sigma_{11} + \left(K_{11\pi} - K_{110}\right)F_{12}\sigma_{21} - \left(K_{110} + K_{11\pi}\right)\sigma_{11}}\right]$$

In a symmetric way, we can derive the zeros from $w_{22}^{0} - w_{22}^{\pi} = 0$, which implies

$$E_{22,1} = \frac{\left(K_{22\pi} - K_{220}\right)\sigma_{22} + F_{11}\sigma_{22}\left(K_{220} + K_{22\pi}\right)}{-F_{21}\sigma_{11}\left(K_{220} + K_{22\pi}\right) + \sigma_{21}\left(K_{22\pi} - K_{220}\right) + F_{11}\sigma_{21}\left(K_{220} + K_{22\pi}\right)} \quad \text{(A5)}$$

$$E_{22,2} = \frac{-\sigma_{22}\left(K_{22\pi} + K_{220}\right) + F_{11}\sigma_{22}\left(K_{220} - K_{22\pi}\right)}{-F_{21}\sigma_{11}\left(K_{220} - K_{22\pi}\right) - \sigma_{21}\left(K_{22\pi} + K_{220}\right) + F_{11}\sigma_{21}\left(K_{220} - K_{22\pi}\right)}. \quad \text{(A6)}$$

where $K_{220}^{2} = \frac{1}{\left[\left(1-F_{11}\right)\left(1-F_{22}\right) - F_{12}F_{21}\right]^{2}}\frac{1}{S_{22}(0)}$ and $K_{22\pi}^{2} = \frac{1}{\left[\left(1+F_{11}\right)\left(1+F_{22}\right) - F_{12i}F_{21}\right]^{2}}\frac{1}{S_{22}(\pi)}$.

In this case, the zeros are $R_2 = \rho_{22,1} = \arctan\left(E_{22,1}\right)$ and $R_4 = \rho_{22,2} = \arctan\left(E_{22,2}\right)$.

Simplify and rearrange (A3) to (A6)

$$E_{12,1} = \frac{\sigma_{22}}{\sigma_{21} + \sigma_{11}\dfrac{\dfrac{K_{110} - K_{11\pi}}{K_{110} + K_{11\pi}} - F_{22}}{F_{12}}}, \quad E_{12,2} = \frac{\sigma_{22}}{\sigma_{21} + \sigma_{11}\dfrac{\dfrac{K_{110} + K_{11\pi}}{K_{110} - K_{11\pi}} - F_{22}}{F_{12}}},$$

$$E_{22,1} = \frac{\sigma_{22}}{\sigma_{21} + \sigma_{11}\dfrac{F_{21}}{\dfrac{K_{220} - K_{22\pi}}{K_{220} + K_{22\pi}} - F_{11}}}, \quad E_{22,1} = \frac{\sigma_{22}}{\sigma_{21} + \sigma_{11}\dfrac{F_{21}}{\dfrac{K_{220} + K_{22\pi}}{K_{220} - K_{22\pi}} - F_{11}}}.$$

Appendix B. Length of the intervals that hold the intertemporal restrictions.

There are two zeros for IR1 ($R_1 = \rho_{12,1}$ and $R_3 = \rho_{12,2}$) and two zeros for IR2 ($R_2 = \rho_{12,1}$ and $R_4 = \rho_{12,2}$) on the interval $\left[-\pi/2, \pi/2\right]$. Let us consider the restriction IR1 for which there are two possible intervals on which the restriction is satisfied:

$\left[-\pi/2, \rho_{12,1}\right] \cup \left[\rho_{12,2}, \pi/2\right]$ or its complementary $\left[\rho_{12,1}, \rho_{12,2}\right]$ if $\rho_{12,1} < \rho_{12,2}$;

$\left[-\pi/2, \rho_{12,2}\right] \cup \left[\rho_{12,1}, \pi/2\right]$ or its complementary $\left[\rho_{12,2}, \rho_{12,1}\right]$ if $\rho_{12,1} > \rho_{12,2}$.

Note that if the length of one of these intervals is $\pi/2$, the length of the complementary is $\pi/2$ as well. So, without loss of generality, we consider the length $\left|\rho_{12,1} - \rho_{12,2}\right|$.

$$\rho_{12,1} - \rho_{12,2} = \arctan\left(E_{12.1}\right) - \arctan\left(E_{12,2}\right) = \arctan\left(\frac{E_{12,1} - E_{12,2}}{1 + E_{12,1}E_{12,2}}\right). \quad \text{(B1)}$$

