Lorentzian Geometry-Benalmádena 2001

Benalmádena (Málaga) Spain

November 14, 15 and 16, 2001

Edited by:
M.A. Cañadas-Pinedo
M. Gutiérrez
A. Romero

Organizing Committee

María A. Cañadas-Pinedo[1]
Manuel Gutiérrez[1]
Alfonso Romero[2]

[1]Dpt. Álgebra, Geometría y Topología
Univ. Málaga

[2]Dpt.Geometría y Topología
Univ. Granada

Scientific Committee

Manuel Gutiérrez, Univ. Málaga
Alfonso Romero, Univ. Granada
Miguel Sánchez, Univ. Granada

PREFACE

This volume contains the proceedings of the meeting “Geometría de Lorentz, Benalmádena 2001” on Lorentzian Geometry and its Application to Mathematical aspects of General Relativity. It was held on November 14th, 15th and 16th, 2001 at Alay Hotel in the very friendly town of Benalmádena, which is located on the Mediterranean coast, in the center of the Costa del Sol, in Málaga, Spain.

The meeting was born as an attempt to assemble mainly Spanish researchers on the area of Lorentzian Geometry and related mathematical ones. Among Spanish geometers, this area has shown to be of special interest. So, the meeting arose as a natural consequence of the state of the research on this topic. We were also very glad with the contributions of several foreign researchers which, in fact, made the meeting of international character. At the same time, the meeting had a vocation to be the first one of a forthcoming series on the same topic. When the edition of these proceedings were finishing, we were glad to know that the second meeting on this topic “Geometría de Lorentz, Murcia 2003” will be held on November 12th, 13th and 14th, 2003, at University of Murcia.

The organizers would like to thank all participants, specially the invited speakers, for their contributions to this meeting. We also would like to thank the referees which, of course, have contributed to increase the quality of this volume.

We would like to thank the Department of Geometry and Topology of University of Granada as well as the Department of Algebra, Geometry and Topology of University of Málaga for all the facilities and help given to the organizers of the meeting. We also would like to thank the support of University of Granada, University of Málaga, Regional government of Junta de Andalucía, Spanish Ministry of Science and Technology, Town of Benalmádena, and Patronato of the Costa del Sol.

We also would like to thank the staff of Alay Hotel for their kindness to the participants, the secretary of the meeting Ms Luisa Gil Aguilar for her patience and friendship with all of us, Alfonso E. Romero López who designed and made the official poster of the meeting, and Benjamín Olea Andrades for his valuable help in the edition of these proceedings. We specially would like to thank Professor Ceferino Ruiz for his continuous and valuable help to the organizers of the meeting.

Finally, we really would like to express our sincere thanks to the Royal Spanish Mathematical Society (RSME) for supporting the young

researchers attending the meeting with several grants, and for publishing these proceedings in its series *Publicaciones no periódicas de la Real Sociedad Matemática Española*.

We are sure to speak on behalf of all participants to dedicate these proceedings to the memory of Professor Luis Santaló.

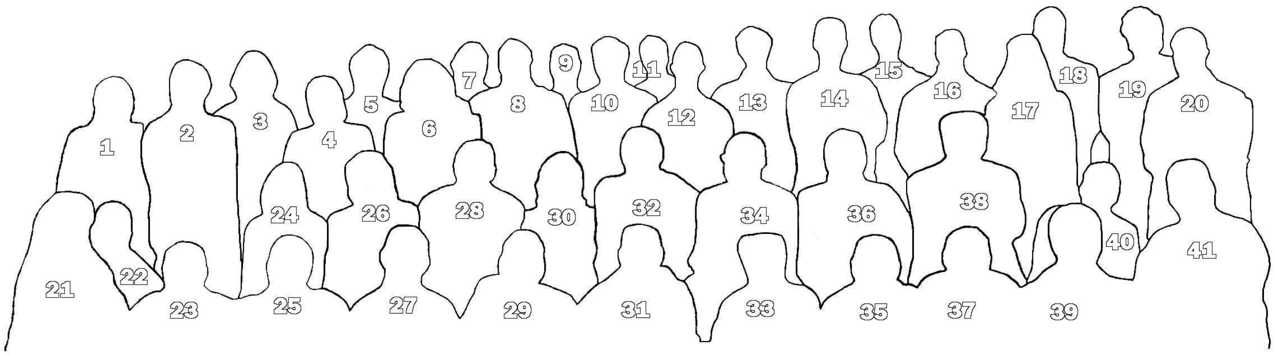
María A. Cañadas-Pinedo
Manuel Gutiérrez
Alfonso Romero

Organizers of the meeting
and editors of the proceedings

List of participants

Aguirre Dabán, Eduardo (14). U. Complutense de Madrid.
Álamo Antúnez, Nieves (17). U. de Málaga.
Aldaya, Victor (25). CSIC-IAA. Granada.
Aledo Sánchez, Juan Ángel (29). U. de Castilla-La Mancha.
Alías, Luis José (31). U. de Murcia.
Amores Lázaro, Ángel Miguel (35). U. Complutense de Madrid.
Barros, Manuel (32). U. de Granada.
Bautista Santa-Cruz, Alfredo (8). U. Complutense de Madrid.
Bergqvist, Göran (5). Mälardalen University (Suecia).
Bernal Navarro, Antonio (10). U. de Granada.
Birman Soto, Graciela Silvia (6). U.N.C.P. Buenos Aires (Argentina).
Candela, Ana María (26). U. de Bari (Italia).
Cañadas Pinedo, María Angustias (21). U. de Málaga.
Criado Cambón, José Carlos (40). U. de Málaga.
Díaz Calle, Ángel (23). U. de Málaga.
Díaz Ramos José Carlos (27). U. de Santiago de Compostela.
Ferrández Izquierdo, Ángel (28). U. de Murcia.
Flores Dorado, José Luis (11). U. de Granada.
Gálvez, José Antonio (9). U. de Granada.
García-Parrado Gómez-Lobo, Alfonso (13). U. del Pais Vasco.
García-Río, Eduardo (2). U. de Santiago de Compostela.
Gil Medrano, Olga (30). U. de Valencia.
Giménez Pastor, Ángel (7). U. de Murcia.
Gutiérrez López, Manuel (41). U. de Málaga.
Hurtado Cortegana, Ana María (24). U. de Valencia.
Izquierdo Barrios, Milagros (4). UNED.
Javaloyes Victoria, Miguel Angel (15). U. de Murcia.
Lafuente López, Javier (12). U. Complutense de Madrid.
Lucas Saorin, Pascual (20). U. de Murcia.
Martín Senovilla, José María (19). U. del Pais Vasco.
Masiello, Antonio (38). U. de Bari (Italia).
Mira Carrillo, Pablo ( ). U. Politécnica de Cartagena.
Olea Andrades, Benjamín ( ). U. de Málaga.
Palomo Ruiz, Francisco José (22). U. de Málaga.
Pastor González, José (34). U. de Murcia.
Ramírez Uclés, Rafael (16). U. de Granada.
Romero Sarabia, Alfonso (18). U. de Granada.

Ruiz Garrido, Ceferino (36). U. de Granada.
San José Martínez, Fernando (39). U. Politécnica de Madrid.
Sánchez Caja, Miguel (37). U. de Granada.
Sánchez Rodríguez, Ignacio ( ). U. de Granada.
Santander Navarro, Mariano (33). U. de Valladolid.
Vázquez Abal, María Elena (3). U. de Santiago de Compostela.
Gil Aguilar, Luisa (1). Secretaria del Congreso.



1. Gil Aguilar, Luisa.
2. García-Río, Eduardo.
3. Vázquez Abal, María Elena.
4. Izquierdo Barrios, Milagros.
5. Bergqvist, Göran.
6. Birman Soto, Graciela Silvia.
7. Giménez Pastor, Ángel.
8. Bautista Santa-Cruz, Alfredo.
9. Gálvez, José Antonio.
10. Bernal Navarro, Antonio.
11. Flores Dorado, José Luis.
12. Lafuente López, Javier.
13. García-Parrado Gómez-Lobo, Alfonso.
14. Aguirre Dabán, Eduardo.
15. Javaloyes Victoria, Miguel Angel.
16. Ramírez Uclés, Rafael.
17. Álamo Antúnez, Nieves.
18. Romero Sarabia, Alfonso.
19. Martín Senovilla, José María.
20. Lucas Saorin, Pascual.
21. Cañadas Pinedo, María Angustias.
22. Palomo Ruiz, Francisco José.
23. Díaz Calle, Ángel.
24. Hurtado Cortegana, Ana María.
25. Aldaya, Victor.
26. Candela, Ana María.
27. Díaz Ramos José Carlos.
28. Ferrández Izquierdo, Ángel.
29. Aledo Sánchez, Juan Ángel.
30. Gil Medrano, Olga.
31. Alías, Luis José.
32. Barros, Manuel.
33. Santander Navarro, Mariano.
34. Pastor González, José.
35. Amores Lázaro, Ángel Miguel.
36. Ruiz Garrido, Ceferino.
37. Sánchez Caja, Miguel.
38. Masiello, Antonio.
39. San José Martínez, Fernando.
40. Criado Cambón, José Carlos.
41. Gutiérrez López, Manuel.

List of contributions

V. Aldaya*, J.L. Jaramillo and J. Guerrero
The Einstein-Lorentz geometry revisited 1
Juan A. Aledo* and José A. Gálvez
Complete spacelike surfaces with a constant principal curvature in the 3-dimensional de Sitter Space 13
Luis J. Alías* and Pablo Mira*
On the Calabi-Bernstein theorem for maximal hypersurfaces in the Lorentz-Minkowski space 23
Manuel Barros
Dynamics and Geometry of order one Rigidity Relativistic Particles . 57
Göran Bergqvist
Causal tensors in Lorentzian geometry 73
Graciela Silvia Birman
Integral formulas in semi-Riemannian manifolds 85
Anna Maria Candela
Normal geodesics in Space–Times 95
M. Fernández-López, E. García-Río* and D. N. Kupeli
Characterizing certain product semi-Riemannian structures by differential equations 109
Angel Ferrández, Angel Giménez* and Pascual Lucas
Geometry of lightlike submanifolds in Lorentzian space forms 125
J. L. Flores* and M. Sánchez
A topological technique for the geodesic connectedness in some Lorentzian manifolds 141
Alfonso García-Parrado* and José M. M. Senovilla
Causal relations and their applications 155
Manuel Gutiérrez, Francisco J. Palomo* and Alfonso Romero
A new approach to the study of conjugate points along null geodesics on certain compact Lorentzian manifolds 171
Antonio Masiello
Geodesics on Lorentzian manifolds: a variational approach 183
José A. Pastor
A uniqueness result for Willmore surfaces in the Minkowski space .. 205
Miguel Sánchez
A survey on geodesic connectedness 215
Ignacio Sánchez Rodríguez
On the intersection of geometrical structures 239

Mariano Santander
The Hyperbolic-AntiDeSitter-DeSitter triality *247*
José M. M. Senovilla
Superenergy tensors and their applications *261*

Proceedings of the meeting
Lorentzian Geometry-Benalmádena 2001
Benalmádena, Málaga, Spain
Pub. de la RSME, Vol. 5 (2003), 1-11

The Einstein-Lorentz geometry revisited

V. Aldaya[1], J.L. Jaramillo[2] and J. Guerrero[3]

[1] *Instituto de Astrofísica de Andalucía, Apartado Postal 3004, Granada 18080, Spain*

[2] *Instituto de Física Teórica y Computacional Carlos I, Facultad de Ciencias, Universidad de Granada, Campus de Fuentenueva, Granada 18002, Spain*

3 *Departamento de Matemática Aplicada, Facultad de Informática, Campus de Espinardo, 30100 Murcia, Spain*

We should start by explaining the actual meaning of the title of this talk. Lorentz geometry is precisely the title of the Conference and everybody associates with it the structure of geodesics and Killing vectors in a manifold endowed with a metric that locally corresponds to that of Minkowski space. Adding the adjective Einstein suggests the study of the kinematics of physical particles that move along geodesics affecting the value of the metric due to their presence and, maybe more, the dynamics of the connection itself considered as gravitational fields derived from the metric as a potential. Well then, we shall keep this meaning and add to the adjective Lorentz the sense of an electromagnetic force, that is, the Lorentz force.

Our proposal here is to revisit the concept of Lorentz geometry, in the usual sense, by considering a background manifold which generalizes the standard Lorentz manifold according to the Quantum Kinematics. In fact, Quantum Mechanics suggests a $U(1)$-extended configuration space as well as an $U(1)$-extended kinematical symmetry. Geodesics in such a generalized geometry are governed by some Christoffel simbols accounting for a gravitational force, an electromagnetic potential and, that which turns out to be really new, some sort of electromagnetic potential originated by the mass of the particles rather than the particle charges.

Firstly, let us remind you the origin of this fundamental $U(1)$ or Phase Invariance in Quantum Mechanics. To this end we shall consider the behaviour of the Schrödinger equation corresponding to the free quantum particle

$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\nabla^2\Psi \quad , \tag{1.1}$$

under the Galilei transformations:

$$\begin{aligned} x' &= x + a + vt \\ t' &= t + b \quad . \end{aligned} \tag{1.2}$$

The equation (1.1) acquires an extra term,

$$i\hbar\frac{\partial}{\partial t'}\Psi + i\hbar v\frac{\partial\Psi}{\partial x'} = -\frac{\hbar^2}{2m}\nabla'^2\Psi, \tag{1.3}$$

which can be compensated by also transforming the wave function. Allowing for a non-trivial phase factor in front of the transformed wave function of the form

$$\Psi' = e^{\frac{im}{\hbar}(vx+\frac{1}{2}v^2t)}\Psi\,, \tag{1.4}$$

the Schrödinger equation becomes strictly invariant, i.e.

$$i\hbar\frac{\partial}{\partial t'}\Psi' = -\frac{\hbar^2}{2m}\nabla'^2\Psi'. \tag{1.5}$$

The need for a transformation like (1.4) accompanying the space-time transformation (1.2) to accomplish full invariance strongly suggests the adoption of a central extension of the Galilei group as the basic (quantum-mechanical) space-time symmetry for the free particle [1]. The constant $\hbar$ is required to keep the exponent in (1.4) dimensionless.

The successive composition of two transformation in the extended Galilei group G immediately leads to the group law:

$$\begin{aligned} b'' &= b' + b \\ a'' &= a' + a + v'b \\ v'' &= v' + v \\ e^{i\phi''} &= e^{i\phi'}e^{i\phi}e^{\frac{im}{\hbar}[a'v+b(v'v+\frac{1}{2}v'^2)]} \end{aligned} \tag{1.6}$$

The main feature of the central extension is that of making non-trivial the commutator between the generators associated with translations $X_a \equiv P$ and boosts $X_v \equiv K$: $[P,K] = -mX_\phi$, just *mimicking* the Poisson bracket between p and x provided that we impose on the wave function the $U(1)$-function condition $X_\phi\Psi \equiv \frac{\partial\Psi}{\partial\phi} = i\Psi$ (i.e. homogeneous of degree one on the new variable $e^{i\phi}$).

Even though the previous analysis is conceptually purely quantum mechanical, we point out that this phenomenon of extension of the space-time symmetry can also be recasted within a semi-classical formalism (keeping the quantum insights), by requiring the simultaneous extension of the classical space-time by a new variable ϕ, transforming in a non-trivial way under the $U(1)$-extended symmetry group. The extension of space-time is required to represent faithfully the extended symmetry by means of first-order differential operators. This is the presentation we shall follow here.

The usual way of introducing an interaction in Physics is through the *gauge principle*, which requires the invariance of the Lagrangian of free matter under a *gauge* group obtained from an original symmetry group, the *rigid* group, by making the group parameter to depend on space-time variables.

In standard Lagrangian formalism, promoting a given underlying rigid symmetry to "local" requires the introduction of a connection which is eventually interpreted as a potential providing the corresponding gauge interaction. This is essentially the formulation of the so-called Minimal Coupling Principle, which culminates in Utiyama's theorem [2]. Internal gauge invariance had originally led successfully to electromagnetic interaction associated with $U(1)$, then to Yang-Mills associated with isospin $SU(2)$ (valid only at the very strong limit), electroweak with $SU(2) \otimes U(1)$, and finally to strong interaction associated with colour $SU(3)$. And, more recently, there have been attempts to unify all of these into gauge groups such as $SU(5)$. On the other hand, the "local" invariance under external (space-time) symmetries, such as a subgroup of the Poincaré group, has been used to provide a gauge framework for gravity [3], although fully disconnected from the other (internal) interactions. In fact, a unification of gravity and the other interactions would have required the non-trivial mixing of the space-time group and some internal symmetry, a task explicitely forbidden by the so-called *no-go theorems* by O'Raifeartaigh and McGlinn [4, 5] (see also [6]) long ago, which stated that there is no finite-dimensional Lie group containing the Poincaré group and any internal $SU(n)$ group except for the direct product. It is worth mentioning that supersymmetry was originally developed in the 70's by Salam and Strathdee [7] in an unsuccessful attemp to invalidate the no-go theorems.

However, the current skill in dealing with infinite-dimensional Lie groups tempts us into revisiting the question of the mixing of symmetries and, accordingly, the unification of interactions in terms of ordinary (though infinite-dimensional) Lie groups. In fact, there is an extremely simple, yet non-trivial, way of constructing a Lie group containing the Poincaré group and some unitary symmetry which accomplishes the above-mentioned task. This consists in looking at the $U(1)$ phase invariance of Quantum Theory as a 1-dimensional Cartan subgroup of a larger internal symmetry. Then, turning the space-time translation subgroup of the Poincaré group into a "local" group automatically promotes the original rigid internal symmetry to the gauge level in a non-direct product way. This of course entails a non-trivial mixing of gravity and the involved internal interaction associated with the given unitary symmetry.

In this talk, we shall approach the problem in the simplest and most economical way, in a Particle Mechanics (versus Field Theoretical) framework, leaving the more mathematically involved field formulation for the near future. To be precise, we face the situation that arises when promoting to the "local" level the space-time translations of the centrally extended space-time symmetry (either Galilei or Poincaré group), rather than the space-time symmetry itself.

The way of associating a physical dynamics with a specific symmetry can be accomplished by means of the rather standard co-adjoint orbits method of Kirillov [8], where the Lagrangian is seen as the local potential of the corresponding symplectic form, or through a generalized group approach to quantization which is directly related to the co-homological structure of the symmetry and leads directly to the quantum theory (see [9] and references there in).

To illustrate technically the present revisited Minimal Coupling Theory, let us consider the simpler case of the non-relativistic pure Lorentz force, keeping rigid the space-time translations. For this aim, we consider the Lie algebra $\mathcal{G}$ of the centrally extended Galilei group G (only non-zero commutators):

$$\begin{array}{rclrcl} [X_{v^i},\, X_t] &=& X_{x^i} & [X_{v^i},\, X_{x^j}] &=& m\delta_{ij}X_\phi \\ [X_{J^i},\, X_{J^j}] &=& \varepsilon_{ij.}^{\ \ k}X_{J^k} & [X_{J^i},\, X_{v^j}] &=& \varepsilon_{ij.}^{\ \ k}X_{v^k} \\ [X_{J^i},\, X_{x^j}] &=& \varepsilon_{ij.}^{\ \ k}X_{x^k} & & & \end{array} \tag{1.7}$$

which leaves strictly invariant the extended Poincaré-Cartan form $\Theta = p_i dx^i - \frac{p^2}{2m}dt + d\phi$, $L_{X_\alpha}\Theta = 0, \forall X_\alpha \in \mathcal{G}$. This 1-form is defined on the extended phase space parametrized by (x^i, p_j, ϕ), where $e^{i\phi} \in U(1)$ is the phase transforming non-trivially under the Galilei group. It generalizes the Lagrangian and constitutes a potential for the symplectic form ω on the solution manifold (on trajectories $s(t)$,

$$(p_i dx^i - \frac{p^2}{2m}dt)|_{s(t)} = (p_i \dot{x}^i - \frac{p^2}{2m})|_{s(t)}dt).$$

Local $U(1)$ transformations generated by $f \otimes X_\phi$, f being a real function $f(\vec{x}, t)$, are incorporated into the scheme by adding to (1.7) the extra commutators [1]:

$$[X_a,\, f \otimes X_\phi] = (L_{X_\alpha} f) \otimes X_\phi \tag{1.8}$$

Keeping the invariance of Θ under $f \otimes X_\phi$ requires modifying Θ by adding a connection piece $A = A_i dx^i + A_0 dt$ whose components transform under $U(1)(\vec{x}, t)$ as the space-time gradient of the function f.

The algebra (1.7)+(1.8) is infinite-dimensional but, if the functions f are real analytic, the dynamical content of it is addressed by the (co-homological) structure of the finite-dimensional subalgebra generated by $\mathcal{G}$ and the generators $f \otimes X_\phi$ with only linear functions. Thus, a very economical trick (eventually supported on unitarity grounds) for dealing with this sort of infinite-dimensional algebra consists in proceeding with the above mentioned 15-dimensional electromagnetic subgroup and then imposing the generic constraint $A^\mu = A^\mu(\vec{x}, t)$ on the symplectic structure. Let us call this group G_E, and the generators associated with linear functions in $f \otimes X_\phi$, X_{A^μ}.

The commutation relations of $\mathcal{G}_E$ are (we omit rotations, which operate in the standard way):

$$\begin{array}{rclrclrcl} [X_t,\, X_{x^i}] &=& 0 & [X_t,\, X_{v^i}] &=& -X_{x^i} & [X_{x^i},\, X_{v^j}] &=& m\delta_{ij}X_\phi \\ [X_t,\, X_{A^i}] &=& 0 & [X_t,\, X_{A^0}] &=& -qX_\phi & [X_{x^i},\, X_{A^j}] &=& q\delta_{ij}X_\phi \\ [X_{x^i},\, X_{A^0}] &=& 0 & [X_{v^i},\, X_{A^i}] &=& \delta_{ij}X_{A^0} & [X_{v^i},\, X_{A^0}] &=& 0 \end{array}$$

[1] In general, $[f \otimes X_a, g \otimes X_b] = (fL_{X_a}g) \otimes X_b - (gL_{X_b}f) \otimes X_a + (fg) \otimes [X_a, X_b]$.

where we have performed a new central extension parametrized by what proves to be the electric charge q.

The co-adjoint orbits of the group G_E with non-zero electric charge have dimension 4+4 as a consequence of the Lie algebra cocycle piece $\Sigma(X_t, X_{A^0}) = -q$, which lends dynamical (symplectic) content to the time variable. This is a property inherited from the (centrally extended) conformal group from which $\mathcal{G}_E$ is an Inönü-Wigner contraction. In the case of the conformal group [10] the symplectic character of time is broken by means of a dynamical constraint (or by choosing a Poincaré vacuum) and the dimension 3+3 of the phase space is restored. Here the constraint $A^\mu = A^\mu(\vec{x}, t)$ also accomplishes this task at the same time as it introduces the notion of electromagnetic potential.

On a general orbit with non-zero q the extended Poincaré-Cartan form acquires the expression:

$$\Theta = m\vec{v}\cdot d\vec{x} - \frac{1}{2}m\vec{v}^2 dt + q\vec{A}\cdot d\vec{x} - qA^0 dt + d\phi \tag{1.9}$$

After imposing the above-mentioned constraint on A^μ, we compute the kernel of the presymplectic form $d\Theta$, i.e. the vector field (up to a multiplicative function) X such that $i_X d\Theta = 0$:

$$\begin{aligned} X = \frac{\partial}{\partial t} + \vec{v}\cdot\frac{\partial}{\partial \vec{x}} + \frac{q}{m}\left[\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}\right)v^j - \frac{\partial A^0}{\partial x^i} - \frac{\partial A_i}{\partial t}\right]\frac{\partial}{\partial v_i} \\ -[\frac{1}{2}m\vec{v}^2 + q(\vec{v}\cdot\vec{A} - A^0)]\frac{\partial}{\partial \phi}, \end{aligned} \tag{1.10}$$

where Latin indices are raised and lowered by the metric δ_{ij}. It defines the equations of motion of a charged particle moving in an electromagnetic field:

$$\begin{aligned} \frac{d\vec{x}}{dt} &= \vec{v} \\ m\frac{d\vec{v}}{dt} &= q\left[\vec{v}\wedge(\vec{\nabla}\wedge\vec{A}) - \vec{\nabla}A^0 - \frac{\partial\vec{A}}{\partial t}\right], \end{aligned} \tag{1.11}$$

which are nothing more than the standard Lorenz force equations. The same can be repeated with the centrally extended Poincaré group P (see [10] and references therein) by promoting to "local" the $U(1)$ transformations and considering the finite-dimensional subgroup P_E analogous to G_E.

Let us consider now the gravitational interaction. To this end, we start directly with the centrally extended Poincaré group P and see how the fact that the translation generators produce the central term under commutation with some other generators (boosts) plays a singular role in the relationship between local space-time translations and local $U(1)$ transformations. Symbolically denoting the generators of translations by P, P_0, those of boosts by K and the central one by X_ϕ, we find:

$$[K,\, f\otimes P] \simeq (L_K f)\otimes P + f\otimes (P_0 + X_\phi), \tag{1.12}$$

which means that turning the translations into local symmetry entails also the local nature of the $U(1)$ phase. We expect, in this way, a non-trivial mixing of gravity and electromagnetism into an infinite-dimensional electro-gravitational group.