In the denominator:

$$E_{12,1}E_{12,2} = \frac{-\left(K_{110}+K_{11\pi}\right)F_{12}\sigma_{22}}{\left(K_{110}+K_{11\pi}\right)F_{22}\sigma_{11}-\left(K_{110}+K_{11\pi}\right)F_{12}\sigma_{21}+\left(K_{11\pi}-K_{110}\right)\sigma_{11}}\times$$

$$\times\frac{\left(K_{11\pi}-K_{110}\right)F_{12}\sigma_{22}}{-\left(K_{11\pi}-K_{110}\right)F_{22}\sigma_{11}+\left(K_{11\pi}-K_{110}\right)F_{12}\sigma_{21}-\left(K_{110}+K_{11\pi}\right)\sigma_{11}}=$$

$$=\frac{-\left(K_{110}+K_{11\pi}\right)\left(K_{11\pi}-K_{110}\right)\left(F_{12}\sigma_{22}\right)^2}{\left\{-K_{110}\left[\left(1-F_{22}\right)\sigma_{11}+F_{12}\sigma_{21}\right]+K_{11\pi}\left[\left(1+F_{22}\right)\sigma_{11}-F_{12}\sigma_{21}\right]\right\}\times\left\{-K_{110}\left[\left(1-F_{22}\right)\sigma_{11}+F_{12}\sigma_{21}\right]-K_{11\pi}\left[\left(1+F_{22}\right)\sigma_{11}-F_{12}\sigma_{21}\right]\right\}}=$$

$$=\frac{K_{110}^2\left(F_{12}\sigma_{22}\right)^2-K_{11\pi}^2\left(F_{12}\sigma_{22}\right)^2}{K_{110}^2\left[\left(1-F_{22}\right)\sigma_{11}+F_{12}\sigma_{21}\right]^2-K_{11\pi}^2\left[\left(1+F_{22}\right)\sigma_{11}-F_{12}\sigma_{21}\right]^2}$$

Note that the spectrum of the process for the first variable at 0 frequency is given by

$$S_{11}(0)=\frac{\left\{\left[\left(1-F_{22}\right)\sigma_{11}+F_{12}\sigma_{21}\right]^2+\left[F_{12}\sigma_{22}\right]^2\right\}}{\left[\left(1-F_{11}\right)\left(1-F_{22}\right)-F_{12}F_{21}\right]^2}, \tag{B2}$$

which implies that

$$\frac{\left\{\left[\left(1-F_{22}\right)\sigma_{11}+F_{12}\sigma_{21}\right]^2+\left[F_{12}\sigma_{22}\right]^2\right\}}{\left[\left(1-F_{11}\right)\left(1-F_{22}\right)-F_{12}F_{21}\right]^2 S_{11}(0)}=1.$$

In the same way, the spectrum at frequency π is

$$S_{11}(\pi)=\frac{\left\{\left[\left(1+F_{22}\right)\sigma_{11}-F_{12}\sigma_{21}\right]^2+\left[-F_{12}\sigma_{22}\right]^2\right\}}{\left[\left(1+F_{11}\right)\left(1+F_{22}\right)-F_{12}F_{21}\right]^2}, \tag{B3}$$

which implies that

$$\frac{\left\{\left[\left(1+F_{22}\right)\sigma_{11}-F_{12}\sigma_{21}\right]^2+\left[-F_{12}\sigma_{22}\right]^2\right\}}{\left[\left(1+F_{11}\right)\left(1+F_{22}\right)-F_{12}F_{21}\right]^2 S_{11}(\pi)}=1.$$

Let us call

$$\frac{\left[F_{12}\sigma_{22}\right]^2}{\left[\left(1-F_{11}\right)\left(1-F_{22}\right)-F_{12}F_{21}\right]^2 S_{11}(0)}=M_0,\quad \frac{\left[\left(1-F_{22}\right)\sigma_{11}+F_{12}\sigma_{21}\right]^2}{\left[\left(1-F_{11}\right)\left(1-F_{22}\right)-F_{12}F_{21}\right]^2 S_{11}(0)}=1-M_0,$$

$$\frac{\left[-F_{12}\sigma_{22}\right]^2}{\left[\left(1+F_{11}\right)\left(1+F_{22}\right)-F_{12}F_{21}\right]^2 S_{11}(\pi)}=M_\pi, \text{ and } \frac{\left[\left(1+F_{22}\right)\sigma_{11}-F_{12}\sigma_{21}\right]^2}{\left[\left(1+F_{11}\right)\left(1+F_{22}\right)-F_{12}F_{21}\right]^2}\frac{1}{S_{11}(\pi)}=1-M_\pi.$$

Turning back to the expression for $E_{12,1}E_{12,2}$ and substituting expressions for K_{110} and $K_{11\pi}$, we get:

$$E_{12,1}E_{12,2}=\frac{\frac{\left(F_{12}\sigma_{22}\right)^2}{\left[(1-F_{11i})(1-F_{22i})-F_{12i}F_{21i}\right]^2}\frac{1}{S_{11}(0)}-\frac{\left(F_{12}\sigma_{22}\right)^2}{\left[(1+F_{11i})(1+F_{22i})-F_{12i}F_{21i}\right]^2}\frac{1}{S_{11}(\pi)}}{\frac{\left[(1-F_{22})\sigma_{11}+F_{12}\sigma_{21}\right]^2}{\left[(1-F_{11i})(1-F_{22i})-F_{12i}F_{21i}\right]^2}\frac{1}{S_{11}(0)}-\frac{\left[(1+F_{22})\sigma_{11}-F_{12}\sigma_{21}\right]^2}{\left[(1+F_{11i})(1+F_{22i})-F_{12i}F_{21i}\right]^2}\frac{1}{S_{11}(\pi)}}=$$

$$=\frac{M_0-M_\pi}{(1-M_0)-(1-M_\pi)}=\frac{M_0-M_\pi}{-M_0+M_\pi}=-1$$

Using this result, it is easy to see that $\rho_{12,1}-\rho_{12,2}=\arctan\left(\frac{E_{12,1}-E_{12,2}}{1-1}\right)=\arctan\left(\frac{E_{12,1}-E_{12,2}}{0}\right)=\frac{\pi}{2}$.

The proof of the length of the intervals that hold the intertemporal restriction IR2 follows the same logic, and the length of the interval is $\pi/2$ as well.

Appendix C. Identification set for sign restrictions.

The identifying sign restrictions are the normalization $-\sigma_{21}\sin\rho+\sigma_{22}\cos\rho\geq 0$ and $-\sigma_{11}\sin\rho\geq 0$ when $b\geq 0$ or $-\sigma_{11}\sin\rho<0$ when $b<0$. Depending on the signs of b and σ_{21}, there are four different possible identified sets for ρ.

When $b\geq 0$, the restriction $-\sigma_{11}\sin\rho\geq 0$ only admits rotations of base shocks with negative angles in the interval $[-\pi/2,0]$. If $\sigma_{21}\geq 0$, the normalization $-\sigma_{21}\sin\rho+\sigma_{22}\cos\rho\geq 0$ implies that $\rho\leq\arctan(\sigma_{22}/\sigma_{21})$, which is always true in the interval $[-\pi/2,0]$ and the length of the identified set is $\pi/2$. If $\sigma_{21}<0$, the normalization implies $\rho\geq\arctan(\sigma_{22}/\sigma_{21})$, which is true only in the interval $[\arctan(\sigma_{22}/\sigma_{21}),0]$ and the length of the identified set is $|\arctan(\sigma_{22}/\sigma_{21})|<\pi/2$.

When $b<0$, the restriction $-\sigma_{11}\sin\rho<0$ only admits rotations of base shocks with negative angles in the interval $[0,\pi/2]$. If $\sigma_{21}<0$, the normalization implies $\rho\geq\arctan(\sigma_{22}/\sigma_{21})$, which is true in the interval $[0,\pi/2]$ and the length of the identified set is $\pi/2$. If $\sigma_{21}\geq 0$, the normalization $-\sigma_{21}\sin\rho+\sigma_{22}\cos\rho\geq 0$ implies that $\rho\leq\arctan(\sigma_{22}/\sigma_{21})$, which is true only in the interval $[0,\arctan(\sigma_{22}/\sigma_{21})]$ and the length of the identified set is $|\arctan(\sigma_{22}/\sigma_{21})|<\pi/2$.

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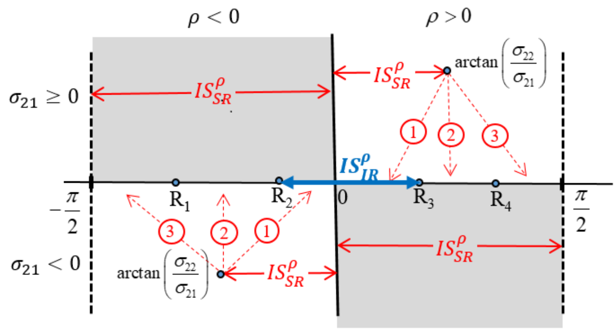
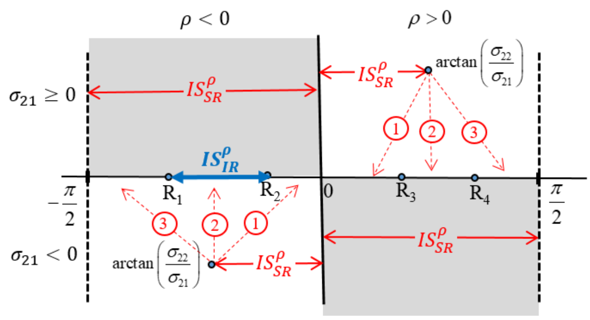
Figure 1. Identified sets.