We shall follow identical steps as those given in the former example. The generators of local space-time translations associated with linear functions will be called $X_{h^{\mu\nu}}$, and the corresponding parameters $h^{\mu\nu}$ will also be constrained, in the form $h^{\mu\nu} = h^{\mu\nu}(\vec{x}, x^0)$, on the symplectic orbits. However, the co-homological structure of this finite-dimensional electro-gravitational subgroup, P_{EG} is richer than that of P_E and the exponentiation of the Lie algebra $\mathcal{P}_{EG}$ must be made, for the time being at least, order by order. Then, the explicit calculations will be kept up to order 3 in the group law. This will be enough to recognize the standard part of the interaction, i.e. the ordinary Lorentz force and the geodesic equations, although the latter in a quasi-linear approximation in terms of the metric $g^{\mu\nu} \equiv \eta^{\mu\nu} + h^{\mu\nu}$. But in addition, and associated with a new Lie algebra co-homology constant, κ, different from m and q and mixing both interactions, a new term appears in the Lorentz force made of the gravitational potential $h^{\mu\nu}$.

Let us write the algebra $\mathcal{P}_{EG}$ in an almost covariant way (the central extensions and induced deformations are necessarily non-covariant). To this end, we parametrize the Lorentz transformations with $L^{\mu\nu}$ as usual. The proposed explicit algebra is (we write for short x_μ, $L_{\mu\nu}, \ldots$ instead of X_{x^μ}, $X_{L^{\mu\nu}}, \ldots$, and let Greek indices run from 0 to 3):

$$\begin{aligned}
[x_\mu, L_{\nu\rho}] &= \eta_{\nu\mu}x_\rho - \eta_{\rho\mu}x_\nu + (m+\kappa q)c(\eta_{\rho\mu}\delta^0_\nu - \eta_{\nu\mu}\delta^0_\rho)\phi \\
[x_\mu, h_{\nu\rho}] &= \eta_{\nu\mu}x_\rho + \eta_{\rho\mu}x_\nu + mc(\eta_{\rho\mu}\delta^0_\nu + \eta_{\nu\mu}\delta^0_\rho)\phi \\
[x_\mu, A_\nu] &= q\eta_{\nu\mu}\phi
\end{aligned}$$

$$
\begin{array}{rcl}
[L_{\mu\nu}, L_{\alpha\beta}] &=& \eta_{\alpha\nu}L_{\mu\beta} - \eta_{\beta\nu}L_{\mu\alpha} - \eta_{\alpha\mu}L_{\nu\beta} + \eta_{\mu\beta}L_{\nu\alpha} \\
{[L_{\mu\nu}, h_{\alpha\beta}]} &=& \eta_{\alpha\nu}h_{\mu\beta} + \eta_{\beta\nu}h_{\mu\alpha} - \eta_{\alpha\mu}h_{\nu\beta} - \eta_{\mu\beta}h_{\nu\alpha} - \\
&& \kappa c(\eta_{\alpha\nu}\delta^{\rho}_{\beta}\delta^{0}_{\mu} - \eta_{\mu\alpha}\delta^{\rho}_{\beta}\delta^{0}_{\nu} + \eta_{\nu\beta}\delta^{\rho}_{\alpha}\delta^{0}_{\mu} \\
&& -\eta_{\mu\beta}\delta^{\rho}_{\alpha}\delta^{0}_{\nu})A_{\rho} \\
{[L_{\mu\nu}, A_{\rho}]} &=& \eta_{\rho\nu}A_{\mu} - \eta_{\rho\mu}A_{\nu} \\
{[h_{\mu\nu}, h_{\alpha\beta}]} &=& \eta_{\alpha\nu}L_{\mu\beta} + \eta_{\beta\nu}L_{\mu\alpha} + \eta_{\alpha\mu}L_{\nu\beta} + \eta_{\mu\beta}L_{\nu\alpha} + \\
&& \kappa c\left[\eta_{\alpha\nu}\delta^{0\rho}_{\beta\mu} + \eta_{\beta\nu}\delta^{0\rho}_{\alpha\mu} + \eta_{\alpha\mu}\delta^{0\rho}_{\beta\nu} + \eta_{\beta\mu}\delta^{0\rho}_{\alpha\nu}\right]A_{\rho} \\
{[h_{\mu\nu}, A_{\rho}]} &=& \eta_{\rho\nu}A_{\mu} + \eta_{\rho\mu}A_{\nu}
\end{array}
\tag{1.13}
$$

where $\delta^{0\rho}_{\beta\mu} \equiv \delta^{0}_{\beta}\delta^{\rho}_{\mu} - \delta^{0}_{\mu}\delta^{\rho}_{\beta}$ is the Kronecker tensor.

It should be remarked that a consequence of having extended the Poincaré group prior to turning "local" space-time translations is the apearance of a term proportional (κ is a new co-homology constant) to A_{ρ} on the r.h.s. of the commutator $[L_{\mu\nu}, h_{\alpha\beta}]$, which will be responsible for a piece in the Lorentz force of gravitational origin. But even more, the term in A_{ρ} on the r.h.s of the commutator $[h_{\mu\nu}, h_{\alpha\beta}]$ could provide a "mixing vertex" at the Field Theory level.

We shall not dwell on explicit calculations in this letter and simply give the resulting equations of motion. Even more, we restrict ourselves to the "non-relativistic" limit stated by the Inönü-Wigner contraction with respect to the subgroup generated by (x_0, L_{ij}, A_k) (the standard $c \to \infty$ limit on the Poincaré group is an I-W contraction with respect to the subgroup (x_0, L_{ij})). The contracted algebra reads:

$$
\begin{array}{rclrcl}
[x_0, L_{0i}] &=& x_i & [x_0, h_{00}] &=& -2m\phi \\
{[x_0, h_{0i}]} &=& x_i & [x_0, A_0] &=& q\phi \\
{[x_i, L_{0j}]} &=& -(m+\kappa q)\delta_{ij}\phi & [x_i, L_{jk}] &=& -\delta_{ij}x_k + \delta_{ik}x_j \\
{[x_i, h_{0j}]} &=& m\delta_{ij}\phi & [x_i, A_j] &=& -q\delta_{ij}\phi \\
{[L_{0i}, L_{jk}]} &=& -\delta_{ij}L_{0k} + \delta_{ki}L_{0j} & [L_{0i}, h_{0j}] &=& -\delta_{ij}h_{00} + \kappa\delta_{ij}A_0 \\
{[L_{0i}, A_j]} &=& -\delta_{ij}A_0 & [L_{ij}, L_{kl}] &=& -\delta_{kj}L_{il} + \delta_{lj}L_{ik} \\
&&&&& +\delta_{ik}L_{jl} - \delta_{il}L_{jk} \\
{[L_{ij}, h_{0k}]} &=& -\delta_{kj}h_{0i} + \delta_{ik}h_{0j} & [L_{ij}, h_{kl}] &=& -\delta_{kj}h_{il} - \delta_{lj}L_{ik} \\
&&&&& +\delta_{ik}L_{jl} + \delta_{il}L_{jk} \\
{[L_{ij}, A_k]} &=& -\delta_{kj}A_i + \delta_{ki}A_j & [h_{0i}, A_j] &=& -\delta_{ij}A_0
\end{array}
\tag{1.14}
$$

Writing $\vec{h}$ for (h^{0i}), we finally derive from this algebra the following Lagrangian and equations of motion, which at this contraction limit are indeed exact:

$$L = \frac{1}{2}(m+\kappa q)\, \dot{\vec{x}}^2 - q(A^0 - \frac{\kappa}{8}\vec{h}^2) + q(\vec{A} - \frac{\kappa}{2}\vec{h})\cdot\dot{\vec{x}} \\ + m(h^{00} + \frac{1}{4}\vec{h}^2) - m\vec{h}\cdot\dot{\vec{x}} \qquad (1.15)$$

$$\begin{aligned} \frac{dx^i}{dt} &= v^i \\ (m+\kappa q)\frac{d\vec{v}}{dt} &= q\left[\vec{v}\wedge(\vec{\nabla}\wedge\vec{A}) - \vec{\nabla}A^0 - \frac{\partial\vec{A}}{\partial t}\right] \qquad (1.16) \\ &- m\left[\vec{v}\wedge(\vec{\nabla}\wedge\vec{h}) - \vec{\nabla}h_{00} - \frac{\partial\vec{h}}{\partial t}\right] + \frac{m}{4}\nabla(\vec{h}\cdot\vec{h}) \\ &- \frac{\kappa q}{2}\left[\vec{v}\wedge(\vec{\nabla}\wedge\vec{h}) - \frac{1}{4}\nabla(\vec{h}\cdot\vec{h}) - \frac{\partial\vec{h}}{\partial t}\right] \end{aligned}$$

The first three lines in (1.17) correspond to the standard motion of a particle in the presence of an electromagnetic and gravitational field (note that $h^{0i} = g^{0i} - \eta^{0i}$), except for the value of the inertial mass, which is corrected by κq, and within a quasi-linear approximation in the gravitational field. In fact, the third line contains one more order than the approximation in which the gravitational field looks like an electromagnetic one (standard gravito-electromagnetism [11]). The fourth, however, is quite new and represents another Lorentz-like force (proportional to q) generated by the gravitational potential and which must not be confused with the previous one. It is worth mentioning that the constant m in front of the term $\vec{\nabla}h_{00}$ in (1.17), naturally interpreted as a gravitational coupling, could acquire a different constant value, let us say g, allowed by the Lie algebra co-homology. Nevertheless, it must be made equal to m to recover the standard physics when switching the constant κ off. In this way, the equivalence principle between inertial and gravitational mass, in this co-homological setting, follows from the natural requirement of absence of a pathological mixing between electromagnetism and gravity when $\kappa = 0$.

It should also be noticed that in the standard formulation, and according to a reasoning not completely clear, the non-relativistic limit of the pure gravitational theory leads to just the term $\vec{\nabla}h^{00}$. Here, the non-relativistic limit, in general, appears as a clean Lie algebra contraction and permits forces derived from $\vec{h}$. As far as the magnitude of the new Lie algebra co-homology constant κ, it is limited by experimental clearance for the difference between particle and anti-particle mass, which for the electron is about $10^{-8}m_e$. Even though this is a small value, extremely dense rotating bodies could be able to produce measurable forces. In the other way around, a mixing of electromagnetism and gravity predicts a mass difference between charged particles and anti-particles, which could be experimentally tested.

Since the present theory has been formulated on symmetry grounds, it can be quantized on the basis of the group approach to quantization referred in [9]. Also, a natural yet highly non-elementary extension of the present theory to Quantum Field Theory is in course.

Finally, and as commented above, considering the group $U(1)$ as a Cartan subgroup of a larger internal symmetry group, for instance $SU(2) \otimes U(1)$ would result in additional phenomenology. Then, and in a QFT version, the production of Z_0 particles out of gravity might be permitted.

Acknowledgments

This work has been partially supported by the DGICYT.

References

[1] V. Bargmann, Ann. Math. **48**, 568 (1947).

[2] R. Utiyama, Phys. Rev. **101**, 1597 (1956).

[3] T.W.B. Kibble, J. Math. Phys. **2**, 212 (1961).

[4] L. O'Raifeartaigh, Phys. Rev. **B139**, 1052 (1965).

[5] W.D. McGlinn, Phys. Rev. Lett. **12**, 467 (1964).

[6] F.J. Dyson, *Symmetry Groups in Nuclear and Particle Physics*, Benjamin N.Y. (1966).

[7] A. Salam and J. Strathdee, Fortschr. der Phys. **26**, 57 (1978).

[8] A.A. Kirillov, *Elements of the Theory of Representations*, Springer-Verlag (1976).

[9] V. ALDAYA, J. NAVARRO-SALAS AND A. RAMÍREZ, Commun. Math. Phys., **121**, 541 (1989).

[10] V. ALDAYA, M. CALIXTO AND J.M. CERVERÓ, Commun. Math. Phys. **200**, 325 (1999).

[11] R.M. WALD, *General Relativity*, The University of Chicago Press (1984).

Proceedings of the meeting
Lorentzian Geometry-Benalmádena 2001
Benalmádena, Málaga, Spain
Pub. de la RSME, Vol. 5 (2003), 13-21

Complete Spacelike Surfaces with a Constant Principal Curvature in the 3-dimensional de Sitter Space

Juan A. Aledo[1] and José A. Gálvez[2]

[1] *Departamento de Matemáticas, Universidad de Castilla-La Mancha, 02071 Albacete, Spain*
e-mail: JuanAngel.Aledo@uclm.es

[2] *Departamento de Geometría y Topología, Universidad de Granada, 18071 Granada, Spain*
e-mail: jagalvez@ugr.es

Abstract

In this work we study complete spacelike surfaces with a constant principal curvature R in the 3-dimensional de Sitter space $\mathbf{S}_1^3$, proving that if $R^2 < 1$, such a surface is either totally umbilical or umbilically free. Moreover, in the second case we prove that the surface can be described in terms of a complete regular curve in $\mathbf{S}_1^3$. We also give examples which show that the result is not true when $R^2 \geq 1$.

1 Introduction and Statement of the Main Result

Spacelike surfaces in the de Sitter space $\mathbf{S}_1^3$ have been of increasing interest in the recent years from different points of view. That interest is motivated, in part, by the fact that they exhibit nice Bernstein-type properties. For instance, Ramanathan [7] proved that every compact spacelike surface in $\mathbf{S}_1^3$ with constant mean curvature is totally umbilical.

This result was generalized to hypersurfaces of any dimension by Montiel [6]. On the other hand, Li [5] obtained the same conclusion when the compact spacelike surface has constant Gaussian curvature. More recently, the first author jointly Romero [3] have proved that the totally umbilical round spheres are the only compact spacelike surfaces in the de Sitter space such that the Gaussian curvature of the second fundamental form is constant.

As a natural generalization of Ramanathan and Li results, the authors [1] have recently proved that the only compact linear Weingarten spacelike surfaces in $\mathbf{S}_1^3$, (that is, surfaces satisfying that a linear combination of their mean and Gaussian curvatures is constant) are the totally umbilical round spheres. In the quoted paper, we also study compact spacelike surfaces with a constant principal curvature, proving that such surfaces are totally umbilical round spheres.

In this work we extend the last result to complete spacelike surfaces in the following terms:

Theorem 1.1 *Let $\psi : M \longrightarrow \mathbf{S}_1^3$ be a complete spacelike surface with a constant principal curvature R such that $R^2 < 1$. Then $\psi(M)$ is either totally umbilical or umbilically free. In the second case $R > 0$ and the surface is not compact and can be described as*

$$\psi(x, y) = \frac{1}{\sqrt{1 - R^2}} \left(R\alpha(y) + \cos(x)v_1(y) + \sin(x)v_2(y)\right), \tag{1.1}$$

where α is a C^∞ complete regular curve in $\mathbf{S}_1^3$ and $\{v_1(y), v_2(y)\}$ is an orthonormal frame of the normal plane along α.

Conversely, given a regular curve α in $\mathbf{S}_1^3$, (1.1) defines an umbilically free spacelike immersion in $\mathbf{S}_1^3$ with a constant principal curvature R such that $0 < R^2 < 1$.

To finish, in Section 4, we construct some examples which show that the result is false when $R^2 \geq 1$.

2 Preliminaries

Let $\mathbf{L}^4$ be the 4-dimensional *Lorentz-Minkowski space*, that is, the real vector space $\mathbf{R}^4$ endowed with the Lorentzian metric tensor $\langle , \rangle$ given by

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2,$$

where (x_1, x_2, x_3, x_4) are the canonical coordinates of $\mathbf{R}^4$. The 3-dimensional unitary *de Sitter space* is given as the following hyperquadric of $\mathbf{L}^4$,

$$\mathbf{S}_1^3 = \{x \in \mathbf{L}^4 : \langle x, x\rangle = 1\}.$$

As is well known, $\mathbf{S}_1^3$ inherits from $\mathbf{L}^4$ a time-orientable Lorentzian metric which makes it the standard model of a Lorentzian space of constant sectional curvature one. A smooth immersion $\psi : M^2 \longrightarrow \mathbf{S}_1^3 \subset \mathbf{L}^4$ of a 2-dimensional connected manifold M is said to be a *spacelike surface* if the induced metric via ψ is a Riemannian metric on M, which, as usual, is also denoted by $\langle, \rangle$. The time-orientation of $\mathbf{S}_1^3$ allows us to choose a timelike unit normal field N globally defined on M, tangent to $\mathbf{S}_1^3$, and hence we may assume that M is oriented by N. Finally, we will denote by λ_1, λ_2 the principal curvatures of M associated to N.

3 Proof of the Theorem

Let $\psi : M^2 \longrightarrow \mathbf{S}_1^3 \subset \mathbf{L}^4$ be a complete spacelike surface in $\mathbf{S}_1^3$ with a constant principal curvature $0 \leq \lambda_1 = R < 1$ (up to a change of orientation). If there exists a non umbilical point $p \in M$, then we can consider local parameters (u, v) in a neighborhood U of p without umbilical points, such that

$$\begin{aligned}\langle d\psi, d\psi\rangle &= E\ du^2 + G\ dv^2\\ \langle d\psi, -dN\rangle &= RE\ du^2 + \lambda_2 G\ dv^2,\end{aligned}$$

where the principal curvature $\lambda_2 \neq R$. Then, the structure equations are given by

$$\begin{aligned}\psi_{uu} &= \frac{E_u}{2E}\ \psi_u - \frac{E_v}{2G}\ \psi_v - REN - E\psi\\ \psi_{uv} &= \frac{E_v}{2E}\ \psi_u + \frac{G_u}{2G}\ \psi_v\\ \psi_{vv} &= -\frac{G_u}{2E}\ \psi_u + \frac{G_v}{2G}\ \psi_v - \lambda_2 GN - G\psi\\ N_u &= -R\ \psi_u\\ N_v &= -\lambda_2\ \psi_v\end{aligned}$$

and the Mainardi-Codazzi equations for the immersion ψ are

$$\begin{aligned}&(R - \lambda_2)E_v = 0\\ &(R - \lambda_2)\frac{G_u}{2G} + (R - \lambda_2)_u = 0.\end{aligned}$$

Since $\lambda_2 \neq R$, the coefficient E does not depend on v, that is, $E = E(u)$. If we consider the new parameters

$$x = \int \sqrt{E(u)}\, du, \qquad y = v,$$

the structure equations become

$$\begin{aligned}
\psi_{xx} &= -RN - \psi \\
\psi_{xy} &= \frac{G_x}{2G}\, \psi_y \\
\psi_{yy} &= -\frac{G_x}{2}\, \psi_x + \frac{G_y}{2G}\, \psi_y - \lambda_2 GN - G\psi \\
N_x &= -R\, \psi_x \\
N_y &= -\lambda_2\, \psi_y,
\end{aligned} \tag{3.1}$$

and the Mainardi-Codazzi equation is

$$(R-\lambda_2)\frac{G_x}{2G} + (R-\lambda_2)_x = 0. \tag{3.2}$$

On the other hand, the Gauss equation is given by

$$\left(\frac{G_x}{2G}\right)_x + \left(\frac{G_x}{2G}\right)^2 = R\lambda_2 - 1 = R(\lambda_2 - R) + R^2 - 1. \tag{3.3}$$

Thus, if we take

$$\varphi = \frac{1}{R-\lambda_2}$$

we obtain from (3.2) and (3.3) that

$$\varphi_x = \frac{G_x}{2G}\, \varphi$$

$$\varphi_{xx} = \left(\left(\frac{G_x}{2G}\right)_x + \left(\frac{G_x}{2G}\right)^2\right)\, \varphi = -R - (1-R^2)\varphi. \tag{3.4}$$

Let γ_q be the maximal line of curvature passing through a point $q = \psi(x_o, y_o) \in U$ for the principal curvature R. Then, from (3.1) it follows that $\gamma_q(t) = \psi(x_o + t, y_o)$ satisfies

$$\begin{aligned}
(\gamma_q)_{tt} &= -R(N \circ \gamma_q) - \gamma_q \\
(N \circ \gamma_q)_t &= -R(\gamma_q)_t\ ,
\end{aligned}$$

so that γ_q is a geodesic curve, which is a solution of the differential equation

$$(\gamma_q)_{tt} + (1 - R^2)\gamma_q = Rw_o$$

for a constant vector $w_o \in \mathbf{L}^4$. Therefore, taking into account that $0 \leq R < 1$, γ_q is given by

$$\gamma_q = \cos\left(\sqrt{1-R^2}\; t\right) w_1 + \sin\left(\sqrt{1-R^2}\; t\right) w_2 + \frac{R}{1-R^2}\, w_o \tag{3.5}$$

for suitable vectors $w_1, w_2 \in \mathbf{L}^4$.

From (3.4), the principal curvature λ_2 can be calculated on γ_q as

$$R - \lambda_2 = \left(a\cos\left(\sqrt{1-R^2}\; t\right) + b\sin\left(\sqrt{1-R^2}\; t\right) - \frac{R}{1-R^2}\right)^{-1} \tag{3.6}$$

for real constants a, b.

Hence, if $\gamma_q(t_1)$ is the first umbilical point on γ_q, we obtain from (3.6) and the continuity of λ_2 that

$$0 = R - \lambda_2(\gamma_q(t_1)) = \lim_{t\to t_1} R - \lambda_2(\gamma_q(t)) \neq 0,$$

which is a contradiction. Therefore, there is no umbilical point on γ_q.

Observe that, since M is complete, it follows that the geodesic γ_q is defined for all $t \in \mathbf{R}$. Moreover $R \neq 0$, because in that case

$$a\cos\left(\sqrt{1-R^2}\; t\right) + b\sin\left(\sqrt{1-R^2}\; t\right) = 0$$

for some $t \in \mathbf{R}$, which contradicts the continuity of λ_2.

Let $\widetilde{U}$ be the connected component of non umbilical points containing p. Note that $\widetilde{U}$ is an open set, and from the above reasoning, can be parametrized by $(x, y) \in (-\infty, \infty) \times (\beta_1, \beta_2)$ for certain β_1, β_2, where $-\infty \leq \beta_1 < \beta_2 \leq \infty$, so that the immersion can be expressed from (3.5) as

$$\begin{aligned}\psi(x,y) \;=\; & \cos\left(\sqrt{1-R^2}\; x\right) w_1(y) \\ & + \sin\left(\sqrt{1-R^2}\; x\right) w_2(y) + \frac{R}{1-R^2}\, w_o(y).\end{aligned} \tag{3.7}$$

Let us suppose now that there exists an umbilical point $\widetilde{q} \in \partial\psi(\widetilde{U})$. Then there exists a sequence of points $q_n = \psi(x_n, y_n)$ tending to $\widetilde{q}$, being

$(x_n, y_n) \in [0, 2\pi/\sqrt{1-R^2}] \times (\beta_1, \beta_2)$. Therefore the sequence of compact geodesics γ_n of length $2\pi/\sqrt{1-R^2}$ passing through q_n associated to the principal curvature R, converges to a compact geodesic $\gamma_{\widetilde{q}}$ passing through $\widetilde{q}$ which is also a line of curvature for the principal curvature R.

Now, from the above argument, it is sufficient to prove that there exists a non umbilical point on $\gamma_{\widetilde{q}}$. In fact, from (3.6) we are able to choose a point $p_n \in \gamma_n$ such that $\lambda_2(p_n) = 1/R \neq R$. Finally, from an argument of compactness, there exists a subsequence $\{p_k\}$ of $\{p_n\}$ converging to a non umbilical point $\widetilde{p} \in \gamma_{\widetilde{q}}$.

Consequently M is either umbilically free or totally umbilical.

Observe that (3.7) can be rewritten as

$$\psi(x,y) = \frac{\left(R\alpha(y) + \cos\left(\sqrt{1-R^2}\,x\right) v_1(y) + \sin\left(\sqrt{1-R^2}\,x\right) v_2(y)\right)}{\sqrt{1-R^2}}$$

where

$$\alpha = \frac{1}{\sqrt{1-R^2}}\, w_o, \qquad v_1 = \sqrt{1-R^2}\, w_1, \qquad v_2 = \sqrt{1-R^2}\, w_2.$$

From the expressions of ψ and ψ_{xx}, the Gauss map N can be calculated using the first equation in (3.1). Thus, since $\langle \psi, \psi\rangle = 1$, $\langle \psi_x, \psi_x\rangle = 1$, $\langle N, N\rangle = -1$ and they are mutually orthogonal, it follows that α, v_1, v_2 are orthogonal, and $\langle \alpha, \alpha\rangle = -1$, $\langle v_1, v_1\rangle = 1$ and $\langle v_2, v_2\rangle = 1$.

On the other hand, since ψ_y is orthogonal to ψ_x and N, we get using also that $\langle \psi_{xy}, N\rangle = 0$

$$\alpha' = \mu_o P, \qquad v_1' = \mu_1 P, \qquad v_2' = \mu_2 P,$$

where P is the wedge product of α, v_1 and v_2 in $\mathbf{L}^4$, and μ_o, μ_1, μ_2 are C^∞ functions. Moreover, since

$$\langle \psi_y, \psi_y\rangle = \frac{1}{1-R^2}\left(R\mu_o + \cos\left(\sqrt{1-R^2}\,x\right)\mu_1 + \sin\left(\sqrt{1-R^2}\,x\right)\mu_2\right)^2$$

is positive, it follows that $\mu_o \neq 0$ and therefore α is a regular curve with tangent vector P. In particular, $\{v_1(y), v_2(y)\}$ is an orthonormal frame of the normal plane along α. Finally, the completeness of α follows from the completeness of M.