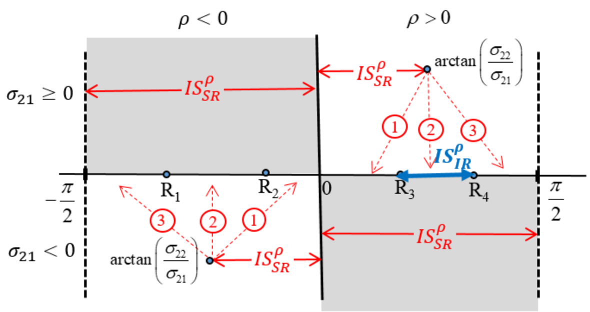
$IS_{IR}^{\rho} = (R_1, R_3) \cap (R_2, R_4) = (R_2, R_3)$



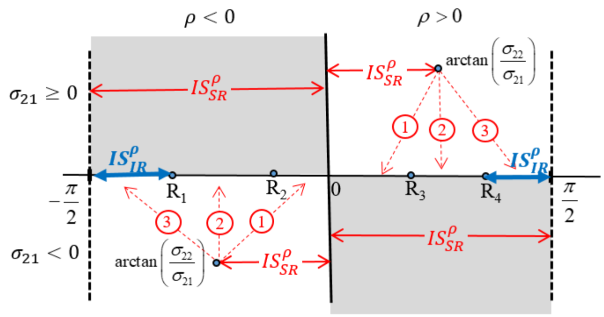
$IS_{IR}^{\rho} = (R_1, R_2)$



$IS_{IR}^{\rho} = (R_3, R_4)$

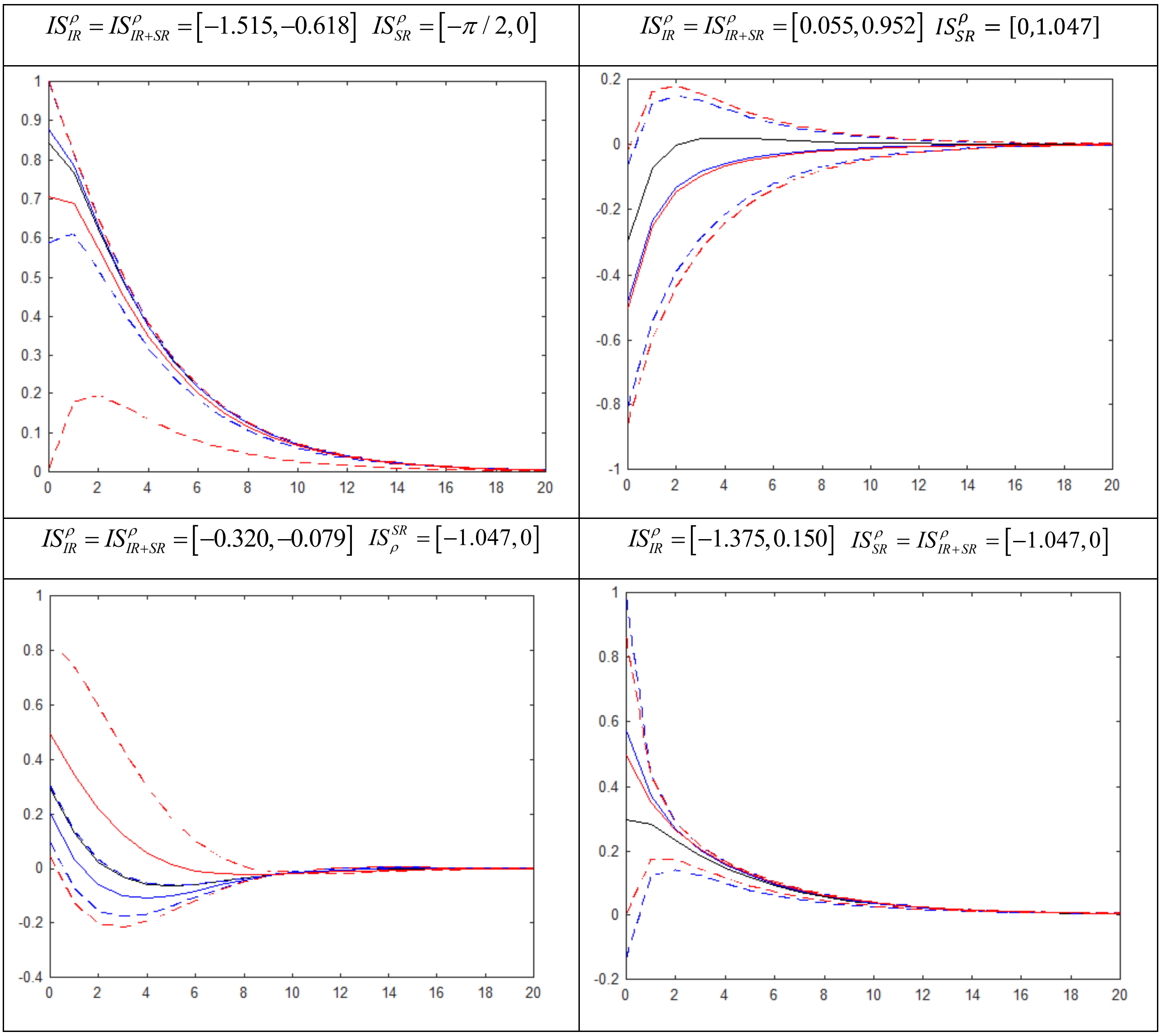


$IS_{IR}^{\rho} = \{(-\pi/2, R_1) \cup (R_4, \pi/2)\}$



Notes. Identified sets (IS) on the interval $\rho \in [-\pi/2, \pi/2]$ provided by the zeros R₁, R₂, R₃, and R₄ of the intertemporal restrictions on $w_{12}^{0} - w_{12}^{\pi}$ and $w_{22}^{0} - w_{22}^{\pi}$ (IR, blue arrows) and by the zeros of the sign restrictions on $-\sigma_{21} \sin\rho + \sigma_{22} \cos\rho$ and $-\sigma_{11} \sin\rho_m$ (SR, red arrows).