The converse is a straightforward computation. Anyway, it is worth pointing out that $R \neq 0$ because in other case from (3.6) and the completeness of the immersion, there would exist a point on γ_q where

$$a\cos\left(\sqrt{1-R^2}\;t\right) + b\sin\left(\sqrt{1-R^2}\;t\right) = 0$$

for some $t \in \mathbf{R}$, which contradicts the continuity of λ_2. Therefore the surface must be a totally umbilical round sphere.

□

Remark 3.1 Observe that we have not assumed that the principal curvatures λ_1, λ_2 *are necessarily ordered*, but

$$(\lambda_1 - R)(\lambda_2 - R) = 0.$$

4 Examples

We use the following well-known result (see, for instance, [4])

> *Given a Riemannian metric I and a symmetric (2,0)-tensor II on a simply-connected 2-dimensional manifold M, if I and II satisfy the Gauss and Mainardi-Codazzi equations of the de Sitter space $\mathbf{S}_1^3$, then there exists an only immersion (up to an isometry) $\psi : M \longrightarrow \mathbf{S}_1^3$ such that I and II are its first and second fundamental forms, respectively.*

First, we construct a family of complete orientable surfaces in $\mathbf{S}_1^3$ with umbilical and non umbilical points, and a constant principal curvature $R = 1$.

Example 4.1 Let us consider $M = \mathbf{R}^2$ and ψ_h the only immersion (up to an isometry) with first and second fundamental forms given by

$$I_h = dx^2 + \frac{1}{4}\left(2 + (x^2 - 2)h(y)\right)^2 dy^2$$

and

$$II_h = dx^2 + \frac{1}{4}\left(2 + x^2 h(y)\right)\left(2 + (x^2 - 2)h(y)\right) dy^2$$

respectively, where $h : \mathbf{R} \longrightarrow \mathbf{R}$ is a C^∞ function which vanishes at some point such that $0 \le h(y) \le c < 1$ for a constant c. Since

$$I = dx^2 + \frac{1}{4}\left(x^2 h(y) + 2(1 - h(y))\right)^2 dy^2 \ge dx^2 + (1 - c)^2 dy^2$$

and $dx^2 + (1-c)^2 dy^2$ is a complete metric, the immersion ψ_h is complete with principal curvatures

$$1 \qquad \text{and} \qquad \frac{2 + x^2 h(y)}{2 + (x^2 - 2)h(y)}.$$

Thus, the set of umbilical points is $\Omega = \{(x,y) : h(y) = 0\}$.

It is worth pointing out that the interior of Ω may be non empty.

$\square$

Now we are going to construct a family of complete orientable surfaces in $\mathbf{S}_1^3$ with umbilical and non umbilical points, and a constant principal curvature $R > 1$.

Example 4.2 Let us consider again $M = \mathbf{R}^2$ and $\psi_{R,h}$ the only immersion (up to an isometry) with first and second fundamental forms

$$I_{R,h} = \frac{1}{R^2}\, dx^2 + \frac{\left(R^3 h(y) + (R^2-1)e^{\sqrt{1-\frac{1}{R^2}}\ x}\right)^2}{R^2(R^2-1)^2}\, dy^2$$

and

$$\begin{aligned} II_{R,h} \ &= \ \frac{1}{R}\, dx^2 + \left(\frac{1}{R}\ e^{\sqrt{1-\frac{1}{R^2}}\ 2x} \right. \\ &\quad \left. + \frac{h(y)}{(R^2-1)^2}\left(R^3 h(y) + (R^4-1)e^{\sqrt{1-\frac{1}{R^2}}\ x}\right)\right)\, dy^2 \end{aligned}$$

respectively, where $h : \mathbf{R} \longrightarrow \mathbf{R}$ is a non negative C^∞ function which vanishes at some point and $R > 1$. Since

$$I_{R,h} \geq \frac{1}{R^2}\left(dx^2 + e^{\sqrt{1-\frac{1}{R^2}}\ 2x} dy^2\right),$$

the immersion ψ_h is complete, with principal curvatures

$$R \qquad \text{and} \qquad R\, \frac{R h(y) + (R^2-1)e^{\sqrt{1-\frac{1}{R^2}}\ x}}{R^3 h(y) + (R^2-1)e^{\sqrt{1-\frac{1}{R^2}}\ x}}\ .$$

Again, the set of umbilical points is $\Omega = \{(x,y) : h(y) = 0\}$.

As above, note that this surface can meet a totally umbilical surface in an open set.

$\square$

Acknowledgments

The first author is partially supported by Fundación Séneca, Grant No PI-3/00854/FS/01, and Junta de Comunidades de Castilla-La Mancha, Grant No PAI-02-027. The second author is partially supported by DGICYT Grant No. BFM2001-3318 and Junta de Andalucía CEC: FQM0203.

References

[1] J.A. Aledo and J.A. Gálvez, *Some Rigidity Results for Compact Spacelike Surfaces in the 3-dimensional de Sitter Space*, Proceedings of the Conference Differential Geometry Valencia 2001, World Scientific P.C., 19-27, (2002).

[2] J.A. Aledo and J.A. Gálvez, *Complete Surfaces in the Hyperbolic Space with a Constant Principal Curvature*, preprint.

[3] J.A. Aledo and A. Romero, *Compact Spacelike Surfaces in the 3-dimensional de Sitter Space with non-Degenerate Second Fundamental Form*, to appear in Differ. Geom. Appl.

[4] R. Greene, *Isometric embeddings of Riemannian and pseudo-Riemannian manifolds*, Mem. Amer. Math. Soc. **97** (1970).

[5] H. Li, *Global Rigidity Theorems of Hypersurfaces*, Ark. Mat. **35**, 327-351, (1997).

[6] S. Montiel, *An Integral Inequality for Compact Spacelike Hypersurfaces in de Sitter Space and Applications to the Case of Constant Mean Curvature*, Indiana Univ. Math. J. **37** 909-917, (1988).

[7] J. Ramanathan, *Complete Spacelike Hypersurfaces of Constant Mean Curvature in de Sitter Space*, Indiana Univ. Math. J. **36**, 349–359, (1987).

Proceedings of the meeting
Lorentzian Geometry-Benalmádena 2001
Benalmádena, Málaga, Spain
Pub. de la RSME, Vol. 5 (2003), 23-55

On the Calabi-Bernstein theorem for maximal hypersurfaces in the Lorentz-Minkowski space

Luis J. Alías [1] and Pablo Mira[2]

[1] *Departamento de Matemáticas, Universidad de Murcia, E-30100 Espinardo, Murcia, Spain*
E-mail: ljalias@um.es

[2] *Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena, E-30203 Cartagena, Murcia, Spain*
E-mail: pablo.mira@upct.es

Abstract

The Calabi-Bernstein theorem in the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ states that the only complete maximal hypersurfaces in $\mathbb{L}^{n+1}$ are the spacelike hyperplanes. The present work surveys on the role that this theorem plays in the global theory of hypersurfaces in $\mathbb{L}^{n+1}$. We shall describe several applications of the Calabi-Bernstein theorem in the study of complete spacelike constant mean curvature hypersurfaces in $\mathbb{L}^{n+1}$ and review some of the known proofs of the theorem. Special attention is paid to the two-dimensional case, in which an Enneper-Weierstrass representation for maximal surfaces is available.

1 Introduction

A maximal hypersurface in the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ is a spacelike hypersurface with zero mean curvature. The importance of maximal hypersurfaces (in general Lorentzian ambient spaces) is well known, not only from the mathematical point of view but also from the physical one, because of their role in different problems in General Relativity (see for instance [32] and references therein). From the mathematical viewpoint, one of the most important global results about maximal hypersurfaces in $\mathbb{L}^{n+1}$ is the Calabi-Bernstein theorem. It states that the only complete maximal hypersurfaces in the Lorentz-Minkowski space are the spacelike hyperplanes. Equivalently, the only maximal entire graphs in $\mathbb{L}^{n+1}$ are the spacelike hyperplanes.

This theorem was first obtained by Calabi [12] for the case where $n \leq 4$, inspired in the classical Bernstein theorem on minimal surfaces in $\mathbb{R}^3$. Later on, Cheng and Yau [13] extended the Calabi-Bernstein theorem to the general n-dimensional case. This shows a deep difference with respect to the Euclidean situation, since the Bernstein theorem for minimal hypersurfaces in $\mathbb{R}^{n+1}$ does not hold for $n > 7$ (see [11]). After the general proof by Cheng and Yau, several authors have approached to the two-dimensional version of the Calabi-Bernstein theorem from different perspectives, providing diverse extensions and new proofs of the result for the case of maximal surfaces in $\mathbb{L}^3$ [25, 17, 45, 8, 9]. On the other hand, some other authors have also developed different related Bernstein-type results on spacelike hypersurfaces in $\mathbb{L}^{n+1}$, looking for the characterization of spacelike hyperplanes among the complete spacelike hypersurfaces with constant mean curvature in $\mathbb{L}^{n+1}$ [44, 1, 48, 4].

In this paper we review on the Calabi-Bernstein theorem and other related results, trying to make the topic comprehensible for a general audience. For that reason, we start by introducing in Section 2 the spacelike hypersurfaces of the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ and establishing some of their basic topological properties. In Section 3 we develop the basic formulas for spacelike hypersurfaces in $\mathbb{L}^{n+1}$. After those preliminaries, we exhibit in Section 4 Cheng and Yau's approach to the Calabi-Bernstein theorem [13]. Their approach is based on a Simons-type formula for spacelike hypersurfaces in $\mathbb{L}^{n+1}$, as well as on an application of a generalized maximum principle due to Omori [40] and Yau [49].

In Section 5 we introduce some other Bernstein-type results on spacelike hypersurfaces in $\mathbb{L}^{n+1}$. These are applications of the above generalized maximum principle, and of Calabi-Bernstein theorem. In particular, we describe two different results of this type. The first one is a characterization of spacelike hyperplanes as the only complete spacelike hypersurfaces with constant mean curvature in the Lorentz-Minkowski space whose hyperbolic image is bounded. This characterization was obtained simultaneous and independently by Aiyama [2] and Xin [48] (see also [44] for a first weaker version given by Palmer). The second one characterizes spacelike hyperplanes as the only complete spacelike hypersurfaces with constant mean curvature in $\mathbb{L}^{n+1}$ which are bounded between two parallel spacelike hyperplanes, and has been recently given by Aledo and Alías [4].

It is interesting to remark that Cheng and Yau's proof of the Calabi-Bernstein theorem was the first application of a Simons-type formula in the context of spacelike hypersurfaces in a Lorentz ambient space. Afterwards, such Simons-type formulas have been applied in several forms by different authors, yielding many interesting results on Bernstein-type problems for spacelike hypersurfaces in Lorentzian spaces [1, 3, 24].

The rest of the paper is devoted to the study of maximal surfaces in $\mathbb{L}^3$ and the two-dimensional version of the Calabi-Bernstein theorem. In Section 6 we introduce the Enneper-Weierstrass representation for maximal surfaces in $\mathbb{L}^3$ and, as an application of it, we exhibit Kobayashi's proof of the theorem [25]. In Section 7 we describe a recent simple approach to the two-dimensional version of the Calabi-Bernstein theorem. It was given by Romero in [45], and is based on the Liouville theorem on harmonic functions on $\mathbb{R}^2$. Section 8 deals with a different approach to the Calabi-Bernstein theorem. This approach is based on finding adequate local upper bounds for the Gaussian curvature of a maximal surface. In particular, we describe two different results of this type. The first one is due to Estudillo and Romero [17], and consists on a pointwise estimate for the Gaussian curvature of a maximal surface in terms of the distance of the point to the boundary of the surface. The second one is a local upper bound for the total curvature of geodesic discs in a maximal surface in $\mathbb{L}^3$. This upper bound involves the local geometry of the surface and its hyperbolic image, and it has been recently obtained by Alías and Palmer [8]. Finally, in Section 9 we have collected some further developments on the topic of maximal surfaces in $\mathbb{L}^3$, including some recent advances by the authors [6, 36], jointly with Chaves in [6], on the so called Björling problem for maximal surfaces in $\mathbb{L}^3$.

Before closing this introduction, it is worth pointing out that the Calabi-Bernstein theorem is no longer true for the case of entire timelike minimal graphs in $\mathbb{L}^{n+1}$, even in the simplest two-dimensional case. Actually, if x_3 stands for the timelike coordinate in $\mathbb{L}^3$, then the graph given by $x_2 = x_3 \tanh x_1$, with $(x_1, x_3) \in \mathbb{R}^2$, is an example of an entire non-planar timelike graph in $\mathbb{L}^3$, having zero mean curvature and positive Gaussian curvature [25, 34].

Nevertheless, in [34] Weinstein (formerly Milnor) obtained a very interesting conformal analogue of the Calabi-Bernstein theorem for timelike surfaces. Specifically, she proved that every timelike entire graph in $\mathbb{L}^3$ with zero mean curvature is conformally equivalent to the Lorentzian plane $\mathbb{L}^2$ (see also [28, 29] for some extensions of this conformal analogue, which have recently been given by Lin and Weinstein). On the other hand, Magid [31] and Weinstein [35] developed independently different approaches to the study of the Calabi-Bernstein problem for timelike surfaces in $\mathbb{L}^3$. In particular, in [31] Magid showed that every timelike entire graph with zero mean curvature over either a timelike or a spacelike plane in $\mathbb{L}^3$ is a global translation surface. This allows him to obtain a standard form for all such graphs and to study some geometric properties of those surfaces. In [35] Weinstein described all entire timelike graphs with zero mean curvature in $\mathbb{L}^3$ via a kind of Weierstrass representation. We also refer the reader to the excellent book [47] by Weinstein, where the author gives a careful, extensive and detailed study of the topic of Lorentz surfaces.

2 Preliminaries

Let $\mathbb{L}^{n+1}$ denote the $(n+1)$-dimensional Lorentz-Minkowski space, that is, the real vector space $\mathbb{R}^{n+1}$ endowed with the Lorentzian metric

$$\langle , \rangle = (dx_1)^2 + \cdots + (dx_n)^2 - (dx_{n+1})^2,$$

where $(x_1, \dots, x_{n+1})$ are the canonical coordinates in $\mathbb{R}^{n+1}$. A smooth immersion $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ of an n-dimensional connected manifold Σ is said to be a *spacelike hypersurface* if the induced metric via ψ is a Riemannian metric on Σ, which, as usual, is also denoted by $\langle , \rangle$. A spacelike hypersurface Σ is said to be *complete* if the Riemannian induced metric is a complete metric on Σ.

As a first interesting property of the topology of such hypersurfaces, let us remark that every spacelike hypersurface in $\mathbb{L}^{n+1}$ is orientable. In fact, observe that $(0, \dots, 0, 1)$ is a unit timelike vector field globally defined on $\mathbb{L}^{n+1}$, which determines a time-orientation on $\mathbb{L}^{n+1}$. This allows us to choose a unique timelike unit normal field N on Σ which is in the same time-orientation as $(0, \dots, 0, 1)$, and hence we may assume that Σ is oriented by N. We will refer to N as the *future-directed Gauss* *map* of the hypersurface Σ. This future-directed normal field N can be regarded as a map $N : \Sigma \longrightarrow \mathbb{H}^n_+$, where $\mathbb{H}^n_+$ denotes the future connected component of the n-dimensional hyperbolic space, that is

$$\mathbb{H}^n_+ = \{x \in \mathbb{L}^{n+1} : \langle x, x\rangle = -1, \quad x_{n+1} \geq 1\}.$$

The image $N(\Sigma) \subset \mathbb{H}^n_+$ will be called the *hyperbolic image* of Σ.

Another interesting remark on the topology of spacelike hypersurfaces in $\mathbb{L}^{n+1}$ is that every complete spacelike hypersurface in the Lorentz-Minkowski space is spatially entire, in the sense that the projection $\Pi : \Sigma \longrightarrow \mathbb{R}^n$ of Σ onto the spacelike hyperplane $x_{n+1} = 0$ is a diffeomorphism between Σ and $\mathbb{R}^n$. In fact, it is not difficult to see that since Σ is spacelike, $\Pi : \Sigma \longrightarrow \mathbb{R}^n$ is a local diffeomorphism which satisfies $\Pi^*(\langle,\rangle_o) \geq \langle,\rangle$, where $\langle,\rangle_o$ stands for the Euclidean metric in $\mathbb{R}^n$. This means that Π increases the distance. The completeness of Σ implies then that $\Pi(\Sigma) = \mathbb{R}^n$ and that Π is a covering map [27, Lemma VIII.1]. Since $\mathbb{R}^n$ is simply connected, Π must be a global diffeomorphism and the hypersurface Σ can be seen as an entire graph over the spacelike hyperplane $(x_1, \dots, x_n)$. As a direct consequence of this, we get the following.

Corollary 2.1 *Let $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ be a complete spacelike hypersurface in the Lorentz-Minkowski space. Then*

1. *Σ is diffeomorphic to $\mathbb{R}^n$.*
2. *The immersion $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ is actually an embedding.*
3. *Its image $\psi(\Sigma)$ is a closed subset in $\mathbb{L}^{n+1}$.*

In particular, there exists no compact (without boundary) spacelike hypersurface in $\mathbb{L}^{n+1}$. It is worth pointing out that no converse of the statements in Corollary 2.1 is true in general. More precisely, there exist examples of spacelike entire graphs in $\mathbb{L}^{n+1}$ which are not complete. For instance, let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be a real function defined by

$$\phi(x) = \int_0^{|x|} \sqrt{1 - e^{-t}}dt$$

when $|x| \geq 1$, and $\phi(x) = f(x)$ when $|x| < 1$, where $f \in \mathcal{C}^\infty(\mathbb{R})$ is a smooth extension satisfying $f'(x)^2 < 1$ for all $x \in (-1, 1)$. Then the entire graph given by $x_{n+1} = \phi(x_1)$ defines a spacelike hypersurface Σ in $\mathbb{L}^{n+1}$ which is not complete. In fact, observe that the curve $\alpha : \mathbb{R} \longrightarrow \Sigma$

given by $\alpha(t) = (t, 0, \ldots, 0, \phi(t))$ is a divergent curve in Σ with finite length, since

$$\begin{array}{rcl} \ell(\alpha) & = & \int_{-\infty}^{+\infty} |\alpha'(t)| dt \\ & = & \int_{-1}^{1} \sqrt{1 - f'(t)^2} dt + 2 \int_1^{+\infty} e^{-t/2} dt \;\; < \;\; 2\left(1 + \frac{2}{\sqrt{e}}\right). \end{array}$$

Therefore, Σ is an entire spacelike graph which is not complete.

This fact points out a curious difference between the behaviour of hypersurfaces in Euclidean space $\mathbb{R}^{n+1}$ and that of spacelike hypersurfaces in the Lorentz-Minkowski space. Actually, let us recall that every embedded hypersurface in Euclidean space which is a closed subset in $\mathbb{R}^{n+1}$ is necessarily complete, while there exist examples of complete embedded hypersurfaces in $\mathbb{R}^{n+1}$ which are not closed. On the other hand, it is also interesting to point out that in the case where the mean curvature is constant, every embedded spacelike hypersurface in the Lorentz-Minkowski space which is a closed subset in $\mathbb{L}^{n+1}$ is necessarily complete [13]. For more details about this topic, we refer the reader to a series of papers by Harris [18, 19, 20].

3 Basic formulas

Throughout this paper we will denote by ∇^{o} the flat Levi-Civita connection of $\mathbb{L}^{n+1}$ and by ∇ the Levi-Civita connection of Σ. Then the Gauss and Weingarten formulas for Σ in $\mathbb{L}^{n+1}$ are given respectively by

$$\nabla^{\text{o}}{}_X Y = \nabla_X Y - \langle AX, Y \rangle N \tag{3.1}$$

and

$$A(X) = -\nabla^{\text{o}}_X N, \tag{3.2}$$

for all tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$, where $A : \mathfrak{X}(\Sigma) \longrightarrow \mathfrak{X}(\Sigma)$ stands for the shape operator of Σ in $\mathbb{L}^{n+1}$ with respect to the future-directed Gauss map N. Associated to the shape operator of Σ there is the *mean curvature* of the hypersurface, which is its main extrinsic curvature and is defined by

$$H = -\frac{1}{n}\text{tr}(A) = -\frac{1}{n}\sum_{i=1}^{n} \kappa_i.$$

Here $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of the hypersurface. The choice of the sign (-1) in our definition of H is motivated by the fact that in that case the mean curvature vector is given by $\vec{H} = HN$. Therefore, $H(p) > 0$ at a point $p \in \Sigma$ if and only if $\vec{H}(p)$ is future-directed in $\mathbb{L}^{n+1}$. A spacelike hypersurface is said to be *maximal* if H vanishes on Σ, $H \equiv 0$.

As is well known, the (intrinsic) curvature tensor R of the hypersurface is described in terms of the shape operator of Σ by the Gauss equation [41, Theorem 4.5]

$$R(X,Y)Z = -\langle AX, Z\rangle AY + \langle AY, Z\rangle AX \tag{3.3}$$

for $X, Y, Z \in \mathfrak{X}(\Sigma)$. Observe that our criterion here for the definition of the curvature tensor is the one in [41],

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z.$$

Thus, the *Ricci curvature* of Σ is written as follows.

$$\mathrm{Ric}(X,Y) = nH\langle AX, Y\rangle + \langle AX, AY\rangle.$$

In particular, every maximal hypersurface in $\mathbb{L}^{n+1}$ has non-negative Ricci curvature, that is, $\mathrm{Ric}(X,X) \geq 0$ for every $X \in \mathfrak{X}(\Sigma)$. More generally,

$$\mathrm{Ric}(X,X) \geq -\frac{n^2H^2}{4}|X|^2 \tag{3.4}$$

for every $X \in \mathfrak{X}(\Sigma)$. On the other hand, the Codazzi equation of the hypersurface is given by

$$\nabla A(X,Y) = \nabla A(Y,X), \tag{3.5}$$

where $\nabla A(X,Y) = (\nabla_Y A)X = \nabla_Y(AX) - A(\nabla_Y X)$ [41, Corollary 4.34].

4 Cheng and Yau's proof of the Calabi-Bernstein theorem

In this section we shall describe Cheng and Yau's approach to the Calabi-Bernstein theorem. One of the main ingredients of their proof is the obtention of a Simons-type formula for spacelike hypersurfaces in $\mathbb{L}^{n+1}$. The idea of this Simons-type formula is to compute the Laplacian of $\mathrm{tr}(A^2)$. To do so, let us introduce the following standard notation. Let $S, T : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \longrightarrow \mathfrak{X}(\Sigma)$ be two self-adjoint endomorphisms. Then

$$\langle S, T\rangle = \mathrm{tr}(S \circ T) = \sum_{i=1}^{n} \langle SE_i, TE_i\rangle,$$

and

$$\langle \nabla S, \nabla T\rangle = \sum_{i,j=1}^{n} \langle (\nabla S)(E_i, E_j), (\nabla T)(E_i, E_j)\rangle,$$

where $\{E_1, \dots, E_n\}$ is a local orthonormal frame tangent to Σ. In our notation, $(\nabla S)(X, Y) = (\nabla_Y S)(X)$. On the other hand, we also define ΔS as the rough Laplacian of the endomorphism field S, that is, $\Delta S : \mathfrak{X}(\Sigma) \longrightarrow \mathfrak{X}(\Sigma)$ is the endomorphism given by

$$\Delta S(X) = \operatorname{tr}\big((\nabla^2 S)(X, \cdot, \cdot)\big) = \sum_{i=1}^{n} (\nabla^2 S)(X, E_i, E_i).$$

Recall again that in our notation, $(\nabla^2 S)(X, Y, Z) = (\nabla_Z \nabla S)(X, Y)$.

Now, a standard tensor computation implies that

$$\frac{1}{2}\Delta \mathrm{tr}(A^2) = \frac{1}{2}\Delta\langle A, A\rangle = |\nabla A|^2 + \langle A, \Delta A\rangle. \tag{4.1}$$

Observe that by the Codazzi equation (3.5) $\nabla^2 A$ is symmetric in the two first variables

$$(\nabla^2 A)(X, Y, Z) = (\nabla^2 A)(Y, X, Z)$$

for every $X, Y, Z \in \mathfrak{X}(\Sigma)$. On the other hand, it is not difficult to show that

$$\begin{aligned} (\nabla^2 A)(X, Y, Z) &= \nabla_Z(\nabla_Y(AX)) - \nabla_{\nabla_Z Y}(AX) \\ &\quad + A(\nabla_Y \nabla_Z X) + A(\nabla_{\nabla_Z Y} X) \\ &\quad - \nabla_Z(A(\nabla_Y X)) - \nabla_Y(A(\nabla_Z X)), \end{aligned}$$

and

$$\begin{aligned} (\nabla^2 A)(X, Z, Y) &= \nabla_Y(\nabla_Z(AX)) - \nabla_{\nabla_Y Z}(AX) \\ &\quad + A(\nabla_Z \nabla_Y X) + A(\nabla_{\nabla_Y Z} X) \\ &\quad - \nabla_Y(A(\nabla_Z X)) - \nabla_Z(A(\nabla_Y X)). \end{aligned}$$

Therefore, $\nabla^2 A$ is not symmetric in the two last variables (unless $R = 0$ or $A = 0$) but

$$(\nabla^2 A)(X, Y, Z) = (\nabla^2 A)(X, Z, Y) - R(Z, Y)AX + A\left(R(Z, Y)X\right).$$

By using now the Gauss equation (3.3), we can conclude from here that

$$\begin{aligned}\Delta A(X) &= \sum_{i=1}^{n}(\nabla^2 A)(E_i,E_i,X)-\sum_{i=1}^{n}R(E_i,X)AE_i\\&\quad+\sum_{i=1}^{n}A\left(R(E_i,X)E_i\right)\\&=\nabla_X(\mathrm{tr}(\nabla A))+\mathrm{tr}(A^2)AX-\mathrm{tr}(A)A^2X\\&=-n\nabla_X(\nabla H)+\mathrm{tr}(A^2)AX+nHA^2X.\end{aligned}$$

In particular, if the mean curvature is constant (not necessarily zero), then

$$\Delta A(X)=\mathrm{tr}(A^2)AX+nHA^2X$$

for every $X\in\mathfrak{X}(\Sigma)$. Using this in (3.6) we arrive at the following Simons-type formula, which holds for spacelike hypersurfaces with constant mean curvature H in the Lorentz-Minkowski space,

$$\frac{1}{2}\Delta\mathrm{tr}(A^2)=|\nabla A|^2+\left(\mathrm{tr}(A^2)\right)^2+nH\mathrm{tr}(A^3). \tag{4.2}$$

On the other hand, Cheng and Yau's proof is also an application of the following generalized maximum principle for complete manifolds due to Omori [40] and Yau [49].

Theorem 4.1 (A generalized maximum principle) *Let Σ be a complete Riemannian manifold whose Ricci curvature is bounded from below and let $u:\Sigma\longrightarrow\mathbb{R}$ be a smooth function bounded from below on Σ (resp. bounded from above on Σ). Then, for each $\varepsilon>0$ there exists a point $p_\varepsilon\in M$ such that*

1. $|\nabla u(p_\varepsilon)|<\varepsilon$.
2. $\Delta u(p_\varepsilon)>-\varepsilon$ *(resp.* $\Delta u(p_\varepsilon)<\varepsilon$*).*
3. $\inf u\le u(p_\varepsilon)<\inf u+\varepsilon$ *(resp.* $\sup u-\varepsilon<u(p_\varepsilon)\le\sup u$*).*

Here ∇u and Δu denote, respectively, the gradient and the Laplacian of u.

As an application of this generalized maximum principle we have the following Liouville-type theorem [13, 37]. We also refer the reader to [15] for a modern accesible treatment of this Liouville-type result and generalizations.

Lemma 4.2 *Let Σ be a complete Riemannian manifold whose Ricci curvature is bounded from below and let $u : \Sigma \longrightarrow \mathbb{R}$ be a non-negative smooth function on Σ. If there exists a positive constant $c > 0$ such that $\Delta u \geq cu^2$, then u vanishes identically on Σ.*

Now we are ready to obtain the Calabi-Bernstein theorem. In fact, if $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ is a complete maximal hypersurface in the Lorentz-Minkowski space, then $H = 0$ and (4.2) reduces to

$$\frac{1}{2}\Delta \mathrm{tr}(A^2) = |\nabla A|^2 + \left(\mathrm{tr}(A^2)\right)^2. \tag{4.3}$$

Besides, by (3.4) we know that the Ricci curvature of Σ is non-negative, and we may apply Lemma 4.2 to the function $u = \mathrm{tr}(A^2)$, which by (4.3) satisfies $\Delta u \geq 2u^2$. Therefore u vanishes identically on Σ, which means that Σ is totally geodesic in $\mathbb{L}^{n+1}$, and by completeness it must be a spacelike hyperplane.

5 Other Bernstein-type results on spacelike hypersurfaces

In this section we will introduce other Bernstein-type results on spacelike hypersurfaces of constant mean curvature in $\mathbb{L}^{n+1}$. These are obtained as an application of the Calabi-Bernstein theorem together with the maximum principle given in Theorem 4.1. The first result was obtained simultaneous and independently by Aiyama [2] and Xin [48], and a first weaker version of it was given by Palmer in [44]. Specifically, it states what follows.

Theorem 5.1 *The only complete spacelike hypersurfaces with constant mean curvature in the Lorentz-Minkowski space whose hyperbolic image $N(\Sigma) \subset \mathbb{H}^n_+$ is bounded in the hyperbolic space are the spacelike hyperplanes.*

The proof of this theorem needs the following standard computations, which we describe below for our later use. Such computations will be also useful in Section 7 and Section 8. Let $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ be a spacelike hypersurface in $\mathbb{L}^{n+1}$. For each fixed arbitrary vector $a \in \mathbb{L}^{n+1}$, let us consider on Σ the smooth function $\langle a, N\rangle$, whose gradient is given by

$$\nabla\langle a, N\rangle = -A(a^\top), \tag{5.1}$$

where $a^\top = a + \langle a, N\rangle N$ denotes the component of a which is tangent to Σ. From $\nabla^o a = 0$, it follows that

$$\nabla_X a^\top = -\langle a, N\rangle AX \tag{5.2}$$

for every $X \in \mathfrak{X}(\Sigma)$, and using the Codazzi equation we obtain that

$$\nabla_X(\nabla\langle a, N\rangle) = -(\nabla_{a^\top} A)X + \langle a, N\rangle A^2 X.$$

Therefore, the Laplacian of $\langle a, N\rangle$ is given in general by

$$\Delta\langle a, N\rangle = -\mathrm{tr}(\nabla_{a^\top} A) + \mathrm{tr}(A^2)\langle a, N\rangle = n\langle\nabla H, a\rangle + \mathrm{tr}(A^2)\langle a, N\rangle.$$

In particular, if the mean curvature H is constant we obtain

$$\Delta\langle a, N\rangle = \mathrm{tr}(A^2)\langle a, N\rangle. \tag{5.3}$$

Proof of Theorem 5.1: Let us assume that the hyperbolic image of Σ is contained in a geodesic ball $B(a, \varrho)$ in $\mathbb{H}^n_+$ of radius $\varrho > 0$ centered at a point $a \in \mathbb{H}^n_+$. Recall that

$$B(a, \varrho) = \{x \in \mathbb{H}^n_+ : 1 \leq -\langle a, x\rangle \leq \cosh(\varrho)\},$$

so that for every $p \in \Sigma$

$$1 \leq -\langle a, N(p)\rangle \leq \cosh(\varrho).$$

Since H is constant, we know from (3.4) that the Ricci curvature of Σ is bounded from below by the non-positive constant $-n^2H^2/4$. Thus we may apply Theorem 4.1 to the function $u = -\langle a, N\rangle$, which is bounded from above on Σ. Since H is constant, by (5.3) it follows that $\Delta u = \mathrm{tr}(A^2)u$. Applying now Theorem 4.1, we know that for every $\varepsilon > 0$ there exists a point $p_\varepsilon \in \Sigma$ such that

$$\Delta u(p_\varepsilon) = \mathrm{tr}(A^2)(p_\varepsilon)u(p_\varepsilon) < \varepsilon, \tag{5.4}$$

and

$$\lim_{\varepsilon\to\infty} u(p_\varepsilon) = \sup u \leq \cosh(\varrho).$$

Besides, by the Cauchy-Schwarz inequality, $\mathrm{tr}(A^2) \geq nH^2$ on Σ, which jointly with (5.4) yields

$$0 \leq nH^2 u(p_\varepsilon) \leq \mathrm{tr}(A^2)(p_\varepsilon)u(p_\varepsilon) < \varepsilon.$$

Since $u(p) \geq 1$ at every point $p \in \Sigma$, we obtain from here that

$$0 \leq H^2 < \frac{\varepsilon}{nu(p_\varepsilon)},$$

and letting $\varepsilon \to 0$ we conclude that the constant mean curvature H must be zero and, by the Calabi-Bernstein theorem, the hypersurface must be a spacelike hyperplane.

□

Let us recall that every complete spacelike hypersurface in $\mathbb{L}^{n+1}$ is spatially entire. In particular, they cannot be spatially bounded and, for instance, no complete spacelike hypersurface in $\mathbb{L}^{n+1}$ can be contained in the slab determined by two parallel timelike hyperplanes. For the case of spacelike hyperplanes, Aledo and Alías [4] have recently obtained the following Bernstein-type result.

Theorem 5.2 *The only complete spacelike hypersurfaces with constant mean curvature in the Lorentz-Minkowski space which are bounded between two parallel spacelike hyperplanes are the (parallel) spacelike hyperplanes.*

It is interesting to remark that the corresponding result for minimal surfaces in Euclidean space $\mathbb{R}^3$ turns out to be false. Actually, Jorge and Xavier [22] constructed examples of complete non-flat minimal surfaces in $\mathbb{R}^3$ contained between two parallel planes.

Proof. Let $a \in \mathbb{L}^{n+1}$ be the future-directed unit vector such that $\psi(\Sigma)$ is bounded between the parallel hyperplanes $\langle a, x\rangle = c$ and $\langle a, x\rangle = C$, $c \leq C$. Hence, the height function $u = \langle a, \psi\rangle$ is bounded from above and from below on Σ, and using Theorem 4.1 we get that for each $\varepsilon > 0$ there exist points $p_\varepsilon, q_\varepsilon \in \Sigma$ such that

$$\Delta u(p_\varepsilon) < \varepsilon, \quad \text{and} \quad \Delta u(q_\varepsilon) > -\varepsilon. \tag{5.5}$$

Since the gradient of u is $\nabla u = a^\top$, we know from (5.2) that the Laplacian of u is given by

$$\Delta u = -\langle a, N\rangle \mathrm{tr}(A) = nH\langle a, N\rangle.$$

Therefore, (5.5) implies that for each $\varepsilon > 0$

$$\Delta u(p_\varepsilon) = nH\langle a, N(p_\varepsilon)\rangle < \varepsilon,$$

and

$$\Delta u(q_\varepsilon) = nH\langle a, N(q_\varepsilon)\rangle > -\varepsilon.$$

Using now that $|\langle a, N\rangle| = -\langle a, N\rangle \geq 1$, it follows from here that for each $\varepsilon > 0$

$$-\frac{\varepsilon}{n} \leq -\frac{\varepsilon}{n|\langle a, N(p_\varepsilon)\rangle|} < H < \frac{\varepsilon}{n|\langle a, N(q_\varepsilon)\rangle|} \leq \frac{\varepsilon}{n}.$$

Therefore, letting $\varepsilon \to 0$ we get that the constant mean curvature H must be zero and, by the Calabi-Bernstein theorem, the hypersurface must be a spacelike hyperplane.

□

Let $\Omega \subseteq \mathbb{R}^n$ be a domain and $u : \Omega \longrightarrow \mathbb{R}$ a smooth function on Ω. Then, the graph determined by u, $x_{n+1} = u(x_1, \ldots, x_n)$, defines a spacelike hypersurface in $\mathbb{L}^{n+1}$ if and only if the Euclidean gradient of u satisfies $|Du| < 1$ on Ω. In that case, the mean curvature H of the spacelike graph is given by

$$\mathrm{Div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = nH, \quad \text{with} \quad |Du| < 1, \tag{5.6}$$

where Div stands for the divergence in $\mathbb{R}^n$. Therefore, for every real number $H \in \mathbb{R}$, the solutions to the differential equation (5.6) which are globally defined on $\mathbb{R}^n$ represent the spacelike entire graphs with constant mean curvature H in $\mathbb{L}^{n+1}$. In terms of this constant mean curvature differential equation, the Calabi-Bernstein theorem can be paraphrased as follows.

Corollary 5.3 *The only entire solutions to the zero mean curvature differential equation*

$$\mathrm{Div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = 0, \quad \text{with} \quad |Du| < 1, \tag{5.7}$$

are affine functions.

On the other hand, if a complete spacelike hypersurface in $\mathbb{L}^{n+1}$ is contained in the slab determined by two parallel spacelike hyperplanes, then it can be seen as an entire graph over one of those spacelike hyperplanes. Using this, Theorem 5.2 can be also stated in terms of the constant mean curvature differential equation in the following way.

Corollary 5.4 *The only entire solutions to the constant mean curvature differential equation*

$$\mathrm{Div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = nH, \quad \text{with} \quad |Du| < 1,$$

which are bounded on $\mathbb{R}^n$ *are the constant ones.*

Finally, observe that the Gauss map of a spacelike graph

$$x_{n+1} = u(x_1, \ldots, x_n)$$

in $\mathbb{L}^{n+1}$ is given by

$$N = \frac{1}{\sqrt{1-|Du|^2}}\left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, 1\right).$$

This implies that the hyperbolic image of a spacelike graph in $\mathbb{L}^{n+1}$ is bounded in $\mathbb{H}^n_+$ if and only if $\sqrt{1-|Du|^2}$ is bounded away from zero. Thus we can formulate Theorem 5.1 in this way.

Corollary 5.5 *The only entire solutions to the constant mean curvature differential equation*

$$\mathrm{Div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = nH, \quad \text{with} \quad |Du| \leq 1-\varepsilon < 1,$$

are affine functions.

6 The Enneper-Weierstrass representation and its applications

In [25] Kobayashi provided a new approach to the two-dimensional version of the Calabi-Bernstein theorem for maximal surfaces in $\mathbb{L}^3$, which is based on an appropriate Enneper-Weierstrass representation for those surfaces (see also [33]).

The way we introduce here the Enneper-Weierstrass representation for maximal surfaces in $\mathbb{L}^3$ follows the exposition in [10] for the corresponding Euclidean situation. For this, recall that every spacelike surface $\psi : \Sigma^2 \to \mathbb{L}^3$ in $\mathbb{L}^3$ is oriented by its future-directed Gauss map

$N : \Sigma \to \mathbb{H}^2_+$. Therefore, using local isothermal parameters whose changes of coordinates preserve the orientation, Σ has in a natural way a structure of Riemann surface. Define, in terms of a local complex parameter $z = u + \mathrm{i}v$, the complex functions ϕ_1, ϕ_2, ϕ_3 given by

$$\phi_k = \frac{\partial \psi_k}{\partial z} = \frac{1}{2}\left(\frac{\partial \psi_k}{\partial u} - \mathrm{i}\frac{\partial \psi_k}{\partial v}\right).$$

It is not difficult to see that

$$\phi_1^2 + \phi_2^2 - \phi_3^2 = 0,$$

and

$$|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 = \frac{e^{2\rho}}{2} > 0,$$

where $ds^2 = e^{2\rho}|dz|^2$ is the Riemannian metric on Σ induced by the spacelike immersion ψ. Moreover, the 1-forms given by

$$\Phi_k = \phi_k dz, \quad k = 1, 2, 3,$$

are globally defined on Σ, have no real periods, and they are holomorphic if and only if Σ is a maximal surface in $\mathbb{L}^3$. In that case, the immersion can be recovered from the holomorphic 1-forms by

$$\psi = 2\mathrm{Re}\int_{\gamma_z}(\Phi_1, \Phi_2, \Phi_3),$$

where γ_z is any path from a fixed base point to z.

Let us assume from now on that Σ is a maximal surface in $\mathbb{L}^3$. Consider the meromorphic function on Σ defined by

$$g = \frac{\Phi_3}{\Phi_1 - \mathrm{i}\Phi_2},$$

that is, if z is a local complex coordinate such that $\Phi_k = \phi_k dz$, then

$$g = \frac{\phi_3}{\phi_1 - \mathrm{i}\phi_2}.$$

We will see now that g is closely related to the Gauss map of Σ. Define a stereographic projection $\sigma : \mathbb{D} \to \mathbb{H}^2_+$ from the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto $\mathbb{H}^2_+$ as follows

$$\sigma(z) = \left(\frac{2\,\mathrm{Re}(z)}{1 - |z|^2}, \frac{2\,\mathrm{Im}(z)}{1 - |z|^2}, \frac{1 + |z|^2}{1 - |z|^2}\right).$$

Geometrically, σ is a conformal bijection between $\mathbb{D}$ and $\mathbb{H}^2_+$ which assigns to each point $z \in \mathbb{D}$ the point in $\mathbb{H}^2_+$ obtained as the intersection of $\mathbb{H}^2_+$ and the straight line joining $(0,0,-1)$ and $(z,0)$. It then follows that

$$g = \sigma^{-1} \circ N. \tag{6.1}$$

In particular $|g| < 1$, so g is a holomorphic function on Σ and

$$\omega = \Phi_1 - \mathrm{i}\Phi_2$$

defines a holomorphic 1-form without zeroes on Σ. This allows us to state the following Enneper-Weierstrass representation.

Theorem 6.1 *Let $\psi : \Sigma^2 \to \mathbb{L}^3$ be a maximal surface in $\mathbb{L}^3$, endowed with its natural Riemann surface structure. Then there exist a holomorphic 1-form $\omega = f dz$ without zeroes on Σ and a holomorphic function $g : \Sigma \to \mathbb{D}$ such that the immersion ψ is represented (up to a translation) by*

$$\psi = \mathrm{Re} \int \big((1+g^2)\omega, \mathrm{i}(1-g^2)\omega, 2g\omega \big). \tag{6.2}$$

Conversely, let Σ be a Riemann surface, $\omega = f dz$ a holomorphic 1-form without zeroes on Σ and $g : \Sigma \to \mathbb{D}$ a holomorphic function. Besides, assume that the following holomorphic 1-forms

$$(1+g^2)\omega, \quad \mathrm{i}(1-g^2)\omega, \quad g\omega, \tag{6.3}$$

do not have real periods. Then the map $\psi : \Sigma^2 \to \mathbb{L}^3$ defined by (6.2) is a maximal surface in $\mathbb{L}^3$.

The condition that the 1-forms in (6.3) do not have real periods is necessary in order to guarantee that the integral (6.2) depends only on the final point.

The quantities (g, ω) are called the Enneper-Weierstrass data of the maximal surface. Recall that the holomorphic function g is closely related to the future-directed Gauss map of Σ, since $N = \sigma \circ g$. On the other hand, the metric on Σ is written as

$$ds^2 = |f|^2(1-|g|^2)^2|dz|^2, \tag{6.4}$$

where $\omega = f dz$ locally, and its Gaussian curvature is

$$K = \left(\frac{2|g'|}{|f|(1-|g|^2)^2} \right)^2. \tag{6.5}$$

In particular, either $K = 0$ or its zeroes are isolated.

In [42, Theorem 8.1] it is shown that the normals of a complete non-planar minimal surface in $\mathbb{R}^3$ are everywhere dense in $\mathbb{S}^2$. Of course, this has as an immediate consequence the classical Bernstein theorem. Kobayashi's approach to the Calabi-Bernstein theorem is a simple modification of the proof of the above theorem, and makes use of the following auxiliary result due to Osserman [42, 43].

Lemma 6.2 *Let f be a holomorphic function on the unit disc $\mathbb{D}$, and assume that f does not vanish on $\mathbb{D}$. Then there exists a divergent curve Γ in $\mathbb{D}$ such that*

$$\int_\Gamma |f(z)||dz| < \infty.$$

Kobayashi's proof of the Calabi-Bernstein theorem: Since Σ is complete, then it is diffeomorphic to $\mathbb{R}^2$ and, by Koebe uniformization theorem, it must be conformally equivalent to either the complex plane $\mathbb{C}$ or the unit disc $\mathbb{D}$. Let us see that Σ cannot be conformally equivalent to $\mathbb{D}$. In fact, if Σ were conformally equivalent to the unit disc, then the holomorphic 1-form ω would be $\omega = f dz$, with f a holomorphic function without zeroes on $\mathbb{D}$. Applying Lemma 6.2, there exists a divergent curve β in Σ such that

$$\int_\beta |f(z)||dz| < \infty.$$

On the other hand, by (6.4) the length of β is

$$L(\beta) = \int_\beta ds = \int_\beta |f(z)|(1 - |g(z)|^2)|dz| \leq \int_\beta |f(z)||dz| < \infty.$$

But that means that β is a divergent curve in Σ with finite length, which is a contradiction to the fact that Σ is complete.

Therefore, Σ must be conformally equivalent to $\mathbb{C}$. In that case, $g : \mathbb{C} \to \mathbb{D}$ is a holomorphic function which is bounded on $\mathbb{C}$ and, by the Liouville theorem, it must be constant. Equivalently, the Gauss map of Σ is constant and the surface must be a spacelike plane.

$\square$

There are further applications of the Enneper-Weierstrass representation of maximal surfaces. For instance, as we will see in Section 8, Estudillo and Romero [17] use this representation to obtain a universal inequality for the Gaussian curvature of a maximal surface in $\mathbb{L}^3$. On the other hand, Kobayashi [26] and, more recently, Imaizumi [21] apply the Enneper-Weierstrass representation to the study of maximal surfaces with conelike singularities, obtaining interesting characterizations of the Lorentz catenoid. In [30] López, López and Souam, also using this Enneper-Weierstrass representation, classify the family of maximal surfaces in $\mathbb{L}^3$ which are foliated by pieces of circles. Finally, in [5] Aledo and Gálvez extend the Weierstrass formula to the case of spacelike surfaces in $\mathbb{L}^3$ whose mean curvature is a constant multiple of its Gaussian curvature (maximal linear Weingarten surfaces). By using this extended complex representation they show the following generalization of the Calabi-Bernstein theorem: the only complete maximal linear Weingarten surfaces in $\mathbb{L}^3$ of non-negative Gaussian curvature are spacelike planes. Nevertheless it is interesting to remark that, opposite to the case of maximal surfaces, there exist complete non-planar maximal linear Weingarten surfaces in $\mathbb{L}^3$ with negative Gaussian curvature.

7 Romero's proof of the Calabi-Bernstein theorem

In this section we will describe a recent simple approach to the two-dimensional version of the Calabi-Bernstein theorem given by Romero in [45]. His approach is based on the Liouville theorem on harmonic functions on $\mathbb{R}^2$, and it is inspired in a simple proof of the classical Bernstein theorem given by Chern [14].

Let $\psi : \Sigma^2 \to \mathbb{L}^3$ be a maximal surface oriented by its future-directed Gauss map N. For each fixed vector $a \in \mathbb{L}^3$, let us consider on Σ the smooth function $\langle a, N\rangle$, whose gradient is given by (5.1),

$$\nabla\langle a, N\rangle = -A(a^\top), \quad a^\top = a + \langle a, N\rangle N.$$

Thus $|a^\top|^2 = \langle a, N\rangle^2 + \langle a, a\rangle$ and

$$|\nabla\langle a, N\rangle|^2 = \langle A^2(a^\top), a^\top\rangle = K|a^\top|^2 = K(\langle a, N\rangle^2 + \langle a, a\rangle), \tag{7.1}$$

since $A^2X = KX$ for all $X \in \mathfrak{X}(\Sigma)$, because of $H = 0$. Therefore $\mathrm{tr}(A^2) = 2K$, and from (5.3) the Laplacian of $\langle a, N\rangle$ is given by

$$\Delta\langle a, N\rangle = 2K\langle a, N\rangle. \tag{7.2}$$

In particular, if $a \in \mathbb{L}^3$ is chosen to be lightlike ($\langle a, a\rangle = 0$ and $a \neq 0$) with $\langle a, N\rangle > 0$, then from (7.1) and (7.2) we get

$$\Delta\left(\frac{1}{\langle a, N\rangle}\right) = -\frac{\Delta\langle a, N\rangle}{\langle a, N\rangle^2} + \frac{2|\nabla\langle a, N\rangle|^2}{\langle a, N\rangle^3} = 0,$$

that is, $1/\langle a, N\rangle$ is a positive harmonic function globally defined on Σ.

On the other hand, if we choose now a future-directed unit timelike vector $b \in \mathbb{L}^3$, then $\langle b, b\rangle = -1$, $\langle b, N\rangle \leq -1$ and from (7.1) and (7.2) we obtain

$$\Delta\left(\log(1 - \langle b, N\rangle)\right) = \frac{\Delta\langle b, N\rangle}{\langle b, N\rangle - 1} - \frac{|\nabla\langle b, N\rangle|^2}{(\langle b, N\rangle - 1)^2} = K,$$

which means that the conformal metric

$$g^* = (1 - \langle b, N\rangle)^2 g \tag{7.3}$$

is a flat metric on Σ. Here g denotes the original Riemannian metric on Σ induced by the spacelike immersion ψ.

Moreover, since $1 - \langle b, N\rangle \geq 2$ on Σ, we also know that $g^* \geq 4g$, which implies that the flat metric g^* on Σ is also complete if the original metric g is complete. For that reason, if Σ is a complete maximal surface in $\mathbb{L}^3$, then we can conclude that (Σ, g^*) is globally isometric to the flat Euclidean plane $\mathbb{R}^2$. The invariance of harmonic functions under conformal changes of metric and that global isometry between Σ and $\mathbb{R}^2$ allow us to induce $1/\langle a, N\rangle$ on a positive harmonic function globally defined on $\mathbb{R}^2$, which by the Liouville theorem must be a constant. Hence $\langle a, N\rangle$ is a positive constant on Σ, which implies by (7.2) that K vanishes on Σ and the surface must be a totally geodesic spacelike plane.