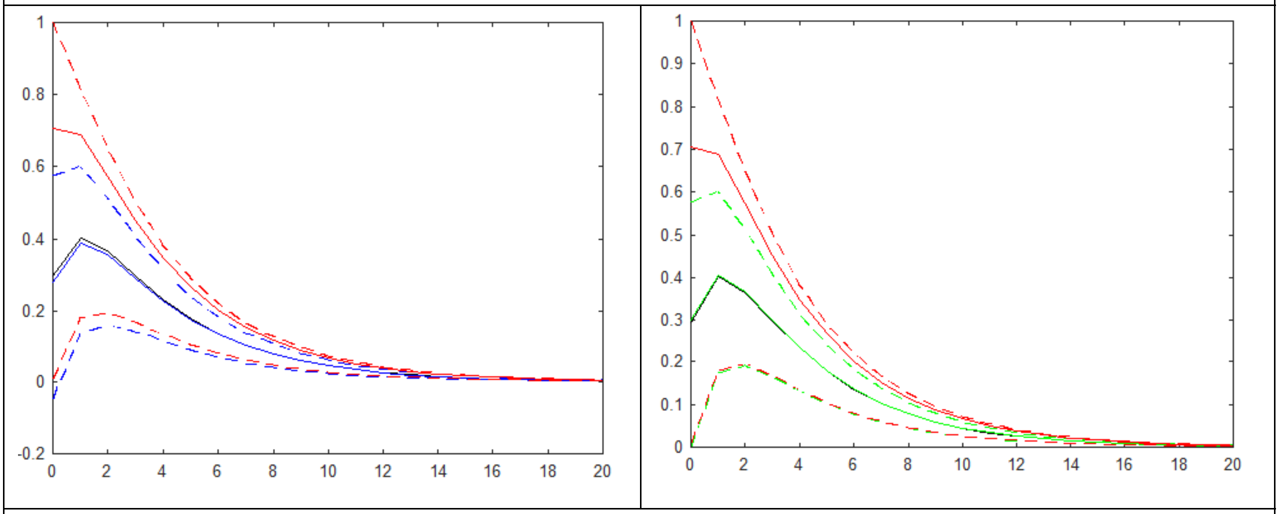
Figure 2. Impulse responses with redundancy of intertemporal or sign restrictions.



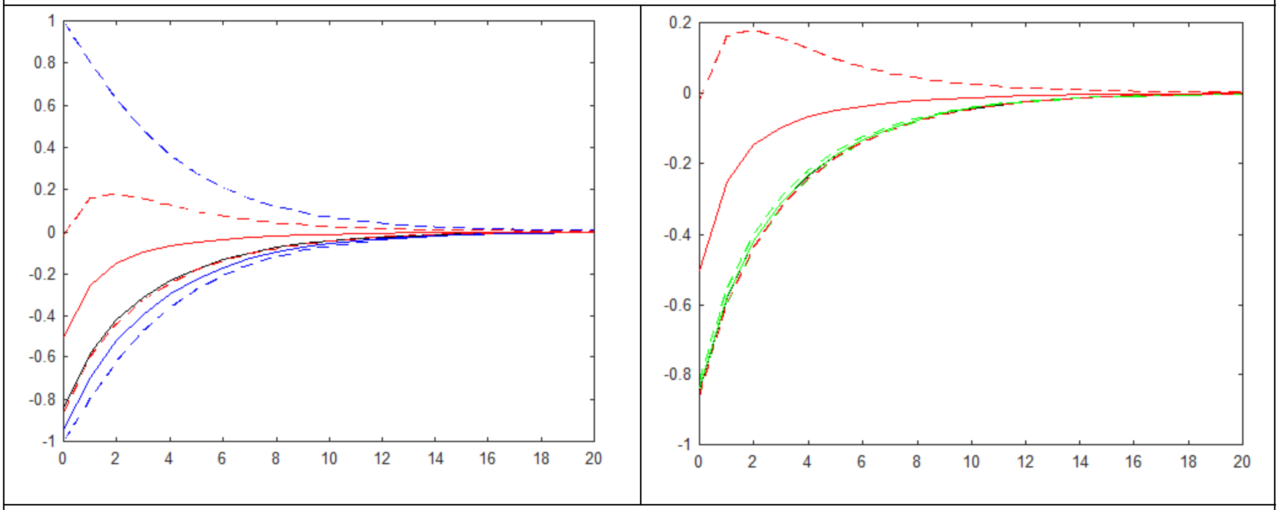
Notes: The graphs display the responses of the first variable to shocks in the second variable. The black lines depict the responses of the data generating process. The blue dashed lines represent the limits of the responses provided by IS_{IR}^{ρ}, with the blue line indicating their median values. Similarly, the red dashed lines represent the limits of the responses provided by IS_{SR}^{ρ}, with the red line indicating their median values.