It is worth pointing out that the completeness of the maximal surface $\psi : \Sigma^2 \to \mathbb{L}^3$ is used here only to assure the completeness of the conformal metric g^* and the simply-connectedness of Σ. Moreover, this second condition can be derived from the first one in the following way. Let $\psi : \Sigma^2 \to \mathbb{L}^3$ be a maximal surface in $\mathbb{L}^3$ such that the conformal flat metric g^* is complete on Σ and consider the universal covering projection $\Pi : \widehat{\Sigma} \to \Sigma$. Then $\Pi \circ \psi : \widehat{\Sigma} \to \mathbb{L}^3$ is a simply connected maximal surface for which the Riemannian metric $\widehat{g} = \Pi^*(g^*)$ is flat and complete. Then $(\widehat{\Sigma}, \widehat{g})$ is globally isometric to the flat Euclidean plane $\mathbb{R}^2$, and so the harmonic function $f = \Pi \circ (1/\langle a, N\rangle)$ on $\widehat{\Sigma}$ induces a positive harmonic function globally defined on $\mathbb{R}^2$. This tells us that $1/\langle a, N\rangle$ is constant on Σ, and hence the maximal surface $\psi : \Sigma^2 \to \mathbb{L}^3$ is totally geodesic. Finally, since N is constant and g^* is complete, we find that our maximal surface must be a spacelike plane.

All of this shows that Romero's proof also works under any other assumption assuring the completeness of g^*. This occurs, for instance, if one assumes the maximal surface to have closed image in $\mathbb{L}^3$, or more generally, if one assumes that the surface is complete with respect to the metric induced on Σ by the Euclidean metric in $\mathbb{R}^3$. In fact, if we denote by g_0 this last metric, some computations lead to

$$g^* = (1 - \langle b, N\rangle)^2 g \geq \langle b, N\rangle^2 g \geq \frac{1}{2} g_0. \tag{7.4}$$

Hence, g^* is complete on Σ if g_0 complete. Therefore, Romero's approach also allows us to obtain a simple direct proof of the following results, without using the general results by Cheng and Yau and Harris on the completeness of spacelike hypersurfaces (see the last paragraph of Section 2).

Corollary 7.1 *The only maximal surfaces in $\mathbb{L}^3$ which are complete with respect to the metric induced by the Euclidean metric in $\mathbb{R}^3$ are the spacelike planes.*

Corollary 7.2 *The only maximal surfaces in $\mathbb{L}^3$ whose image is closed in $\mathbb{L}^3$ are the spacelike planes.*

8 On the Gaussian curvature of maximal surfaces

A different approach to the Calabi-Bernstein theorem can be made by finding adequate local upper bounds for the Gaussian curvature of a maximal surface. In this section we shall describe two different results of this type. The first one is due to Estudillo and Romero [17], and consists on a pointwise estimate for the Gaussian curvature in terms of the distance of the point to the boundary of the surface. This result makes use of the Enneper-Weierstrass representation and the Schwarz lemma of complex analysis, and it is inspired in a paper by Osserman [43], where he proves an analogous result for minimal surfaces in $\mathbb{R}^3$. Actually the results in [17] are more elaborated that the one presented here, and involve the study of the Gauss map of the surface. Nevertheless, the result we give here suffices to prove the Calabi-Bernstein theorem as well as some of its generalizations.

Let $\psi : \Sigma^2 \to \mathbb{L}^3$ be a maximal surface in $\mathbb{L}^3$, and let us denote by (ω, g) its Enneper-Weierstrass data. Now consider on Σ the metric given by

$$d\sigma^2 = |\omega|^2.$$

This is a flat metric. In fact, since $\omega = f dz$ locally and the metric $|dz|^2$ is flat, then the Gaussian curvature K^* of $d\sigma^2$ satisfies

$$|f|^2 K^* = -\frac{1}{2}\Delta \log |f|^2 = 0,$$

where we have used that f is holomorphic. Thus $K^* = 0$ and $d\sigma^2$ is flat.

Now choose $p \in \Sigma$. We shall asume that $N(p) = (0,0,1)$, and thus $g(p) = 0$, by composing with a rigid motion in $\mathbb{L}^3$ if necessary. Next let us consider $\exp_p$, the exponential map of $(\Sigma, d\sigma^2)$ at p. It follows from the Cartan theorem that

$$\exp_p : V \subset T_p\Sigma \longrightarrow \Sigma$$

is a local isometry with respect to $d\sigma^2$. Here V is any neighbourhood of $0 \in T_p\Sigma$ over which $\exp_p$ is defined.

Let us choose a disc $\mathcal{D}(0,r) \subset V \subset T_p\Sigma$, and denote $W = \exp_p(\mathcal{D}(0,r))$. Then W is covered by coordinate systems

$$\{(V_\alpha, \exp_p^{-1})\} \equiv \{(V_\alpha, z_\alpha)\},$$

all of which are in the conformal atlas of Σ. This happens because every z_α is an isometry between the metric $|dz|^2$ of $T_p\Sigma \equiv \mathbb{C}$ and the flat metric $d\sigma^2$, which is conformal to the induced metric on Σ. In this way we can consider on every (V_α, z_α) its Enneper-Weierstrass data (f_α, g_α). Thus $f_\alpha dz_\alpha = \omega$ and $g_\alpha = g$ on V_α. Since z_α is an isometry we get $|dz_\alpha|^2 = |\omega|^2$, and from there $|\omega|^2 = |f_\alpha|^2 |dz_\alpha|^2 = |f_\alpha|^2|\omega|^2$. This shows that $|f_\alpha| = 1$ on V_α.

Under the identification $T_p\Sigma \equiv \mathbb{C}$, we can define the map

$$\widehat{g} : D(0,r) \longrightarrow \mathbb{D}$$

given by

$$\widehat{g}(w) = g(\exp_p(w)).$$

It is obvious that $\widehat{g}$ is holomorphic, with $\widehat{g}(0) = g(p) = 0$ and $\widehat{g}'(0) = g'(p)$. Once here let us define $G : \mathbb{D} \longrightarrow \mathbb{D}$ as

$$G(\eta) = \widehat{g}(r\eta).$$

Then G is holomorphic and satisfies $G(0) = 0$. Thus it follows from the Schwarz lemma of complex analysis that $|G'(0)| \leq 1$. Since $G'(0) = r\widehat{g}'(0)$ we find that

$$|g'(p)|^2 = |\widehat{g}'(0)|^2 \leq \frac{1}{r^2}. \tag{8.1}$$

Finally, taking into account that $g(p) = 0$ and $|f_\alpha| \equiv 1$ we obtain by means of (6.5) that

$$0 \le K(p) = 4|g'(p)|^2 \le \frac{4}{r^2}. \tag{8.2}$$

Of course, if the metric $d\sigma^2$ is complete we can choose r as large as we desire. This implies that $K(p) = 0$ at every point $p \in \Sigma$, and hence the surface must be a spacelike plane. Besides, it follows directly from (6.4) that the flat metric $d\sigma^2$ satisfies $ds^2 \le d\sigma^2$, where ds^2 is the induced metric of the surface. This provides us with an alternative proof of the Calabi-Bernstein theorem.

From (8.2) we can also find the following estimate for the Gaussian curvature of a maximal surface.

Theorem 8.1 *Let $\psi : \Sigma^2 \to \mathbb{L}^3$ be a maximal surface in $\mathbb{L}^3$. Then the Gaussian curvature of the surface at an arbitrary point $p \in \Sigma$ satisfies*

$$K(p) \le \frac{4}{d(p, \partial\Sigma)^2}.$$

Proof. We keep the notations of the above arguments. We know that if the flat metric $d\sigma^2$ is complete the surface is a spacelike plane, and hence the result holds trivially. If $d\sigma^2$ is not complete we can choose $\mathcal{D}(0, r)$ as the larger disc over which $\exp_p$ is defined. In this situation there exists some $w_0 \in \partial\mathcal{D}(0, r)$ such that the geodesic $\gamma(t)$ of $(\Sigma, d\sigma^2)$ given by

$$\gamma(t) = \exp_p(tw_0)$$

is defined on $[0, 1)$, but it cannot be extended to 1. This forbids that the geodesic $\gamma(t)$ lies in a compact set of Σ, and thus $\gamma(t)$ must be a divergent curve in Σ. Furthermore,

$$d(p, \partial M) \le L(\gamma) = \int_\gamma ds \le \int_\gamma |\omega| = r, \tag{8.3}$$

where we have used that $d\sigma^2(\gamma'(t), \gamma'(t)) = r^2$ for all t. The result then follows from (8.3) and (8.2).

$\square$

The estimate (8.2) is given in terms of the flat metric $d\sigma^2$. It is interesting to note that $d\sigma^2$ coincides up to a constant with the flat metric introduced in (7.3) for the choice $b = (0, 0, 1)$. In fact, taking into account (6.1) we get

$$\langle N, b\rangle = -\frac{1 + |g|^2}{1 - |g|^2}.$$

Thus

$$g^* = (1 - \langle N, b\rangle)^2 ds^2 = \frac{4}{(1 - |g|^2)^2} ds^2 = 4d\sigma^2.$$

In this way (7.4) turns into $d\sigma^2 \geq (1/8)g_0$, and we can reobtain Corollary 7.1 and Corollary 7.2.

The second approach to the Calabi-Bernstein theorem of this section has been recently obtained by Alías and Palmer [8]. It is based on an upper bound for the total curvature of geodesic discs in a maximal surface in $\mathbb{L}^3$, involving the local geometry of the surface and its hyperbolic image. Specifically, they have proved the following local integral inequality for the Gaussian curvature.

Theorem 8.2 *Let $\psi : \Sigma^2 \to \mathbb{L}^3$ be a maximal surface in the Lorentz-Minkowski space. Let p be a point of Σ and $R > 0$ be a positive real number such that the geodesic disc of radius R about p satisfies $D(p, R) \subset\subset \Sigma$. Then for all $0 < r < R$, the total curvature of a geodesic disc $D(p, r)$ of radius r about p satisfies*

$$0 \leq \int_{D(p,r)} K \mathrm{d}A \leq c_r \frac{L(r)}{r \log(R/r)}, \tag{8.4}$$

where $L(r)$ denotes the length of $\partial D(p, r)$, the geodesic circle of radius r about p, and

$$c_r = \frac{\pi^2}{8} \frac{(1 + \cosh^2 \varrho_r)^2}{\cosh \varrho_r \arctan(\cosh \varrho_r)} > 0.$$

Here ϱ_r denotes the radius of a geodesic disc in $\mathbb{H}^2_+$ containing the hyperbolic image of $D(p, r)$.

This local integral inequality (8.4) clearly implies the global Calabi-Bernstein theorem. Indeed, if Σ is complete, then R can approach to infinity in (8.4) for a fixed arbitrary $p \in \Sigma$ and a fixed r, which gives

$$\int_{D(p,r)} K \mathrm{d}A = 0.$$

Taking into account that the Gaussian curvature of a maximal surface in $\mathbb{L}^3$ is always non-negative, this yields $K = 0$ on Σ, and Σ must be a spacelike plane.

The proof of Theorem 8.2 is an application of the following (intrinsic) local integral inequality on an analytic Riemannian surface with non-negative Gaussian curvature.

Lemma 8.3 *Let Σ^2 be an analytic surface endowed with an analytic Riemannian metric with non-negative Gaussian curvature $K \geq 0$. Let u be a smooth function on Σ which satisfies*

$$u\Delta u \geq 0$$

on Σ. Then for $0 < r < R$

$$\int_{D_r} u\Delta u \leq \frac{2L(r)}{r\log(R/r)} \sup_{D_R} u^2,$$

where D_r denotes the geodesic disc of radius r about a fixed point in Σ, $D_r \subset D_R \subset\subset \Sigma$, and $L(r)$ denotes the length of ∂D_r, the geodesic circle of radius r.

The proof of Lemma 8.3 follows from Lemma 2.1 in [7] and inequality (2.4) in [7].
Proof of Theorem 8.2: Let us assume that the hyperbolic image of $D(p, r)$ is contained in a geodesic disc $\widetilde{D}(a, \varrho_r)$ in $\mathbb{H}^2_+$ of radius ϱ_r centered at the point $a \in \mathbb{H}^2_+$. Recall that

$$\widetilde{D}(a, \varrho_r) = \{x \in \mathbb{H}^2_+ : 1 \leq -\langle a, x\rangle \leq \cosh \varrho_r\},$$

so that $1 \leq -\langle a, N(q)\rangle \leq \cosh \varrho_r$ for all $q \in D(p, r)$.

Observe that since $\psi : \Sigma^2 \to \mathbb{L}^3$ is a maximal surface, Σ is an analytic Riemannian surface with non-negative Gaussian curvature, and we may apply Lemma 8.3 to an appropriate smooth function u. The idea of the proof is to apply it to the smooth function $u = \arctan(-\langle a, N\rangle)$, which by (7.1) and (7.2) satisfies

$$\Delta u = -\frac{1}{1+\langle a, N\rangle^2}\Delta\langle a, N\rangle + \frac{2\langle a, N\rangle}{(1+\langle a, N\rangle^2)^2}|\nabla\langle a, N\rangle|^2 = \frac{-4K\langle a, N\rangle}{(1+\langle a, N\rangle^2)^2},$$

and

$$u\Delta u = \phi(-\langle a, N\rangle)K \geq 0, \tag{8.5}$$

where $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is given by

$$\phi(t) = \frac{4t \arctan(t)}{(1+t^2)^2}$$

The function $\phi(t)$ is strictly decreasing for $t \geq 1$, so that for $t \in [1, \cosh \varrho_r]$ it is bounded from below by

$$\phi(t) \geq \phi(\cosh \varrho_r) = \frac{4 \cosh \varrho_r \arctan(\cosh \varrho_r)}{(1 + \cosh^2 \varrho_r)^2}.$$

Hence, at each point $q \in D(p, r)$ we get from (8.5)

$$u(q)\Delta u(q) \geq \frac{4 \cosh \varrho_r \arctan(\cosh \varrho_r)}{(1 + \cosh^2 \varrho_r)^2} K(q) \geq 0.$$

Integrating now this inequality over $D(p, r)$ and using Lemma 8.3 we conclude that

$$0 \leq \frac{4 \cosh \varrho_r \arctan(\cosh \varrho_r)}{(1 + \cosh^2 \varrho_r)^2} \int_{D(p,r)} K \mathrm{d}A \leq \int_{D(p,r)} u \Delta u \mathrm{d}A$$
$$\leq \frac{\pi^2}{2} \frac{L(r)}{r \log (R/r)},$$

that is,

$$0 \leq \int_{D(p,r)} K \mathrm{d}A \leq c_r \frac{L(r)}{r \log (R/r)}.$$

□

9 Further developments

In this section we include some additional results and topics which are related to the Calabi-Bernstein theorem. First of all, let us remark that the classical Bernstein theorem on minimal surfaces in the Euclidean space $\mathbb{R}^3$ can be seen as a consequence of the Calabi-Bernstein theorem on maximal surfaces in $\mathbb{L}^3$ (and viceversa). This follows from the following duality between solutions to their corresponding differential equations [9].

Theorem 9.1 *There exists an entire, nonlinear C^2 solution to the minimal surface equation*

$$\mathrm{Minimal}[u] = \mathrm{Div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0,$$

on $\mathbb{R}^2$ if and only if there exists an entire, nonlinear C^2 solution to the maximal surface equation

$$\mathrm{Maximal}[w] = \mathrm{Div}\left(\frac{Dw}{\sqrt{1-|Dw|^2}}\right) = 0, \qquad |Dw|^2 < 1$$

on $\mathbb{R}^2$.

Proof. Assume that u is an entire nonlinear solution of $\mathrm{Minimal}[u] = 0$ in the (x, y) plane. Recall that for a vector field X on $\mathbb{R}^2$ it holds that

$$(\mathrm{Div}X)dx \wedge dy = d\omega_{JX},$$

where J denotes the usual almost complex structure in the plane and ω_{JX} denotes the 1-form on $\mathbb{R}^2$ which is metrically equivalent to the field JX. Then, since the plane is simply connected, we can write

$$\frac{Du}{\sqrt{1+|Du|^2}} = J(Dw) \tag{9.1}$$

for a C^2 function w. Since J is an isometry, there follows

$$\frac{|Du|^2}{1+|Du|^2} = |Dw|^2 < 1,$$

and so

$$1 + |Du|^2 = \frac{1}{1-|Dw|^2}. \tag{9.2}$$

Using that $J^2 = -\mathrm{id}$, we obtain from (9.1),

$$J(Du) = -\sqrt{1+|Du|^2}Dw = -\frac{Dw}{\sqrt{1-|Dw|^2}},$$

and so $\mathrm{Maximal}[w] = 0$ follows.

If w were linear, then Dw is a constant vector, $|Dw|^2 \equiv$constant, and then it follows from (9.2) that $|Du|^2$ is a constant also. It then follows from (9.1) that Du is a constant vector, contradicting the assumption that u is nonlinear.

A very similar argument, starting with an entire solution of Maximal$[w] = 0$, produces an entire solution of Minimal$[u]$=0.

□

On the other hand, we introduce here an elementary proof of the Calabi-Bernstein theorem for maximal surfaces in $\mathbb{L}^3$. As we have seen throughout this paper, many of the known proofs of this theorem are non-trivial adaptations to the Lorentz-Minkowski space of proofs of the classical Bernstein theorem. The approach that we present here is inspired by a paper of Nitsche [38], where he proves the classical Bernstein theorem for minimal surfaces in $\mathbb{R}^3$. It is amazing that in our case the proof follows very closely the steps made by Nitsche.

Our proof is a consequence of the following result by Jörgens [23].

Lemma 9.2 (Jörgens theorem) *If $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfies the following differential equation*

$$\det(D^2\phi) = \frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2} - \left(\frac{\partial^2\phi}{\partial x \partial y}\right)^2 = 1,$$

then $\phi = \phi(x, y)$ is a quadratic polynomial in x and y.

Actually, let us assume that $u = u(x, y)$ is an entire solution to the maximal surface equation on $\mathbb{R}^2$, that is,

$$\operatorname{Div}\left(\frac{Du}{\sqrt{1 - |Du|^2}}\right) = 0, \qquad |Du|^2 < 1,$$

or equivalently,

$$(1 - |Du|^2)\Delta u + D^2u(Du, Du) = 0, \tag{9.3}$$

where $\Delta u = \operatorname{Div}(Du)$ is the Euclidean Laplacian of u, with $|Du|^2 < 1$. Then, if we define $W = \sqrt{1 - |Du|^2}$, we can easily see from (9.3) that

$$\frac{\partial}{\partial x}\left(\frac{1}{W}\left(1 - \left(\frac{\partial u}{\partial y}\right)^2\right)\right) + \frac{\partial}{\partial y}\left(\frac{1}{W}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right) = 0,$$

and

$$\frac{\partial}{\partial x}\left(\frac{1}{W}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{W}\left(1 - \left(\frac{\partial u}{\partial x}\right)^2\right)\right) = 0.$$

This assures the existence of certain functions $\alpha, \beta : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that

$$\frac{\partial \alpha}{\partial x} = \frac{1}{W}\left(1 - \left(\frac{\partial u}{\partial x}\right)^2\right), \qquad \frac{\partial \alpha}{\partial y} = -\frac{1}{W}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y},$$
$$\frac{\partial \beta}{\partial x} = -\frac{1}{W}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}, \qquad \frac{\partial \beta}{\partial y} = \frac{1}{W}\left(1 - \left(\frac{\partial u}{\partial y}\right)^2\right).$$

Again, from here we can assure that there exists a function $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfying $\partial\phi/\partial x = \alpha$ and $\partial\phi/\partial y = \beta$. Therefore

$$\begin{aligned} \frac{\partial^2 \phi}{\partial x^2} &= \frac{1}{W}\left(1 - \left(\frac{\partial u}{\partial x}\right)^2\right), \\ \frac{\partial^2 \phi}{\partial x \partial y} &= -\frac{1}{W}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}, \\ \frac{\partial^2 \phi}{\partial y^2} &= \frac{1}{W}\left(1 - \left(\frac{\partial u}{\partial y}\right)^2\right), \end{aligned} \tag{9.4}$$

which implies that $\det(D^2\phi) = 1$. Finally, by applying the Jörgens theorem we obtain that ϕ is a quadratic polynomial in x and y, which implies by (9.4) that u is linear in x and y.

Finally, let us note that the local surface theory in $\mathbb{L}^3$ is quite much more complicate than its Euclidean counterpart. This mainly happens because the presence in $\mathbb{L}^3$ of vectors with different causal characters usually turns into a wider variety of cases to consider. In [6] Alías, Chaves and Mira have recently introduced a complex representation formula for maximal surfaces in $\mathbb{L}^3$, obtained by modifying the Enneper-Weierstrass representation and based in the classical Björling problem for minimal surfaces (see [16, 39]). This representation formula in $\mathbb{L}^3$ is quite adequate for describing the local geometry of maximal surfaces, since it splits this geometric situation into simpler parts, that is, into one-dimensional objects.

To present this new approach to the local theory of maximal surfaces we first formulate the *Björling problem* in $\mathbb{L}^3$:

> Let $\beta : I \longrightarrow \mathbb{L}^3$ be a regular spacelike analytic curve in $\mathbb{L}^3$, and let $V : I \longrightarrow \mathbb{L}^3$ be a unit timelike analytic vector field along β such that $\langle \beta', V\rangle \equiv 0$. Construct a maximal surface in $\mathbb{L}^3$ containing β whose Gauss map along β is given by V.

This Björling problem turns out to have a unique solution, which provides the complex representation formula specified above. The solution to Björling problem for maximal surfaces in $\mathbb{L}^3$ states what follows [6].

Theorem 9.3 *Let $\beta : I \longrightarrow \mathbb{L}^3$ be a regular analytic spacelike curve in $\mathbb{L}^3$, and let $V : I \longrightarrow \mathbb{L}^3$ be a timelike analytic unit vector field along β such that $\langle \beta', V\rangle \equiv 0$. There exists a unique maximal surface whose image contains $\beta(I)$ and such that its Gauss map along β is V. This maximal surface is explicitly given by*

$$\chi(z) = \mathrm{Re}\beta(z) - \mathrm{Im}\int_{s_0}^{z} V(w) \times \beta'(w)dw. \tag{9.5}$$

Here $\beta(z), V(z)$ are holomorphic extensions of $\beta(s), V(s)$ over a certain simply connected open set $\Omega \subseteq \mathbb{C}$ containing I, and $s_0 \in I$ is fixed but arbitrary.

Several applications of this formula have been recently studied in [6, 36]. More specifically, one can use Theorem 9.3 mainly in two directions: the construction of maximal surfaces with prescribed geometric properties, and the obtention of new results on maximal surfaces. As an example of the latter we note the following consequence of Björling problem [36].

Theorem 9.4 *Let $\beta : I \longrightarrow \mathbb{L}^3$ be a regular analytic spacelike curve of a semi-Riemannian analytic surface $\Sigma \subset \mathbb{L}^3$, and choose $\varphi > 0$. There exist exactly two maximal surfaces in $\mathbb{L}^3$ that intersect Σ along β with constant angle φ. Both maximal surfaces can be explicitly constructed as solutions to adequate particular Björling problems.*

If Σ is a non-degenerate plane, those two maximal surfaces are congruent in $\mathbb{L}^3$.

Acknowledgments

The authors were partially supported by Dirección General de Investigación (MCYT), Grant N$\mathbb{S}$ BFM2001-2871 and Consejería de Educación y Universidades (CARM) Programa Séneca, Grant N$\mathbb{S}$ PI-3/00854/FS/01.

References

[1] R. Aiyama, *On complete spacelike surfaces with constant mean curvature in a Lorentzian 3-space,* Tsukuba J. Math. **15**, 235-247, (1991).

[2] R. Aiyama, *On the Gauss map of complete spacelike hypersurfaces of constant mean curvature in Minkowski space,* Tsukuba J. Math. **16**, 353-361, (1992).

[3] K. Akutagawa, *On spacelike hypersurfaces with constant mean curvature in the de Sitter space,* Math. Z. **196**, 13-19, (1987).

[4] J.A. Aledo and L.J. Alías, *On the curvatures of bounded complete spacelike hypersurfaces in the Lorentz-Minkowski space,* Manuscripta Math. **101**, 401-413, (2000).

[5] J.A. Aledo and J.A. Gálvez, *A Weierstrass representation for maximal linear Weingarten surfaces in the Lorentz-Minkowski space,* preprint (2001).

[6] L.J. Alías, R.M.B. Chaves and P. Mira, *Björling problem for maximal surfaces in Lorentz-Minkowski space,* to appear in Math. Proc. Cambridge Philos. Soc. **134**, (2003).

[7] L.J. Alías and B. Palmer, *Zero mean curvature surfaces with non-negative curvature in flat Lorentzian 4-spaces,* R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **455**, 631-636, (1999).

[8] L.J. Alías and B. Palmer, *On the Gaussian curvature of maximal surfaces and the Calabi-Bernstein theorem,* Bull. London Math. Soc. **33**, 454-458, (2001).

[9] L.J. Alías and B. Palmer, *A duality result between the minimal surface equation and the maximal surface equation,* An. Acad. Bras. Cien. **73**, 161-164, (2001).

[10] J.L. Barbosa and A.G. Colares, *Minimal surfaces in* $\mathbb{R}^3$, Lecture Notes in Mathematics **1195**. Springer-Verlag, Berlin, (1986).

[11] E. Bombieri, E. De Giorgi and E. Giusti, *Minimal cones and the Bernstein problem,* Invent. Math. **7**, 243-268, (1969).

[12] E. CALABI, *Examples of Bernstein problems for some nonlinear equations*, Proc. Symp. Pure Math. **15**, 223-230, (1970).

[13] S.Y. CHENG AND S.T. YAU, *Maximal spacelike hypersurfaces in the Lorentz-Minkowski spaces*, Ann. of Math. **104**, 407-419, (1976).

[14] S.S. CHERN, *Simple proofs of two theorems on minimal surfaces,* Enseign. Math. **15**, 53-61, (1969).

[15] S.M. CHOI, J-H KWON AND Y.J. SUH, *A Liouville-type theorem for complete Riemannian manifolds,* Bull. Korean Math. Soc. **35**, 301-309, (1998).

[16] U. DIERKES, S. HILDEBRANDT, A. KÜSTER AND O. WOHLRAB, *Minimal Surfaces I. Boundary value problems*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] **295**. Springer-Verlag, Berlin, (1992).

[17] F.J.M. ESTUDILLO AND A. ROMERO, *On the Gauss curvature of maximal surfaces in the* 3*-dimensional Lorentz-Minkowski space,* Comment. Math. Helv. **69**, 1-4, (1994).

[18] S.G. HARRIS, *Complete codimension-one spacelike immersions,* Class. Quantum Grav. **4**, 1577-1585, (1987).

[19] S.G. HARRIS, *Closed and complete spacelike hypersurfaces in Minkowski space,* Class. Quantum Grav. **5**, 111-119, (1988).

[20] S.G. HARRIS, *Complete spacelike immersions with topology,* Class. Quantum Grav. **5**, 833-838, (1988).

[21] T. IMAIZUMI, *Maximal surfaces with conelike singularities of finite type,* Kobe J. Math. **18**, 51-60, (2001).

[22] L.P.M. JORGE AND F. XAVIER, *A complete minimal surface in* $\mathbb{R}^3$ *between two parallel planes,* Ann. of Math. **112**, 203-206, (1980).

[23] K. JÖRGENS, *Über die Lösungen der Differentialgleichung* $rt - s^2 = 1$*,* Math. Ann. **127**, 130-134, (1954).

[24] U-H. KI, H-J KIM AND H. NAKAGAWA, *On spacelike hypersurfaces with constant mean curvature of a Lorentz space form,* Tokyo J. Math. **14**, 205-216, (1991).

[25] O. KOBAYASHI, *Maximal surfaces in the 3-dimensional Lorentz-Minkowski space* $\mathbb{L}^3$, Tokyo J. Math. **6**, 297-309, (1983).

[26] O. KOBAYASHI, *Maximal surfaces with conelike singularities,* J. Math. Soc. Japan **36**, 609-617, (1984).

[27] S. KOBAYASHI AND K. NOMIZU, *Foundations of Differential Geometry, Vol. II*, Interscience, New York, (1969).

[28] S. LIN AND T. WEINSTEIN, $\mathbb{E}^3$*-complete timelike surfaces in* $\mathbb{E}^3_1$ *are globally hyperbolic,* Michigan Math. J. **44**, 529-541, (1997).

[29] S. LIN AND T. WEINSTEIN, *A better conformal Bernstein's theorem,* Geom. Dedicata **81**, 53-59, (2000).

[30] F.J. LÓPEZ, R. LÓPEZ AND R. SOUAM, *Maximal surfaces of Riemann type in Lorentz-Minkowski space* $\mathbb{L}^3$, Michigan Math. J. **47**, 469-497, (2000).

[31] M.A. MAGID, *The Bernstein problem for timelike surfaces,* Yokohama Math. J. **37**, 125-137, (1990).

[32] J.E. MARSDEN AND F.J. TIPLER, *Maximal hypersurfaces and foliations of constant mean curvature in general relativity,* Phys. Rep. **66**, 109-139, (1980).

[33] L.V. MCNERTNEY, *One-parameter families of surfaces with constant curvature in Lorentz 3-space*, Ph.D. thesis, Department of Mathematics, Brown University, (1980).

[34] T.K. MILNOR, *A conformal analog of Bernstein's theorem for timelike surfaces in Minkowski 3-space,* The legacy of Sonya Kovalevskaya (Cambridge, Mass., and Amherst, Mass., 1985), 123-132, Contemp. Math. **64**, Amer. Math. Soc., Providence, RI, (1987).

[35] T.K. MILNOR, *Entire timelike minimal surfaces in* $\mathbb{E}^3_1$, Michigan Math. J. **37**, 163-177, (1990).

[36] P. MIRA, *Construction of maximal surfaces in the Lorentz–Minkowski space,* Geometry, Integrability and Quantization III, I.M. Mladenov and G.L. Naber (Editors), Coral Press Sci. Publ., Sofia, 337–350, (2001).

[37] S. NISHIKAWA, *On maximal spacelike hypersurfaces in a Lorentzian manifold,* Nagoya Math. J. **95**, 117-124, (1984).

[38] J.C.C. Nitsche, *Elementary proof of Bernstein's theorem on minimal surfaces,* Ann. of Math. **66**, 543-544, (1957).

[39] J.C.C. Nitsche, *Lectures on Minimal Surfaces*, Vol. I. Cambridge University Press, Cambridge, (1989).

[40] H. Omori, *Isometric immersions of Riemannian manifolds,* J. Math. Soc. Japan **19**, 205-214, (1967).

[41] B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, New York, (1983).

[42] R. Osserman, *A survey of minimal surfaces*, Dover Publications, New York, (1986).

[43] R. Osserman, *On the Gauss curvature of minimal surfaces,* Trans. Amer. Math. Soc. **12**, 115-128, (1960).

[44] B. Palmer, *The Gauss map of a spacelike constant mean curvature hypersurface of Minkowski space,* Comment. Math. Helv. **65**, 52-57, (1990).

[45] A. Romero, *Simple proof of Calabi-Bernstein's theorem on maximal surfaces,* Proc. Amer. Math. Soc. **124**, 1315-1317, (1996).

[46] A.E. Treibergs, *Entire spacelike hypersurfaces of constant mean curvature in Minkowski space,* Invent. Math. **66**, 39-56, (1982).

[47] T. Weinstein, *An introduction to Lorentz surfaces*, de Gruyter Expositions in Mathematics, **22**. Walter de Gruyter & Co., Berlin, (1996).

[48] Y.L. Xin, *On the Gauss image of a spacelike hypersurface with constant mean curvature in Minkowski space,* Comment. Math. Helv. **66**, 590-598, (1991).

[49] S.T. Yau, *Harmonic functions on complete Riemannian manifolds,* Comm. Pure Appl. Math. **28**, 201-228, (1975).

Proceedings of the meeting
Lorentzian Geometry-Benalmádena 2001
Benalmádena, Málaga, Spain
Pub. de la RSME, Vol. 5 (2003), 57-72

Dynamics and Geometry of order one RigidityRelativistic Particles

Manuel Barros[1]

[1]*Departamento de Geometría y Topología*
Universidad de Granada
18071, Granada, Spain
e-mail: mbarros@ugr.es

1 Motivation

The classical model describing the free fall relativistic particle, that is motion solely under the influence of gravity, is governed by the Lagrangian

$$\mathcal{F}(\gamma) = m \int_\gamma d\, s,$$

the world trajectories being geodesics of the spacetime. Recently, the interest in constructing models to describe spinning relativistic particles, both massive and massless, has been revived (see for example [6, 14, 15, 16, 17, 18, 20, 21, 22, 24] and references therein).

The conventional approach to these models consists in the extension of the initial spacetime by means of extra spinning degrees of freedom. However, there is another, less developed way to construct these models. The main idea is to consider Lagrangians, which are formulated in the initial spacetime, but depend on higher order derivatives. In other words, the action depend on the geometrical invariants, curvatures, of the particle trajectories,

$$\mathcal{N}(\gamma) = \int_\gamma \Omega(\kappa_1, \cdots, \kappa_{n-1}),$$

where κ_i, $1 \leq i \leq n-1$ denote the curvatures in n dimensions.

Several motivations should be considered when regarding this new approach. Between these, we remember here the following

1. The old idea of modelling the spinning particle without introducing new additional spinning variables.
2. The consideration of the above mentioned Lagrangians is strongly related to the development of the bosonic string theory in the sense of Polyakov.
3. The construction of models for supersymmetric particles.
4. The searching for models of particles with arbitrary fractional spin, anyons.
5. The theory of particles with maximal proper acceleration.

Certainly, the simplest models are those with Lagrangian density involving only the first curvature, that is the curvature κ which plays the role of proper acceleration of particle, of the worldlines. In this paper, I consider models for spinning relativistic particles, both massive and massless, that are governed by actions with densities being linear functions of the particle proper acceleration, that is they have rigidity of order one. In particular, the beautiful Lagrangian system with action measuring the total proper acceleration is known in the literature as the Plyushchay model and describe the spinning massless relativistic particle.

Obviously, my interest in these models come from mathematical reasons such as turning tangent or Fenchel program. Also because my interest in the theories of elastic curves, [1, 2, 4, 8] and in topics related with helices, [3, 4, 5].

When studying these models in backgrounds with constant curvature, I have learned several interesting properties on their dynamical behaviours so as on the geometry of their world trajectories.

1. The dynamics of these Lagrangian systems takes place in dimension three.

2. The geometry of the trajectories in the massive model is completely determined. Particles evolve along helices. The model is consistent no matter if the $D = 3$ background gravitational fields correspond with either Lorentz-Minkowski, De Sitter or anti De Sitter. Moreover, the moduli space of solutions may be determined from three different pairs of dependent moduli.

3. In contrast, the trajectories of the massless model, or the Plyushchay model, are not helices in general. The model is consistent only in anti De Sitter backgrounds. The moduli space of solutions being determined from a unique modulus that moves along the space of smooth functions on either a hyperbolic plane or an anti De Sitter plane.

4. One can obtain nice quantization principles. For example, the moduli subspace of closed trajectories, in the Plyushchay model on a $D = 3$ anti De Sitter space, is quantized via a rational constraint on one of the moduli.

5. An interesting algorithm to get explicit examples of closed solitons in Plyushchay model can be exhibited. This involves the following ingredients.

   - An isoareal mapping such as the hyperbolic Lambert projection.
   - The Hopf mapping from anti De Sitter space onto the hyperbolic plane.
   - A series of classical curves including the elliptic lemniscates (also called Perseo espiricas), the limaçon or snail of Pascal, the Vivianni ovoides in particular the folium simple of Descartes, the roses of Gido-Grandi, the quartics of Ruiz-Castizo...

Many of the results of this talk are contained in [6] (see also [1, 2]).

2 The models and the motion equations

Let $(M, g =<,>)$ be a semi-Riemannian manifold with Levi-Civita connection ∇ and curvature tensor R. In this setting, we consider dynamics associated with Lagrangian that depend linearly from the proper acceleration of particles. The space of elementary fields is a certain space of non-null curves in M, say Λ and the action is $\mathcal{L}_m : \Lambda \to \mathbb{R}$, given by

$$\mathcal{L}_m(\gamma) = \int_\gamma (\kappa(s) + m)\, d\, s,$$

where κ denotes the curvature of the curve $\gamma \in \Lambda$ and m is a coupling mass parameter which serves as a Lagrange multiplier or it is prescribed experimentally. The massless model of Plyushchay correspond with the case $m = 0$.

The field equations can be computed using a standard argument that involves some integrations by parts. Therefore, for $\gamma \in \Lambda$ and $W \in T_\gamma\Lambda$, we have

$$\begin{aligned}
\delta\mathcal{L}_m(\gamma)[W] &= \int_\gamma < \Omega(\gamma), W > d\, s + [\mathcal{B}(\gamma, W)]_0^L \\
&- \sum_{i=1}^{m} < \nabla_T W, N(s_i^+) - N(s_i^-) > \\
&+ \sum_{i=1}^{m} < W(s_i), \nabla_T N(s_i^+) - \nabla_T N(s_i^-) >,
\end{aligned}$$

where $\Omega(\gamma)$ and $\mathcal{B}(\gamma, W)$ stand by the Euler-Lagrange and the boundary operators, given respectively by

$$\Omega(\gamma) = \left(\varepsilon_2\varepsilon_3\tau^2 + \varepsilon_1\varepsilon_2 m\, \kappa\right) N - \varepsilon_3\tau_s B - \varepsilon_3\tau\, \eta - R(N, T)T,$$

$$\mathcal{B}(\gamma, W) = \varepsilon_3 < \nabla_T W, N > +\varepsilon_1 m < W, T > \varepsilon_3\tau < W, B >,$$

T is the unit tangent, N and B are unit normal and binormal respectively. Also, η is a section of the normal sub-bundle of γ which is orthogonal to that generated by $\{T, N, B\}$. Here τ is the torsion and $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ are the causal characters of T, N, B, respectively.

It should be noticed that, a priori, curves could have inflection points, $\gamma(s_i)$, $1 \leq i \leq m$. Therefore, if dimension of M is greater than two, then N is not defined in such points. However, this is not the case of trajectories. The trajectories of relativistic particles in the model $(M, g, \mathcal{L}_m)$ are nothing but the critical points of $\mathcal{L}_m : \Lambda \to \mathbb{R}$, that is, those curves $\gamma \in \Lambda$ that satisfy $\delta\mathcal{L}_m(\gamma)[W] = 0$, $\forall W \in T_\gamma\Lambda$.

On the other hand, we can consider suitable first order boundary conditions to drop out the boundary operator. In this case, it is suffices to take clamped curves, that is, curves that connect two fixed points and are tangent in these two points. Therefore, we have

Theorem 2.1 ***(First order boundary conditions or clamped curves).*** *Let q_1 and q_2 two points in M and choose unit vector $x_i \in T_{q_i}M$, $1 \leq i \leq 2$ to consider the space of curves*

$$\Lambda = \{\gamma : [t_1, t_2] \to M \, : \, \gamma(t_i) = q_i, \, T(t_i) = x_i, \, 1 \leq i \leq 2\}.$$

Then, we have

1. *If $\gamma \in \Lambda$ is the trajectory of a relativistic particle in $(M, g, \mathcal{L}_m)$, then N, B and τ are well defined along γ, even in the inflection points.*

2. *The boundary operator vanishes identically along the trajectories so the relativistic particles of the model $(M, g, \mathcal{L}_m)$ evolve along the curves that are solution of the following field equation*

$$R(N,T)T = \left(\varepsilon_2\varepsilon_3\tau^2 + \varepsilon_1\varepsilon_2 m\,\kappa\right) N - \varepsilon_3\tau_s B - \varepsilon_3\tau\,\eta,$$

Corollary 2.2 ***(Backgrounds with constant curvature.)*** *If (M, g) has constant curvature, say c, then the field equations describing the motion of relativistic particles in the model $(M, g, \mathcal{L}_m)$ are*

$$\begin{aligned} \varepsilon_2\varepsilon_3\tau^2 + \varepsilon_1\varepsilon_2 m\,\kappa &= \varepsilon_1 c, \\ \tau_s &= 0, \\ \eta &= 0. \end{aligned}$$

In particular, we have

1. *The motion of relativistic particles with order one rigidity in spaces with constant curvature takes place in totally geodesic three dimensional submanifolds, [10], which obviously have the same geometry of the big background, for example have constant curvature, c.*

2. *The massive ($m \neq 0$) relativistic particles with order one rigidity in spaces with constant curvature evolve along helices in spaces with dimension three. That is, trajectories have curvature and torsion, both, constant.*

At this point, we will restrict ourselves to $D = 3$ Lorentzian space forms. The curvature and the torsion of trajectories are not independent but obviously, they determine completely the geometry of these curves, up to congruences in the spacetime. Moreover, the particle spin, S and its mass, M, can be determined in terms of those invariants. The converse also holds, that is, one can determine the curvature and the torsion of trajectories in terms of S and M, (see [6]). Then, we obtain

Corollary 2.3 *In $D = 3$ spacetimes with constant curvature, the spinning massive relativistic particles with order one rigidity, evolve along helices. The geometry of a trajectory, (κ, τ) is equivalent to the particle dynamics, (M, S). Therefore, each solution of the field equations can be geometrically determined by the parameters (κ, τ), or equivalently by its dynamics parameters, (M, S). In this sense, the motion equation plays, in the worldline geometry, the role of Regge trajectory in the dynamics of the particle.*

3 The case of $\mathbf{AdS_3}$

The three dimensional anti de Sitter space, $\mathbf{AdS_3}$, with curvature -1 can be regarded as the hyperquadric in $\mathbb{C}_1^2$ given by

$$\mathbf{AdS_3} = \{z \in \mathbb{C}_1^2 \, : \, (z, z) = -1\},$$

endowed with the induced metric.

On the other hand, the hyperbolic plane, $\mathbb{H}^2$, and the pseudo-hyperbolic plane (or anti de Sitter plane), $\mathbb{H}_1^2$, can be viewed as orbit spaces obtained from $\mathbf{AdS_3}$. In fact, just consider the natural action of the unit circles $\mathbb{S}^1$ in $\mathbb{R}^2$ and $\mathbb{H}^1$ in $\mathbb{L}^2$ on $\mathbf{AdS_3}$ defined by

$$a.(z_1, z_2) = (a\, z_1, a\, z_2), \qquad a \in \mathbb{S}^1 \quad \text{or} \quad a \in \mathbb{H}^1.$$

Therefore, we obtain two called Hopf fibrations, [7],

$$\Pi_r : \mathbf{AdS_3} \to \mathbb{H}_r^2, \qquad r = 0, 1,$$

with fibre $\mathbb{S}^1$ and $\mathbb{H}^1$, respectively, where $\mathbb{H}_0^2 = \mathbb{H}^2$. These become into semi Riemannian submersions when $\mathbb{H}_r^2$ is endowed with the metric with curvature -4 (see [9, 19] for details on this topic). In particular, one can prove the following facts, [7],

- The horizontal lifts of regular curves in $\mathbb{H}^2$ are space Frenet curves in $\mathbf{AdS_3}$ which have torsion $\tau = \pm 1$.

- The horizontal lifts of regular curves in $\mathbb{H}^2_1$ are space or time Frenet curves in $\mathbf{AdS_3}$ which have torsion $\tau = \pm 1$.

Next, I will give an algorithm which allows to obtain geometrically all the helices in $\mathbf{AdS_3}$ (see [4, 5] for details).

1. First, notice that if β is a curve in $\mathbb{H}^2_r$, then $\Pi_r^{-1}(\beta)$ is a flat surface in $\mathbf{AdS_3}$. If $r = 0$, then $\Pi_0^{-1}(\beta)$ is Lorentzian, the Hopf tube of β, while if $r = 1$ then $\Pi_1^{-1}(\beta)$ is Riemannian or Lorentzian according to the causal character of β, in this case the surface is called a B-scroll, [13]. In both cases, the surface can be parametrized in $\mathbf{AdS_3}$ through fibres (s =constant) and horizontal liftings, $\bar{\beta}(s)$, of β.

$$\Phi(s,t) = \begin{cases} \cos(t)\bar{\beta}(s) + \sin(t) i\bar{\beta}(s), & \text{if } r = 0, \\ \cosh(t)\bar{\beta}(s) + \sinh(t) i\bar{\beta}(s), & \text{if } r = 1, \end{cases}$$

2. Now, if β has constant curvature, say ρ, in $\mathbb{H}^2_r$, then the non-null geodesics of $\Pi_r^{-1}(\beta)$ are helices in $\mathbf{AdS_3}$. In fact, let g be the slope of a geodesic, γ, in $\mathbb{H}^2_r$. Then, one can compute the curvature and the torsion of γ in $\mathbf{AdS_3}$ to be respectively

$$\kappa = \varepsilon_1 \varepsilon \frac{\rho + 2g}{\varepsilon - (-1)^r g^2},$$

   and

$$\tau = -(-1)^r \varepsilon_1 \varepsilon \frac{\varepsilon(-1)^r + g\rho + g^2}{\varepsilon - (-1)^r g^2}.$$

   where ε and ε_1 are the causal characters of β and γ, respectively.

3. The converse of the above fact also holds. Every helix, γ, of $\mathbf{AdS_3}$ is a geodesic in either a Hopf tube or in a B-scroll. In fact, let κ, τ and ε_1 be the curvature, the torsion and the causal character of γ. In $\mathbb{H}^2_r$, one considers a curve, β, with constant curvature

$$\rho = \varepsilon_1 \frac{\kappa^2 + (-1)^r(1 - \tau^2)}{\kappa},$$

   then, one chooses the geodesic in $\Pi_1^{-1}(\beta)$ determined by the slope

$$g = (-1)^r \frac{\varepsilon_1 + \tau}{\kappa}.$$

It is now easy to see the so obtained curve is congruent in $\mathbf{AdS_3}$ to γ.

Now, the field equation can be written in terms of the cylindrical coordinates, (ρ, g). A geodesic of either a Hopf tube or a B-scroll is a trajectory of a spinning, massive, relativistic particle with order one rigidity if and only if its slope, g, and the modulus curvature, ρ, satisfy the following equation

$$(2g + \rho)\left((\rho - \varepsilon_3\varepsilon(-1)^r m)g^2 + 2\varepsilon(-1)^r g + \varepsilon_3 m\right) = 0. \tag{3.1}$$

This algorithm can be combined with Corollary 3 to describe the complete moduli space of massive, relativistic particles of order one rigidity in $\mathbf{AdS}_n$. They evolve generating trajectories which are helices in $\mathbf{AdS}_3$. Moreover the moduli space of solutions can be described in terms of three couples of dependent parameters either:

- The curvature, κ, and the torsion, τ, of the world line whose dependence define pieces of parabola, or
- The mass, M, and the spin, S, whose dependence gives the Regge trajectory, or
- The cylindrical coordinates, (ρ, g), of the trajectory regarded as a geodesic of a Hopf tube or a B-scroll. In this case the constraint is given by (3.1).

4 Massive closed trajectories, a quantization principle

The Hopf map, $\pi_0 : \mathbf{AdS}_3 \to \mathbb{H}^2$, can be viewed as a principal fibre bundle on the hyperbolic plane with structure group $\mathbb{S}^1$ (a circle bundle). We define a vector potential, ω, on this bundle by assigning to each $z \in \mathbf{AdS}_3$ the horizontal 2-plane

$$\mathcal{H}_z = < iz >^{\perp},$$

the canonical principal connection. The field strength, Ω, of this principal connection is given by

$$\Omega = \pi_0^*(\Theta) \qquad \text{with} \qquad \Theta = -2\, dA,$$

where dA stands for the canonical area form on $\mathbb{H}^2$.

On the other hand, if $\beta : [0, L] \to \mathbb{H}^2$ is an immersed closed curve with length $L > 0$ (we always assume that β is parametrized by its arclength) and $\bar{\beta}$ denotes a horizontal lift of β, then the Lorentzian Hopf tube, $\mathcal{T}_\beta$, generated by β is a flat torus which is embedded in $\mathbf{AdS}_3$ if β is simple. To compute its isometry type, we essentially use the holonomy of the above described connection. In fact, notice that the mapping Φ when it is considered on the whole Lorentzian plane, $\mathbb{L}^2$ is nothing but a semi-Riemannian covering. The lines parallel to the t-axis in $\mathbb{L}^2$ are mapped by Φ onto the fibres of π_0, while the lines parallel to the s-axis in $\mathbb{L}^2$ are mapped by Φ onto the horizontal lifts of β. The later curves are not closed because the non-trivial holonomy of the involved vector potential, which was defined before. However, the non-closedness of the horizontal lifts of closed curves is measured just for the field strength. To see this, we will apply, without major details, a well known argument which is nicely exposited in [12]. According to that, there exists $\delta \in [-\pi, \pi)$ such that $\bar{\beta}(L) = e^{i\delta}\bar{\beta}(0)$, for any horizontal lift. The whole group of deck transformations of Φ is so generated by the translations $(0, 2\pi)$ and (L, δ). Finally, we have $\delta = \int_c \Theta$, where c is any 2-chain in $\mathbb{H}^2$ with boundary $\partial c = \beta$. In particular, we get $\delta = 2A$. Therefore, we have obtained the following result.

Proposition 4.1 *Let β be a closed immersed curve in $\mathbb{H}^2$ of length L and enclosing an area A. Then, the corresponding Hopf torus, $\mathcal{T}_\beta$ is isometric to $\mathbb{L}^2/\Gamma$, where Γ is the lattice in the Lorentzian plane, $\mathbb{L}^2 = \mathbb{R}^2_1$, generated by $(0, 2\pi)$ and $(L, 2A)$.*

Now, we can characterize those geodesics of a Hopf torus that are closed. In particular the closed helices in $\mathbf{AdS}_3$. To do it, suppose β is a closed curve, with constant curvature, ρ, in $\mathbb{H}^2$. This means that β is a geodesic circle of a certain radius, say $\varepsilon > 0$, in $\mathbb{H}^2$. Then its curvature is $\rho = -2\coth 2\varepsilon$, notice that we used suitable orientation to get negative values for curvature. The length of β is $L = \pi \sinh 2\varepsilon$ and the enclosed area in $\mathbb{H}^2$ is $A = \frac{\pi}{2}(\cosh 2\varepsilon - 1)$. Using the isometry type of $\mathcal{T}_\beta$, we see that a geodesic, γ, of $\mathcal{T}_\beta$ is closed if and only if there exists $s_o > 0$ such that $\Phi^{-1}(\gamma(s_o)) \in \Gamma$. Consequently

$$g = \frac{2\pi}{L}\left(q + \frac{A}{\pi}\right),$$

where q is a rational number which we call the rational slope.