Figure 3. Impulse responses without redundancy.

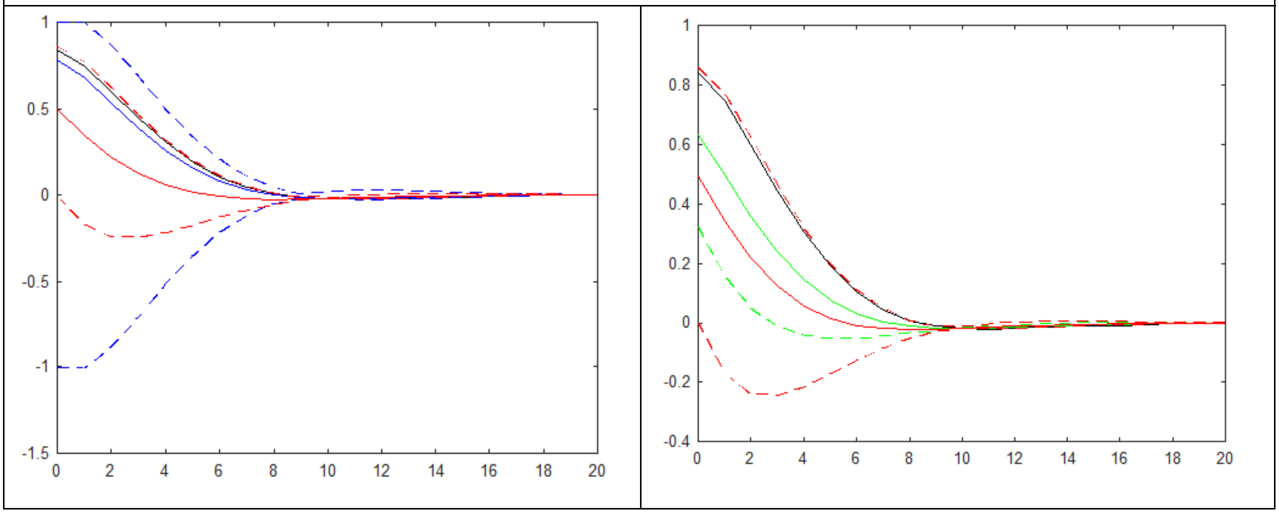
$$IS_{IR}^{\rho}=\left[-0.618,0.055\right]\quad IS_{SR}^{\rho}=\left[-\pi/2,0\right]\quad IS_{IR+SR}^{\rho}=\left[-0.6187,0\right]$$



$$IS_{IR}^{\rho}=\left[-\pi/2,-1.515\right]\cup\left[0.952,\pi/2\right]\quad IS_{SR}^{\rho}=\left[0,1.047\right]\quad IS_{IR+SR}^{\rho}=\left[0.9521,1.0472\right]$$

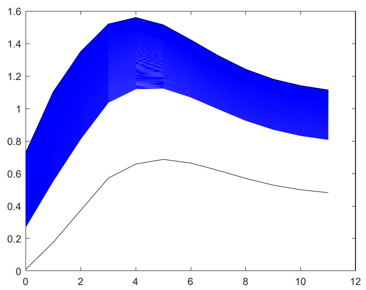
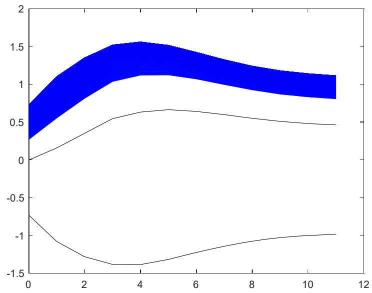
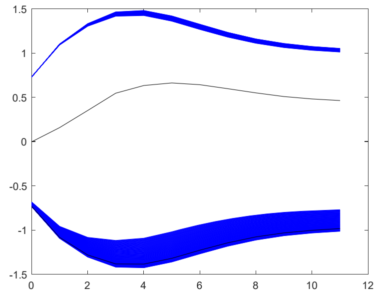


$$IS_{IR}^{\rho}=\left[-\pi/2,-0.320\right]\cup\left[1.491,\pi/2\right]\quad IS_{SR}^{\rho}=\left[-1.047,0\right]\quad IS_{IR+SR}^{\rho}=\left[-1.0472,-0.3206\right]$$



Notes: The green dashed lines represent the limits of the responses provided by IS_{IR+SR}^{ρ}, with the green line indicating their median values. Refer to the notes in Figure 2 for further details.