This condition can be also written in terms of the the cylindrical coordinates, (g, ρ)

$$g = q\sqrt{\rho^2 - 4} - \frac{1}{2}\rho,$$

where $q \in \mathbb{Q} - \{0\}$.

It should be noticed that $\rho^2 > 4$, recall that $\mathbb{H}^2$ was chosen to have constant curvature -4. Hence, the field equation simplifies to

$$(\rho - \varepsilon_3 m)g^2 + 2g + \varepsilon_3 m = 0.$$

Then, we obtain the complete class of solutions that correspond with closed worldlines in the following quantization result

Proposition 4.2 *The complete space of closed trajectories correspond with a rational one-parameter family of helices in* $\mathbf{AdS}_3$*. These lie in Hopf tori on closed curves with constant curvature in* $\mathbb{H}^2$*, moreover they are geodesics in those tori and they are obtained when its slope is quantized via a rational constraint.*

5 Massless spinning particles

The Plyushchay model for massless spinning particle admits a consistent formulation in anti De Sitter backgrounds. The motion equations for Plyushchay's model, in a Loretzian-space-form (or space-time with constant sectional curvature), turn out to be

$$-\tau^2 = c, \qquad \tau' = 0, \qquad \delta = 0,$$

where recall that c is nothing but the background constant sectional curvature. These equations have strong consequences which can be summarized as follows:

The Lorentzian plane, $\mathbb{L}^2$, and the three-dimensional anti De Sitter space, $\mathbf{AdS}_3$, are the only of these backgrounds that a priori could admit a consistent formulation for Plyushchay's model massless spinning particle.

However, the case of the Lorentzian plane is trivial and it could be considered as a preambule of Fenchel's theory, [11]. Therefore, we only need to consider the case of $\mathbf{AdS}_3$. In this case, we consider without loss of generality $c = -1$, the motion equations reduce to

$$\tau = \pm 1,$$

and no information on the proper acceleration of particles is obtained. Therefore, the Plyushchay model lie to put the following problem: *to classify those curves in* $\mathbf{AdS}_3$ *with torsion* $\tau = \pm 1$. This innocent question is not trivial. However its solution involves the nice powerful of the geometry of $\mathbf{AdS}_3$, [23]. In particular, the main tools to solve this problem are both: (i) The high rigidity of the standard gravitational field on AdS_3 and (ii) The nice geometry associated with the Hopf mappings. This allow us to generate a beautiful argument to obtain the whole moduli space of massless spinning particles for the Plyushchay model.

First, it should be observed that any horizontal lift via π_r of any curve in $\mathbb{H}^2_r$, $r = 0, 1$, has torsion $\tau = \pm 1$ and so automatically gives a worldline of a massless spinning particle evolving in $\mathbf{AdS}_3$. Conversely, let assume that α is the worldline of a massless spinning particle in $\mathbf{AdS}_3$, then its torsion is $\tau = \pm 1$. Denote by κ^* its curvature function and take γ to be a curve in $\mathbb{H}^2_r$ so that its curvature function is $\kappa = \pi_r \circ \kappa^*$. Finally, choose a horizontal lift, say $\bar{\gamma}$, of γ. Since α and $\bar{\gamma}$ have the same curvature, κ^*, and torsion, $\tau = \pm 1$, then they must be congruent in $\mathbf{AdS}_3$.

It should be noticed that, in contrast with the massive models where two dependent real moduli describe the space of solutions, now the only modulus moves along the space of smooth functions from, say $\mathbb{R}$, in $\mathbb{H}^2_r$.

Proposition 5.1 *For any* $\varrho \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{H}^2_r)$, *denote* α^ϱ_r *the curve (up to congruence) with curvature function* ϱ. *Then the moduli space of trajectories in the Plyushchay model is*

$$\mathcal{MP} = \bigcup_{r=0}^{1} \{\bar{\alpha}^\varrho_r \, : \, \varrho \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{H}^2_r)\}.$$

6 Closed trajectories in the Plyushchay model

The moduli subspace made up of closed solutions can be nicely determined as follows. Let γ be a closed curve in the hyperbolic plane with length L and enclosing an area A. If *gam̄ma* is any horizontal lift of γ, then $\bar{\gamma}(L) = e^{i\delta}\bar{\gamma}(0)$, recall that δ is the holonomy number of the above described vector potential on the circle principal bundle given by the Hopf map $\pi_0 : AdS_3 \to \mathbb{H}^2$. Now, $\bar{\gamma}$ closes up if and only if there exists $n \in \mathbb{N}$ such that after n consecutive liftings of γ (that means, we lift the n-fold cover of γ) we get $\bar{\gamma}(n.L) = e^{in\delta}\bar{\gamma}(0) = \bar{\gamma}(0)$. Then $\delta = \frac{2\pi p}{n}$ for a certain integer p. On the other hand, we already know that $\delta = 2A$. Consequently, we obtain $A = \frac{p}{n}\pi$. Hence, we have the following quantization condition to obtain the moduli subspace of closed solutions in the Plyushchay model for massless spinning particle

Proposition 6.1 *The subspace of closed worldlines is obtained when we lift, some fold cover of closed curves in $\mathbb{H}^2$ which bounded an area that is a rational multiple of π.*

To illustrate the above result, we will exhibit some explicit examples.

Example 6.2 A rational one-parameter class of closed helices. The enclosed area of a geodesic circle, γ, with radius $\varepsilon > 0$ in $\mathbb{H}^2$ is given by $A = (\cosh 2\varepsilon - 1)\frac{\pi}{2}$. Thus a horizontal lift of the n-fold cover of γ closes if and only if $\frac{1}{2}(\cosh 2\varepsilon - 1) = \frac{p}{n}$. We now solve this equation in ε to obtain a rational one-parameter family of radii whose circles lift to closed worldlines of massless spinning particles in $\mathbf{AdS}_3$. Notice that these world trajectories are helices in anti De Sitter background.

To better understand the next examples, let consider

$$\mathbb{H}^2 = \{(x, y, z) \in \mathbb{L}^3 \,/\, x^2 + y^2 - z^2 = -\frac{1}{4} \quad \text{and} \quad z > o\}.$$

Pseudo-spherical coordinates, (φ, θ) can defined on $\mathbb{H}^2$ by putting $x = -\frac{1}{2}\cos\varphi\cosh\theta$, $y = -\frac{1}{2}\sin\varphi\cosh\theta$, $z = \cosh\theta$. On the other hand, by considering cylindrical coordinates, (φ, z) in the Euclidean plane, $\mathbb{R}^2$, we can define a kind of hyperbolic Lambert map, $L : \mathbb{H}^2 \to \mathbb{R}^2$, by $L(\varphi, \theta) = (\varphi, \frac{1}{2}\cosh\theta)$. A simple computation shows that this map preserve the areas of domains, in other words, it is an iso-areal mapping.

Example 6.3 The hyperbolic elliptic lemniscate. In pseudo-spherical coordinates, (φ, θ) on $\mathbb{H}^2$, we consider the curve given by

$$\gamma : \frac{1}{4}(\varphi^2 + \cosh^2\theta)^2 = a^2\cosh^2\theta + b^2\varphi^2,$$

with parameters a and b satisfying $b^2 \geq 2a^2$. This curve is nothing but the image under an appropriate hyperbolic Lambert map of an elliptic lemniscate in the Euclidian plane (that is the inverse curve of an ellipse, of axis $2a$ and $2b$, with respect to its centre). The area enclosed by γ in $\mathbb{H}^2$ is $A = \frac{1}{2}(a^2 + b^2)\pi$. Therefore, if we choose the axis such that $a^2 + b^2$ is a rational number, say $\frac{p}{q}$, with $a^2 + b^2 \leq 1$, then, a horizontal lift of the $2q$-fold cover of γ gives a closed worldline of a massless spinning particle evolving in $\mathbf{AdS}_3$.

Example 6.4 The hyperbolic limaçon or the hyperbolic snail of Pascal. In $\mathbb{H}^2$, we consider the curve that in pseudospherical coordinates is defined by

$$\gamma : (\frac{1}{2}\varphi^2 + \frac{1}{2}\cosh^2\theta - 2a\varphi^2)^2 = h^2(\varphi^2 + \cosh^2\theta),$$

for suitable parameters a and h. This curve closes because it is obtained as the image, under a hyperbolic Lambert mapping, of a limaçon of Pascal (the inverse curve of an ellipse with respect to a focus). Hence, this curve encloses, in $\mathbb{H}^2$ the area $A = (h^2 + \frac{1}{2}a^2)\pi$. Again, for a suitable choice of parameters and by lifting to $\mathbf{AdS}_3$, we get closed worldlines of the Plyushchay model for massless spinning particle.

Example 6.5 The hyperbolic folium. This element, of the moduli subspace of closed solutions, is obtained, via a hyperbolic Lambert map, from the folium simple in the Euclidean plane. In pseudospherical coordinates it is defined by

$$\gamma : \frac{1}{2}(\varphi^2 + \cosh^2\theta)^2 = a\varphi^3.$$

This curve is closed ant it encloses an area, $A = \frac{5}{32}a^2\pi$. For example, if $a = 1$, then a horizontal lift of the 32-fold cover is closed and so it provides a member of the above mentioned moduli space of massless spinning particles.

Example 6.6 The hyperbolic roses. Let n be an integer and define, in the Euclidean plane, the curve

$$\beta : \varepsilon = c \sin n\psi,$$

where (ε, ψ) stand for polar coordinates in the plane and c denotes a real number. This curve defines a rose in the plane with n petals if n is odd and $2n$ petals when n is even. These curves are also called clover curves (for example, if $n = 2$ we obtain the four-leaved clover, while if $n = 3$ we get the three-leaved clover or trefoil, which are very important when plotting tensor properties of quaternary and ternary cristals): The enclosed area in the plane by a clover curve is $A = \frac{1}{4}c^2\pi$ if n is odd and $A = \frac{1}{2}c^2\pi$ if n is even. Now, we take the image, under a hyperbolic Lambert map, of a suitable clover curve, to obtain closed curves in $\mathbb{H}^2$ enclosing the same area. Finally, we choose c^2 to be a rational number and lift them to $\mathbf{AdS}_3$, via the Hopf mapping, to get nice examples of solutions for Plyushchay model of massless spinning particles.

Acknowledgments

This work has been partially supported by the DGICYT.

References

[1] J.Arroyo, M.Barros and O.J.Garay, *Some examples of critical points for the total mean curvature functional.* Proc. Edinburgh Math. Soc. 43 (2000), 587-603.

[2] J.Arroyo, M.Barros and O.J.Garay, *Holography and total charge.* Journal of Geometry and Physics 41 (2002), 65-72.

[3] M.Barros, *General helices and a theorem of Lancret.* Proc.A.M.S. 125 (1997), 1503.

[4] M.Barros, *Free elasticae and Willmore tori in warped product spaces.* Glasgow Math.J. 40 (1998), 265-272.

[5] M.Barros, *Willmore-Chen branes and Hopf T-duality.* Class. Quantum Grav. 17 (2000), 1979-1988.

[6] M.Barros, *Geometry and Dynamics of relativistic particles with rigidity.* General Relativity and Gravitation 34 (2002), 837-852.

[7] M.Barros, A.Ferrández, P.Lucas and M.A.Meroño, *Hopf cylinders, B-scrolls and solitons of the Betchov-Da Rios equation in the three-dimensional anti De Sitter space.* C.R.Acad.Sci. Paris 321 (1995), 505.

[8] M.Barros, J.L.Cabrerizo and M.Fernández, *Elasticity and conformal tension via the Kaluza-Klein mechanism.* Journal of Geometry and Physics 34 (2000), 111.

[9] A.L.Besse, *Einstein manifolds.* Springer-Verlag, 1987.

[10] J.Erbacher, *Reduction of the codimension of an isometric immersion.* J. Differential Geometry 5 (1971), 333.

[11] W.Fenchel, *Über die Krümmung und Windung geschlossenen Raumkurven.* Math.Ann. 101 (1929), 238.

[12] W.Greub, S.Halperin and R.Vanstone, *Connections, Curvature and cohomology.* 3 Vols. Academic Press, New-York, 1972,1973,1976.

[13] L.Graves, *Codimension one isometric immersions between Lorentz spaces.* Trans.Amer.Math.Soc. 252 (1979), 367.

[14] A.Nersessian, *Massless particles and the geometry of curves. Classical pictures.* QFTHEP'99, Moscow 1999. hep-th/9911020.

[15] A.Nersessian, *D-dimensional massless particle with extended gauge invariance.* Czech.J.Phys. 50 (2000), 1309.

[16] A.Nersessian and E.Ramos, *A geometrical particle model for anyons.* Mod.Phys.Lett. A 14 (1999), 2033.

[17] V.V.Nesterenko, A.Feoli and G.Scarpetta, *Dynamics of relativistic particle with Lagrangian dependent on acceleration.* J.Math.Phys. 36 (1995), 5552-5564.

[18] V.V.Nesterenko, A.Feoli and G.Scarpetta, *Complete integrability for Lagrangian dependent on acceleration in a spacetime of constant curvature.* Class.Quant.Grav. 13 (1996), 1201-1212.

[19] B.O'Neill, *Semi-Riemannian geometry.* Academic Press, 1983.

[20] M.S.PLYUSHCHAY, *Massless point particle with rigidity.* Mod.Phys.Lett. 4 (1989), 837.

[21] M.S.PLYUSHCHAY, *Massless particle with rigidity as a model for the description of bosons and fermions.* Phys.Lett. B 243 (1990), 383-388.

[22] M.S.PLYUSHCHAY, *Comment on the relativistic particle with curvature and torsion of world trajectory.* hep-th/9810101.

[23] S.F.PROKUSKIN AND M.A.VASILIEV, *Currents of arbitrary spin in AdS_3.* Phys.Lett. B 464 (1999), 53.

[24] E.RAMOS AND J.ROCA, *W-symmetry and the rigid particle.* Nuclear Physics B 436 (1995), 529.

Proceedings of the meeting
Lorentzian Geometry-Benalmádena 2001
Benalmádena, Málaga, Spain
Pub. de la RSME, Vol. 5 (2003), 73-83

Causal tensors in Lorentzian geometry

Göran Bergqvist[1]

[1] *Matematiska Institutionen, Linköpings Universitet*
SE-58183 Linköping, Sweden

Abstract

We discuss the use and construction of positive geometric quantities in Lorentzian geometry. Basic results for causal tensors and superenergy tensors are presented, and applications to classical and generalised Rainich theory, a way of geometrising physics, is described.

1 Introduction

Positive quantities are fundamental in many parts of mathematics. In this paper we discuss their constructions and use in Lorentzian geometry. Sometimes the geometry is used as a model in a physical theory, and in this case the quantities may correspond to a physical quantity. We will here however emphasise that the positive geometric quantities we study do not need to have physical interpretations themselves, even if they are useful to prove results about other quantities which do have physical meaning.

We will first briefly present some examples of well known and very important mathematical results in general relativity. Common for these is that positivity properties of certain geometric quantities are used or assumed. The examples are the singularity theorems, the positive energy theorem, the stability of Minkowski spacetime, and the Penrose inequality.

Causal tensors are tensors with a certain natural positivity property. A fundamental way of constructing a causal tensor $T\{A\}$ from any tensor A on a Lorentzian manifold is the so-called superenergy tensor construction, developed in full generality by Senovilla [21]. We review some general results on causal tensors and superenergy tensors and present some recent generalisations.

As an application we then discuss the algebraic Rainich theory. This was originally presented as a theory which gave necessary and sufficient conditions on the Ricci curvature tensor to correspond via Einstein's equations to a spacetime describing an Einstein-Maxwell field (an electromagnetic field on a curved spacetime). We present various generalisations, which are most naturally expressed in terms of superenergy tensors, and discuss some open problems and possibilities for future work.

We use the index free notation as far as possible, thus adopting to the tradition of Riemannian and Lorentzian geometry rather than to the tradition of general relativity. However, we sometimes find that the use of indices simplifies an expression, such indices may however, unless explicitly stated otherwise, be considered as abstact labels which tells us what type of tensors we are using or how mappings should be applied [16]. They do not refer to any basis or system of coordinates, not even implicitly. We will assume that our Lorentzian manifolds have metrics g of signature $+, -, \ldots, -$ and our sign conventions for the curvature tensors are those of Penrose and Rindler [16]. We denote the Ricci curvature tensor by Ric, the scalar curvature by R and the Weyl curvature by C.

2 Positive geometric quantities and some important results in mathematical relativity

2.1 The singularity theorems

We first consider one version of the singularity theorems, first developed by Penrose and Hawking in the 1960's [8, 13].

Theorem 2.1 *Assume that M is a globally hyperbolic spacetime and that $-Ric(u, u) \geq 0$ for any timelike vector u. If H is a Cauchy hypersurface with $K \leq c < 0$, where K is the trace of the extrinsic curvature and c is a constant, then no past directed geodesic can have length greater than $3/|c|$ from H.*

We note that the spacetime M is a four dimensional Lorentzian manifold, the dimension being essential for the proof. The positivity condition $-Ric(u,u) \geq 0$ is called the strong energy condition. By Einstein's equations $Ric - \frac{1}{2}Rg = kT$, k a constant and T the energy-momentum tensor, this condition can be expressed $T(u,u) \geq \frac{1}{2}tr(T)g(u,u)$ and it is an *assumption* on the matter fields on the spacetime. The condition is used to control the focusing of nearby geodesics as given by the Raychaudhuri equation [8]. For a discussion on the physical aspects on the assumption, see [20]. The interpretation of the assumption $K \leq c < 0$ is that the universe is expanding and the conclusion that its age is finite. No symmetry assumptions are made. There are other versions of the theorem, some with weaker assumptions, see [8, 20]. If the time direction is reversed one obtains theorems predicting singularities in black holes. For a recent comprehensive review on the singularity theorems we refer to Senovilla [20].

2.2 The positive energy theorem

The next example is the positive energy theorem [19, 22]. Here we must again be very brief and refer to the literature for definitions of various concepts. An asymptotically flat spacetime is a model for an isolated object and has a curature which in a precise sense tends to zero at "large distances" from some point, something which does not have an obvious formulation for the curvature along null directions [16]. The total energy or momentum of the spacetime can be expressed as limits of certain curvature expressions and can be thought of as the energy measured by an observer at a large (infinite) distance from the object. At spacelike infinity one defines the ADM momentum and at null infinity the Bondi momentum.

Theorem 2.2 *Let M be an asymptotically flat spacetime and suppose that $T(u,v) \geq 0$ for all causal future directed vectors u and v. Then the total momentum P (ADM or Bondi) is causal and future directed.*

The conclusion means that any observer at infinity measures a positive total energy.

The condition $T(u,v) \geq 0$ on the energy-momentum tensor is called *the dominant energy condition* and is an *assumption* on the matter fields on the spacetime. It is considered to be a generally accepted condition physically.

Penrose [15] uses a different assumption.

2.3 Stability of Minkowski spacetime

Christodoulou and Klainerman [5] have proved a global existence result for Einstein's vacuum equations $Ric = 0$, which is interpreted as a stability result for Minkowski spacetime. The statement is essentially the following:

Given the initial data $C = 0$ outside a compact subset H_1 of a Cauchy hypersurface H and C "small" on H_1, then there exists a global non-singular solution to $Ric = 0$ in the future of H.

The proof is very long and techically extremely complicated [5, 10]. The basic ingredient used to define norms and to find energy estimates is the Bel-Robinson (superenergy) tensor, which is defined as

$$\mathcal{T}_{abcd}\{C\} = C_{aecf}C_b{}^e{}_d{}^f + {}^*C_{aecf}{}^*C_b{}^e{}_d{}^f, \tag{2.1}$$

where ${}^*C$ is the dual of the Weyl tensor C (it has only one dual in four dimensions).

$\mathcal{T}$ is completely symmetric, trace-free, and satisfies

$$\mathcal{T}(u^{(1)}, u^{(3)}, u^{(3)}, u^{(4)}) \geq 0 \tag{2.2}$$

for all future directed vectors $u^{(1)}, \ldots, u^{(4)}$. This is not an assumption but a *property* of $\mathcal{T}$, as well as of any superenergy tensor in any dimension [1, 17, 21].

General superenergy tensors $T\{A\}$ have also been used to find criteria for causal propagation of fields A on Lorentzian manifolds [3].

2.4 The Penrose inequality

Our last example is the Penrose inequality. If S is a marginally trapped surface (two dimensional spacelike) in a spacetime of total mass m, then Penrose [14] has conjectured that

$$Area(S) \leq 16\pi m^2. \tag{2.3}$$

The conjecture has its origin in the fact that the area of a black hole is increasing and that the final (stationary) state is given by a Kerr spacetime where the inequality always holds.

Special cases have been proven [9, 11], but the general case is still unproven, see Fraundiener [6].

A common idea in the proofs of these special cases is to use Hawking's quasilocal mass, $m_H(S) = c(Area(S))^{1/2}(2\pi - \int_S \rho\mu)$, where c is a constant, and ρ and μ the expansions of outgoing and incoming null geodesics orthogonal to S [7]. The Hawking mass has monotonicity properties along families of 2-surfaces from S to infinity, and it approaches the total mass at infinity.

In this case we want to prove a postivity property of the total mass by using that a derivative of the Hawking mass is positive.

A related problem in general relativity is that is impossible to define a pointwise energy or mass density. The total mass is well defined for an asymptotically flat spacetime but it is not known if the mass contained in a finite 3-volume is well defined, or rather the mass within a spacelike 2-surface [14].

3 Causal tensors

In Lorentzian geometry a natural positivity concept, which generalises properties in the examples above, is the following.

Definition 3.1 *A tensor T of order r is said to have the <u>dominant property</u>, $T \in DP$, if*

$$T(u^{(1)}, \ldots, u^{(r)}) \geq 0 \tag{3.1}$$

for all future directed vectors $u^{(1)}, \ldots, u^{(r)}$. T is <u>causal</u> if $T \in DP$ or $-T \in DP$.

For symmetric tensors of order 2, $T \in DP$ is equivalent to the dominant energy condition.

Some basic properties of causal tensors are [4]:

(i) $T \in DP \iff T(k^{(1)}, \ldots, k^{(r)}) \geq 0$ for all future directed *null* vectors $k^{(1)}, \ldots, k^{(r)}$.

(ii) $0 \neq T \in DP \iff T(v^{(1)}, \ldots, v^{(r)}) > 0$ for all future directed *timelike* vectors $v^{(1)}, \ldots, v^{(r)}$.

(iii) $T \neq 0$ antisymmetric in two positions $\Longrightarrow T \notin DP$.

(iv) $T \in DP \iff C_i^j(T \otimes t) \in DP$ for any $t \in DP$, where C_i^j is a contraction over one index in T and one in t. Especially for tensors of order 2 we have $T \in DP \Longrightarrow T^2 \in DP$.

The superenergy tensor $T\{\Omega\}$ of a p-form Ω, $1 \leq p \leq N$, is a symmetric tensor of order 2 which, in arbitrary dimension N, can be written

[4, 21]

$$T_{ab}\{\Omega\} = k(\Omega_{aa_2\ldots a_p}\Omega_b{}^{a_2\ldots a_p} - \frac{1}{2p}\Omega_{a_1\ldots a_p}\Omega^{a_1\ldots a_p}g_{ab}), \tag{3.2}$$

where k is a constant depending on p and the signature of the metric. Note that $T_{ab}\{\Omega\} = T_{ab}\{{}^*\Omega\}$, where ${}^*\Omega$ is the Hodge dual of Ω.

As mentioned above, $T\{\Omega\} \in DP$ as any superenergy tensor is in DP [1, 17, 21]

A well known result is the following [16].

Theorem 3.2

$$T\{\Omega\}^2 = h^2 g \textit{ for any } p\textit{-form } \Omega \textit{ if } N \leq 4. \tag{3.3}$$

This does not hold if $N \geq 5$ [4].

Recall that a p-form $T\{\Omega\}$ is *simple* if $T\{\Omega\} = \omega^{(1)} \wedge \cdots \wedge \omega^{(p)}$ where $\omega^{(j)}$ are 1-forms.

Theorem 3.3

$$T\{\Omega\}^2 = h^2 g \textit{ for any simple } p\textit{-form } \Omega \textit{ for any } N. \tag{3.4}$$

We know that $T\{\Omega\} \in DP$, conversely we can represent any symmetric tensor in DP with a sum of superenergy tensors of simple forms [4].

Theorem 3.4 *$S \in DP$ symmetric $\Longrightarrow$*

$$S = \sum_{p=r}^{N} T\{\Omega_{[p]}\}, \tag{3.5}$$

where $T\{\Omega_{[p]}\}$ are superenergy tensors of simple p-forms and r is the number of null eigenvectors (except that $r = 1$ also in case there is no null eigenvector). $\Omega_{[p]}$ can be constructed explicitly from the null eigenvectors.

4 Rainich theory and generalisations

The problem of finding all four dimensional Lorentzian manifolds with $Ric = 0$ may of course be considered as a purely mathematical problem. However, this problem also has a fundamental meaning in physics; such manifolds describe all vacuum spacetimes in general relativity.