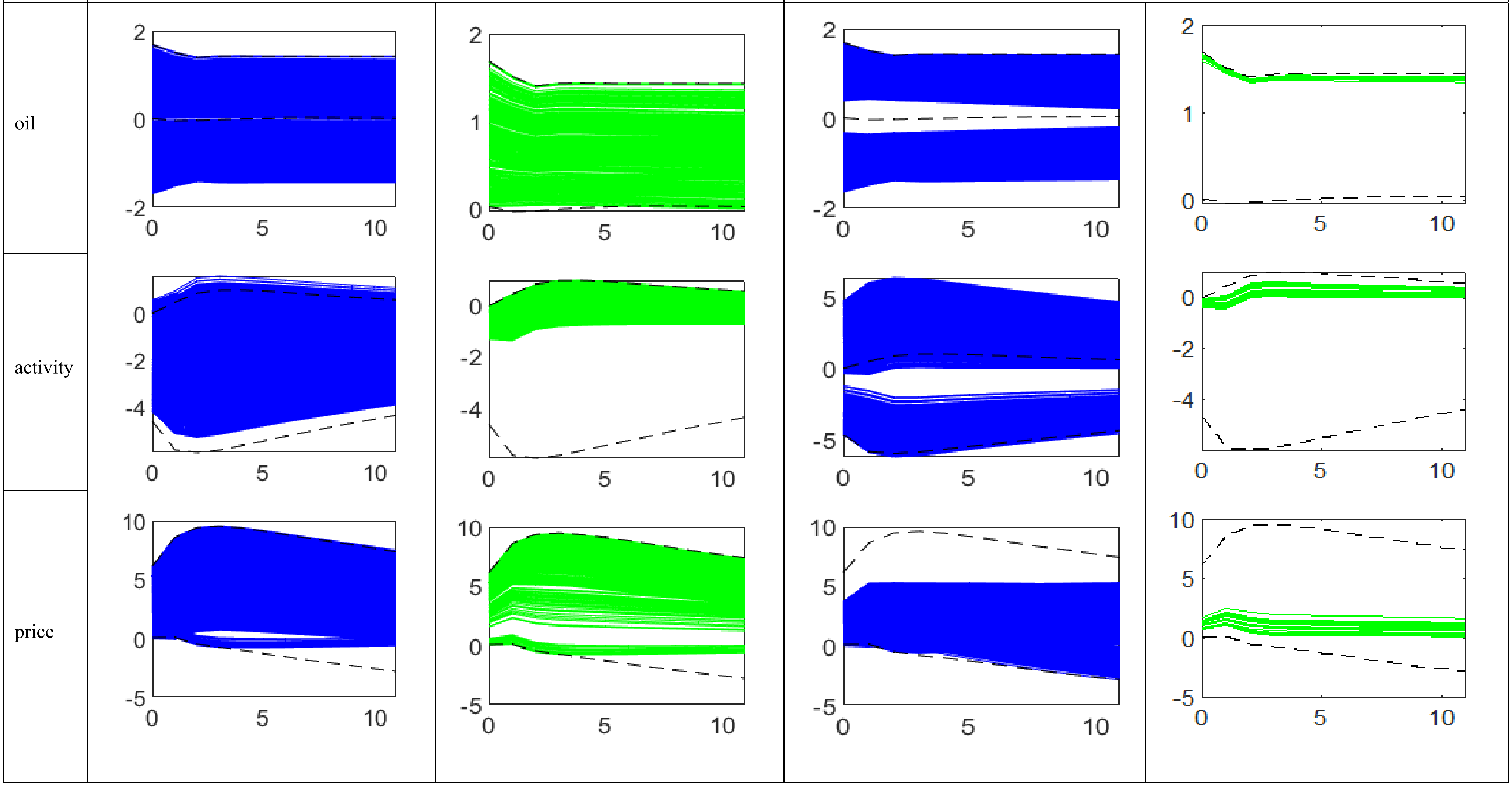
Figure 4. Technology shocks.

IR: $w_{12}^{0} - w_{12}^{\pi} \geq 0, w_{22}^{0} - w_{22}^{\pi} \geq 0$ SR: $b \geq 0, d \geq 0$	IR: $w_{12}^{0} - w_{12}^{\pi} \geq 0, w_{22}^{0} - w_{22}^{\pi} \geq 0$ SR: $b < 0, d \geq 0$	IR: $w_{12}^{0} - w_{12}^{\pi} < 0, w_{22}^{0} - w_{22}^{\pi} \geq 0$ SR: $b < 0, d \geq 0$
		

Notes: The figure displays the response of hours worked to technology shocks. The shocks are identified with both intertemporal restrictions (resulting in the blue identified set of responses) and sign restrictions (limiting the identified set of responses represented by the black lines).

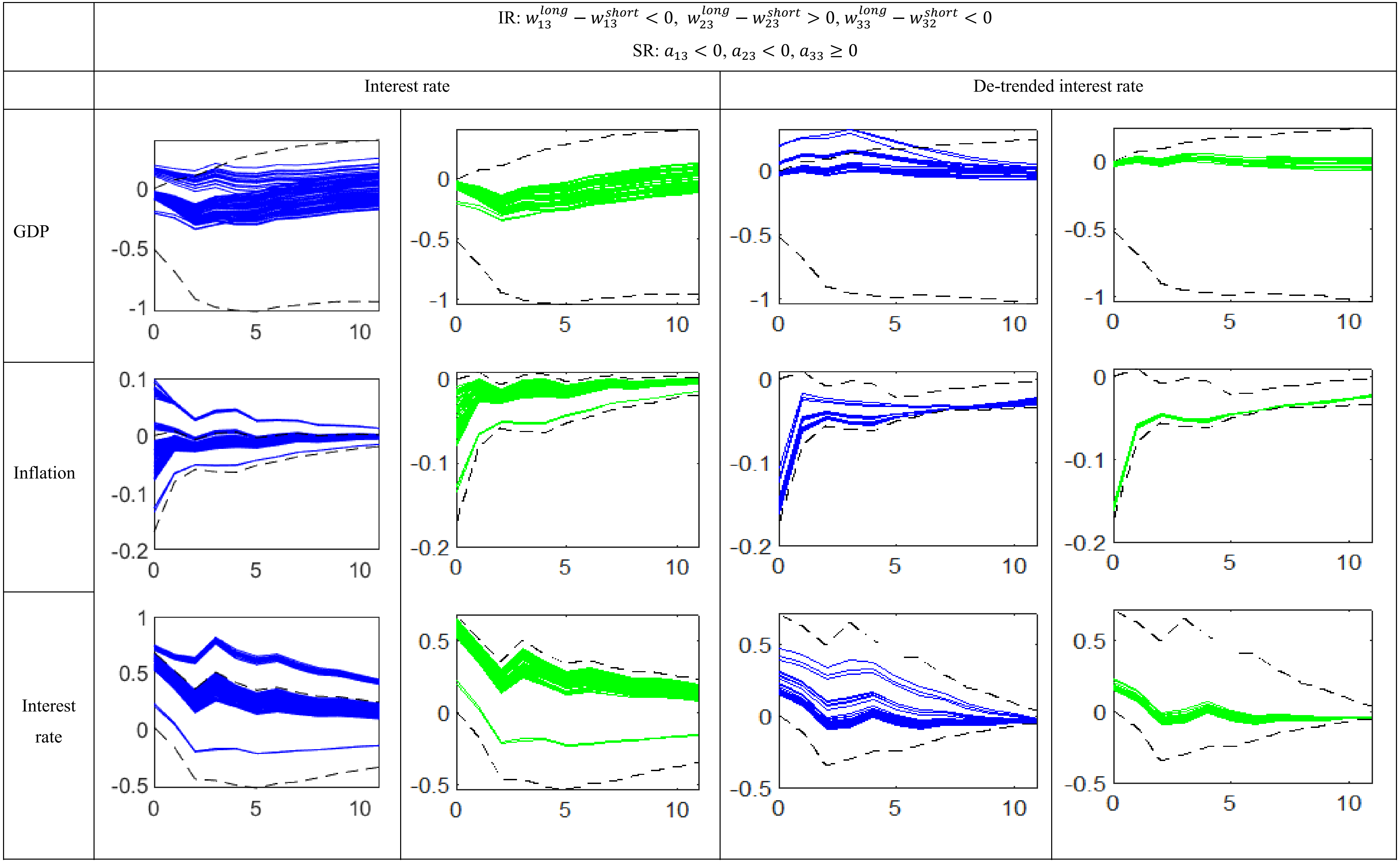
Figure 5. Oil-market-specific shocks

	IR: $w_{13}^{0} - w_{13}^{\pi} < 0, w_{23}^{0} - w_{23}^{\pi} < 0, w_{33}^{0} - w_{33}^{\pi} < 0$ SR: $a_{13} \geq 0, a_{23} < 0, a_{33} \geq 0$	IR: $w_{13}^{0} - w_{13}^{\pi} < 0,\ w_{23}^{0} - w_{23}^{\pi} < 0, w_{33}^{0} - w_{33}^{\pi} \geq 0$ SR: $a_{13} \geq 0, a_{23} < 0, a_{33} \geq 0$



Notes: The figure displays the response of global crude oil production (oil), the index of real economic activity, and real prices of oil (price) to an oil-market specific demand shock. The shocks are identified with both intertemporal restrictions (resulting in the blue identified set of responses) and sign restrictions (limiting the identified set of responses represented by the black lines). The responses identified with intertemporal and sign restrictions are displayed in green.

Figure 6. Monetary policy shocks



Notes: The figure displays the response of GDP, inflation, and interest rates to a monetary policy shock. Refer to Figure 5 for further details.