One can weaken the conditions on the Ricci curvature in the following way [12].

Definition 4.1 *A* <u>*Riemann-Rainich space*</u> *is a four dimensional Lorentzian manifold on which the Ricci curvature tensor satisfies*

$$(alg.) \quad R = tr(Ric) = 0, \quad (Ric)^2 = \frac{1}{4} tr(Ric^2) g, \quad (Ric)_{00} \le 0 \quad (4.1)$$

$$(diff.) \quad curl(w/tr(Ric^2)) = 0; \quad w_a = e_{ca}{}^{pq}(Ric)_{pq} \nabla_q (Ric)^{bc} \quad (4.2)$$

Here e is completely antisymmetric of order 4 [16], and $(Ric)_{00}$ is the pure time component of Ric with respect to some ON basis.

Again this may be seen as a purely mathematical concept. Its importance is explained by an old result of Rainich [18]:

Theorem 4.2 *If $tr(Ric^2) \neq 0$ then M is a Riemann-Rainich space if and only if $Ric - \frac{1}{2}Rg = -T\{F\}$, where F is a 2-form which satisfies Maxwell's equations (and note that in fact $tr(Ric) = 0$).*

Therefore, Riemann-Rainich spaces describe gravitation *and* electromagnetism as pure geometry, an *already* unified theory according to Misner and Wheeler [12].

It is easy to prove that if F is a 2-form, then $tr(T\{F\}) = 0$ and $T\{F\}^2$ is proportional to the metric g. Rainich proved the converse but it was not widely known before Misner and Wheeler published their paper. Soon thereafter, other physical fields were geometrised (see [4] for references). In order to have conditions $(alg.)$ or $(diff.)$ that are independent of the field equations, the conditions are usually expressed in terms of the energy-momentum tensor T (in the case of an electromagnetic field the conditions for T and Ric are identical).

We will now show some ways of generalising the algebraic part of the above theory. The complete solution of the equation $T^2 = fg$ with T symmetric is given by the following theorem.

Theorem 4.3 *Suppose that T is symmetric of index 2. Then, for any dimension N, we have $T^2 = fg \iff$ if $f > 0$: $T = \pm T\{\Omega_{[p]}\}$, where $\Omega_{[p]}$ is a simple p-form and $(2p-N)\sqrt{f} = \pm tr(T)$; if $f = 0$: $T = \pm k \otimes k$ with k null.*

This was first proved in [4] using the representation theorem. A more direct proof is given in [2]. Note that the condition $T^2 = fg$ automatically implies that T is causal. The special case $N = 4$, $p = 2$ and $tr(T) = 0$ is the classical result by Rainich, Misner and Wheeler.

Corollary 4.4 *Any involutive (symmetric) conformal Lorentz transformation in any dimension is a superenergy tensor of a simple form.*

Pozo and Parra [17] define superenergy tensors of elements in the Clifford algebra, such superenergy tensors are not symmetric in general, and they prove that any conformal Lorentz transformation is such a superenergy tensor.

Theorem 4.3 gives necessary and sufficient conditions for a geometry to correspond to a physical field in several cases, and it generalises all the algebraic results of this type known previously.

As $T\{\Omega\}^2 = fg$ for any p-form Ω (also non-simple ones) if $N = 4$, there is a freedom, $F \to F\cos\alpha + {}^*F\sin\alpha$, a so-called duality rotation, which transforms the non-simple 2-form F into a simple 2-form without changing $T\{\Omega\}$.

Recall that the *rank* of a p-form Ω is the dimension of the subspace spanned by $\Omega(\cdot, u, \ldots, v)$ when the vectors $u, \ldots, v$ vary. Thus the rank of a 2-form is always even and a simple 2-form has rank 2. For 2-forms of rank 4 we have the following result [2].

Theorem 4.5 *Suppose that T is symmetric of order 2. Then, if $N > 4$, we have that T is the superenergy tensor of a 2-form of rank at most 4 if and only if*

$$a) \qquad T \text{ satisfies the dominant energy condition} \tag{4.3}$$

$$b) \quad (T^2 - \frac{1}{4}tr(T^2) + \frac{1}{4(N-4)}tr(T)^2)(T - \frac{1}{N-4}tr(T)) = 0 \tag{4.4}$$

In dimension $N = 5$ the rank of a 2-form is at most 4. Theorem 4.5 therefore implies

Theorem 4.6 *Suppose that T is symmetric of order 2 and $N = 5$. Then T is the superenergy tensor of a 2-form if and only if*

$$a) \quad T \text{ satisfies the dominant energy condition} \tag{4.5}$$

$$b) \quad (T^2 - \frac{1}{4}tr(T^2) + \frac{1}{4}tr(T)^2)(T - tr(T)) = 0 \tag{4.6}$$

As only 2-forms and their dual 3-forms can be non-simple if $N = 5$, this theorem together with theorem 4.3 give a complete generalisation of the classical algebraic Rainich theory in dimension 5. Note that the condition *b*) in theorem 4.6 does not imply that $\pm T \in DP$, a different result than in theorem 4.3.

We also remark that the equation *b*) in theorem 4.6 can be obtained as a necessary condition from a so-called dimensionally dependent identity. In dimension 5 any 6-form is zero, hence $A(F \otimes F \otimes F) = 0$ if F is a 2-form and A denotes the antisymmetric part. Contracting this with another $F \otimes F \otimes F$ gives the equation.

The complexity of algebraic Rainich theory grows with the rank of the forms and the dimension of the space. We have seen that in dimensions up to 4 any superenergy tensor satisfies a polynomial of degree 2, and in dimension 5 a polynomial of degree 3. In dimension 6, however, the degree of the polynomial can be 6 [2].

For tensors of higher order no Rainich theory has been developed. One can prove that in dimension $N = 4$, the Bel-Robinson tensor $\mathcal{T}$ satisfies $\mathcal{T}_{abcd}\mathcal{T}^{abce} = fg_d{}^e$. It is not known, however, if the equation $T_{abcd}T^{abce} = fg_d{}^e$ together with the assumptions that the tensor T of order 4 is completely symmetric and trace-free, implies that T is the superenergy tensor of a tensor with the symmetries of a Weyl tensor.

Acknowledgments

A grant from the Ministerio de Educación, Cultura y Deporte in Spain for a sabatical year at the University of the Basque Country in Bilbao is acknowledged.

References

[1] G. Bergqvist, *Positivity of general superenergy tensors*, Commun. Math. Phys., **207**, 467-479, (1999).

[2] G. Bergqvist and A. Höglund, *Algebraic Rainich theory and antisymmertrisation in higher dimensions*, preprint (gr-qc/0202092), (2002).

[3] G. Bergqvist and J.M.M. Senovilla, *On the causal propagation of fields*, Class. Quant. Grav., **16**, L55-61, (1999).

[4] G. Bergqvist and J.M.M. Senovilla, *Null cone preserving maps, causal tensors and algebraic Rainich theory*, Class. Quant. Grav., **18**, 5299-5325, (2001).

[5] D. Christodoulou and S. Klainerman, *The global non-linear stability of Minkowski space*, Princeton University Press, Princeton (1993).

[6] J. Frauendiener, *On the Penrose inequality*, Phys. Rev. Lett., **87**, 101101,1-4, (2001)

[7] S.W. Hawking, *Gravitational radiation in an expanding universe*, J. Math. Phys., **9**, 598-604, (1998)

[8] S.W. Hawking and G.F.R. Ellis, *The large-scale structure of space-time*, Cambridge University Press, Cambridge (1973).

[9] G. Huisken and T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose conjecture*, J. Diff. Geom., to appear

[10] S. Klainerman and F. Nicolò, *On local and global aspects on the Cauchy problem in general relativity*, Class. Quant. Grav., **16**, R73-157, (1999)

[11] M. Ludvigsen and J. A. G. Vickers, *An inequality relating the total mass and the area of a trapped surface in general relativity*, J. Phys. A: Math. Gen., **16**, 3349-3353, (1983)

[12] C.W. Misner and J.A. Wheeler, *Classical physics as geometry*, Ann. Phys. NY, **2**, 525-603, (1957).

[13] R. Penrose, *Gravitational collapse and space-time singularities*, Phys. Rev. Lett., **14**, 57-59, (1965)

[14] R. Penrose, *Some unsolved problems in classical general relativity*, in *Seminar on Differential Geometry*, ed S.-T. Yau, Princeton University Press, Princeton (1982).

[15] R. Penrose, *Light rays near spacelike infinity: A new mass-positivity theorem*, Twist. Newslett., **30**, 1-5, (1990)

[16] R. Penrose and W. Rindler, *Spinors and space-time*, vol. 1-2, Cambridge University Press, Cambridge (1986).

[17] J.M. POZO AND J.M. PARRA, *Positivity and conservation of superenergy tensors*, Class. Quant. Grav., to appear (2002), (gr-qc/0110050)

[18] G.Y. RAINICH, *Electrodynamics in the general relativity theory*, Trans. Am. Math. Soc., **18**, 5299-5325, (1925).

[19] R. SCHOEN AND S.-T. YAU, *Positivity of the total mass of a general space-time*, Phys. Rev. Lett., **43**, 369-371, (1979).

[20] J.M.M. SENOVILLA, *Singularity theorems and their consequences*, Gen. Rel. Grav., **30**, 701-848, (1998).

[21] J.M.M. SENOVILLA, *Super-energy tensors*, Class. Quant. Grav., **17**, 2799-2842, (2000).

[22] E. WITTEN, *A new proof of the positive energy theorem*, Commun. Math. Phys., **80**, 381-402, (1981).

Proceedings of the meeting
Lorentzian Geometry-Benalmádena 2001
Benalmádena, Málaga, Spain
Pub. de la RSME, Vol. 5 (2003), 85-93

Integral formulas in semi-riemannian manifolds

Graciela Silvia Birman[1]

[1] *NUCOMPA-Dep. of Mathematics-UNCPBA*
Pinto 399-(7000) Tandil-Argentina
gbirman@exa.unicen.edu.ar

Abstract

Let R_s^n be the semi-riemannian space of signature $(s, n-s)$. In this ambient, we define pure and piece-pure manifolds according to the causality of their tangent spaces. Moreover, we obtain some integral formulas, named of Crofton´s type, using the invariant measure of indefinite Grassmann manifolds. In the last section we show in L^2 the reverse of the well known isoperimetric formula.

1 Introduction

The causality is a jealous condition demanding to choose be spacelike or timelike. As a consequence of that, we introduce in [2] the notion of pure curves, and since it deals with right lines in $R\,{}_1^2$ a metric characterization is given.

Now, we think it is interesting and useful to characterize the causal condition of differentiable oriented manifolds. For higher dimensions the metrical criterion does not seem appropiate and we define the causality of any r-subspace R^r in R_s^n with respect to the unitary sphere. We also define the causality of any hyperplane R^{n-1} in R_1^n in terms of its normal vector.

From these concepts we generalize [2] for piece-pure manifolds and pure and piece-pure indefinite Grassmann manifolds.

In the third section we generalize the, so called, reproductive formulas by Banchoff & Pohl using the concepts defined in the previous section and the density or invariant measure for pure and piece-pure indefinite Grassmann manifolds.

In the last section we study the classical isoperimetric problem in R_1^2 (or L^2) and as we find an inequality reversed from the usual one in the euclidean plane, we call it anti-isoperimetric inequality.

2 Preliminaries and Definitions

Let R_s^n be the semi-euclidean space of signature $(s, n-s)$ with inner product

$$\langle v_p, w_p \rangle = -\sum_{i=1}^{s} v^i w^i + \sum_{j=s+1}^{n} v^j w^j$$

and S_s^{n-1} be the unit sphere of R_s^n. Let $f : M^m \longrightarrow \mathrm{R}_s^n$ be an immersion of a compact oriented differentiable semi-riemannian manifold M^m.

In the following we want to define subspaces of R_s^n according to its causality.

In this ambient we will consider the set of r-subspaces through the origin and we will name them $G_{r;s,n}$.

As well as R_s^n becomes a semi-riemannian manifold, an analogous structure is induced on $G_{r;s,n}$ which we can call the real indefinite Grassmann manifold. The signature of a given r-subspace may be (p, q) in such a way that $p + q = r$, where p indicates the number of "minus "and q indicates the number of "plus "of the metric tensor of the r-subspace.

Naturally, for $0 \le p \le s, 0 \le q \le n-s$ they verify

$$0 \le p \le \min(r, s) \text{ and } r - p = q \le n - s.$$

Now, fixing p and q, $G_{p,q;s,n}$ will denote the Grassmann manifold of subspaces of signature (p, q) in R_s^n. This concept corresponds to the $G_{p,s-p;q,n-q}$ of Wolf, [7].

Definition 2.1 *Let R_s^n and S_s^{n-1} be the n-dimensional semi-riemannian space of signature (s,n-s) (for $s > 0$) and the unit sphere in it, respectively.*

a) For $n \geqslant 3, s \leq n-2$ we will say that the subspace R^r is timelike if $R^r \cap S_s^{n-1} = \begin{cases} \emptyset \text{ for } r \leq n-2, r \leq s \\ S_s^{n-2} \text{ for } s < r = n-1 \end{cases}$ where S_s^{n-2} is the unit sphere in R_s^{n-1}.

b) For $r \leq n-1$ we will say that the subspace R^r is spacelike if $R^r \cap S_s^{n-1} \backsimeq S^{r-1}$ where S^{r-1} is the (r-1)-euclidean sphere.

For n =2, the case is trivial, and thus its measure is finite and of easy computing.

In particular, for $r = n-1, s = 1$, we add

Definition 2.2 *Let R_1^n be the semi-euclidean space of signature $(1, n-1)$ with the corresponding inner product and R^{n-1} be an hyperplane in R_1^n. Let N be its normal vector (respect to the lorentzian metric) then R^{n-1} is a spacelike (timelike) hyperplane if and only if N is a timelike (spacelike) vector.*

Now, we are going to define the notion of pure manifold; following the idea of [5] and [2] we want to characterize the causal condition of a differentiable manifold in terms of the causality of its tangent space.

Definition 2.3 *We will say that a differentiable semi-riemannian manifold M^r is pure if at every point $p \in M$, its tangent space $T_pM(\backsimeq R^r)$ is timelike or spacelike. We can be more explicit saying pure timelike or pure spacelike.*

We also generalize the concept of piece-pure curve:

Definition 2.4 *We will say that a differentiable semi-riemannian manifold M^r is piece-pure if the measure of its set of null points vanish.*

We call $G_{p,0;s,n}$ the indefinite Grassman manifold of p-timelike subspaces in R_s^n. Analogously, $G_{0,q;s,n}$ is the indefinite Grassman manifold of q-spacelike subspaces in R_s^n.

If $\mid G_{p,0;s,n} \mid$ (*) and $\mid G_{0,q;s,n} \mid$ (**) are the invariant measures of these disjoint manifolds, we can not express the invariant measure $\mid G_{p,q;s,n} \mid$ in terms of (*) and (**). But we can assert that upon a set of measure zero we have

$$\int_{G_{p,q;s,n}} \mid G_{p,q;s,n} \mid = \int_{G_{p,0;s,n}} \mid G_{p,0;s,n} \mid + \int_{G_{0,q;s,n}} \mid G_{0,q;s,n} \mid . \quad (2.1)$$

3 Crofton'style formulas

Now, following Banchoff &Pohl, we will show that some formulas presented in [1] are still valid for indefinite Grassmann manifolds.

[1] asserts ¨Let $f : M^m \to R^n$ be an immersion of a compact oriented differentiable manifold M^m. Let $A(M^m)$ be

$$A(M^m) = K_{m,n} \int_{G_{q,n}} \lambda^2(h) \mid dH_{n-m-1,n} \mid \tag{3.1}$$

where $H_{n-m-1,n}$ denotes the Grassmann manifold of all (n-m-1)-planes in R^n, $h \in H_{n-m-1,n}$, $\lambda(h)$ is the linking number of h, and

$$K_{m,n} = \prod_{j=m+2}^{n} \pi^{-(m+1)n/2} \frac{\Gamma((j+1)/2)}{\Gamma((j-m)/2)} \ .$$

If $q > n - m - 1$, then

$$l_{m,n,q} \int_{H_q} A(M^m \cap H_q) \mid dH_{q,n} \mid = A(M^m), \tag{3.2}$$

where

$$l_{m,n,q} = \frac{K_{m,n}}{K_{q-n+m,q}} \frac{\prod_{j=0}^{q-n+m} S_j}{\prod_{k=0}^{n} S_k} \prod_{i=0}^{n-q-1} S_i \ ,$$

and $S_j = \frac{2\pi^{(j+1)/2}}{\Gamma((j+1)/2)}$ is the surface area of the unit j-sphere. ¨

$A(M^m)$ generalizes the volume bounded by a simple subspace ($K_{m,n} = 1$) and, in particular, for m=1 q=1, n=2, (3.1) is one of the classical Crofton's formulas. For n=2 and q=1, we show, in [2], that the Crofton's formulas are valid in the Lorentzian plane.

The result (3.2) is called the reproductive property of A. Such a property hold for pure manifolds, we mean, if M^m is pure timelike (spacelike) the property is still valid. Comparing notations we have that $G_{q,n} = G_{0,q;0,n}$, for $q > n - m - 1$, and $\mid dG_{q,n} \mid$ is the invariant measure or density.

Let $f : M^m \to R^n_s$ be an immersion of a compact oriented differentiable manifold M^m. As in [1], let $A(M^m)$ be

$$A(M^m) = K_{m,n} \int_{G_{q,n}} \lambda^2(h) \mid dG_{q,n} \mid, \tag{3.3}$$

where $h \in G_{q,n}$, $\lambda(h)$ is the linking number of h, and

$$K_{m,n} = \prod_{j=m+2}^{n} \pi^{-(m+1)n/2} \frac{\Gamma((j+1)/2)}{\Gamma((j-m)/2)} .$$

Theorem 3.1 *Let $f, A(M^m), K_{m,n}$ and $l_{m,n,q}$ be as above.*

a) If M^m is a pure timelike manifold and G_q is a timelike q-subspace $s \geqslant q \geqslant n-m-1$, then

$$l_{m,n,q} \int_{H_q} A(M^m \cap G_q) \mid dG_{q,n} \mid = A(M^m),$$

where $l_{m,n,q} = \frac{K_{m,n}}{K_{q-n+m,q}} \frac{\prod_{j=0}^{q-n+m} S_j}{\prod_{k=0}^{n} S_k} \prod_{i=0}^{n-q-1} S_i$.

b) If M^m is a pure spacelike manifold and G_{q+} is a spacelike q-subspace,

$s > n-q$, $q > n-m-1$, then

$$l_{m,n,q} \int_{H_q} A(M^m \cap G_q) \mid dG_{q,n} \mid = A(M^m),$$

where $l_{m,n,q} = \frac{K_{m,n}}{K_{q-n+m,q}} \frac{\prod_{j=0}^{q-n+m} S_j}{\prod_{k=0}^{n} S_k} \prod_{i=0}^{n-q-1} S_i$.

c) If M^m is an oriented compact piece-pure semi-riemannian manifold in R_s^n and G_{q+}, G_{q-} are spacelike and timelike subspaces, respectively $(q = q_+ + q_-)$ $q_- \leq s$, then

$$A(M^m) = l_{m,n,q-} \int_{G_{q-}} A(M^m \cap G_{q-}) \mid dG_{q-,0;p,n} \mid +$$
$$+ l_{m,n,q+} \int_{H_{q+}} A(M^m \cap G_{q+}) \mid dG_{0,q+;p,n} \mid .$$

Proof. a) and b) Let $f : M^m \to R_s^n$ be an immersion of a compact oriented differentiable manifold M^m

The q-planes in $G_{q,n}$ can be thought as euclidean. Then, from Definition 2.3, when M^m and G_q have the same causality, we are under the hypothesis of Banchoff & Pohl´s theorem, [1], and the formulas (3.1) and (3.2) are still valid.

c) As the properties of being timelike or spacelike is a disjoint condition for manifolds, if M^m is an oriented compact piece-pure semi-riemannian manifold in R_s^n then we can state from a), b) and Definition 4 that

$$A(M^m) = l_{m,n,q-} \int_{G_{q-}} A(M^m \cap G_{q-}) \mid dG_{q-,0;p,n} \mid +$$
$$+ l_{m,n,q+} \int_{H_{q_+}} A(M^m \cap G_{q+}) \mid dG_{0,q+;p,n} \mid .$$

□

4 Anti-isoperimetric inequality

Let L^2 be the Lorentzian plane provided with metric of signature (+,-); it is well known that in L^2 there are timelike lines, L_t, and spacelike lines, L_s, [2]. The general equation of the timelike lines is

$$L_t \equiv xchv - yshv - p = 0, \tag{4.1}$$

and of the spacelike lines is

$$L_s \equiv xshv - ychv - p = 0, \tag{4.2}$$

where $p = p(v)$ is the distance from the right line to the origin and $v \in R$ is such that $v = (1, ctghv)$.

From (4.1), notating with ' the derivation with respect to v, we have,

$$p = xchv - yshv$$

$$p' = xshv - ychv.$$

From the preceding equations we can not find the radius of curvature of the evolvent, since ds in terms of p and p' vanish.

This situation is different from that in R^2, [6], we mean that in L^2 the function $p = p(v)$ is not a support function of the lines L_t (analoguosly L_s).

Also from [2] we know that the density for timelike lines can be expresssed by

$dL_t = dp \wedge dv$, (and analogously for dL_s) .

Let D be a convex domain in L^2 which area is F, its border $\partial D = \mathcal{C}$ is a pure, closed, simple curve of length l, [2] and [3]. We assume that L_t intersects D in a timelike (spacelike) point.

Naming $u = u(s)$ the angle between the tangent to $\mathcal{C}$ in the point given by the arc-length parameter s and the y-axis and taking $w = v - u(s)$ we can rewrite, [2],

$$dL_t = sh \mid w \mid ds \wedge dw$$

and analogously for dL_s.

Classically, [6], we name σ_t, (σ_s), the length of the chord $L_t \cap D$, ($L_s \cap D$).

Considering the timelike lines L_t, (or spacelike ones L_s), which intersect the curve $\mathcal{C}$ in two points $T_1, T_2, (S_1, S_2)$, both timelike, (both spacelike) we have

$$\sigma_t dL_t = shw_1 shw_2 ds_1 ds_2 \tag{4.3}$$

analogously $\sigma_s dL_s$, where w_i is the angle between the line and the tangent to the curve at the intersection point .

From [3] it is known that the curve $\mathcal{C}$ has four vertex, which will be named P, Q, R, S, running the curve in counter-clock sense. In consequence, the arcs PQ and RS are part of timelike curves and the arcs QR and SP are part of spacelike curves.

Applying (4.3) to the non-oriented lines L_t, analogously L_s, we have, [6],

$$\int_{L_t \cap \mathcal{C}} \sigma_t dL_t = \int_{L_s \cap \mathcal{C}} \sigma_s dL_s = F\pi/2. \tag{4.4}$$

If the right lines are oriented

$$\int_{L_t \cap \mathcal{C}} \sigma_t dL_t = \int_{L_s \cap \mathcal{C}} \sigma_s dL_s = F\pi. \tag{4.5}$$

Without loss of generality, we know that, up to a set of measure zero

$$\begin{aligned} \int_{L \cap \mathcal{C}} \sigma dL = \int_{L_t \cap \mathcal{C}} \sigma_t dL_t + \int_{L_s \cap \mathcal{C}} \sigma_s dL_s = \\ = A + \int_{L_t(PQ) \cup L_t(RS)} \sigma_t dL_t + \int_{L_s(QR) \cup L_s(SP)} \sigma_s dL_s \ , \end{aligned} \tag{4.6}$$

where A is the area of the cuadrangle $PQRS$. Now, we can apply formulas (4.5) and (4.6) because the right lines intersect the border in points with the same causality, then we have

$$\begin{aligned} \int_{L \cap \mathcal{C}} \sigma dL = A + \int_{L_t(PQ) \cup L_t(RS)} shw_1 shw_2 ds_1 ds_2 + \\ + \int_{L_s(QR) \cup L_s(SP)} shu_1 shu_2 ds_1 ds_2 \ . \end{aligned}$$

Having in mind that $shw_1 shw_2 = 1/2\{ch(w_1 + w_2) - ch(w_1 - w_2)\}$ and that the integral

$$\int_L ch(w_1 - w_2)ds_2$$

is equal to the projection of $\mathcal{C}$ on the right line which is tangent to the other intersection point, (and in consequence is equal to zero), we obtain

$$\int_{L\cap\mathcal{C}} \sigma dL = A + 1/2 \int_{L_t(PQ)\cup L_t(RS)} ch(w_1 + w_2)ds_1 ds_2 +$$

$$+1/2 \int_{L_s(QR)\cup L_s(SP)} ch(u_1 + u_2)ds_1 ds_2 \geq A + 1/2l^2.$$

Sustituting (4.5) in the preceding expression and using the corresponding analogous, we get

$$2\pi F \geq A + 1/2l^2$$

or equivalently

$$4\pi F - l^2 \geqslant 2A \geqslant 0 \tag{4.7}$$

which we call anti-isoperimetric inequality.

Acknowledgments

This work has been partially supported by CONICET

The author is grateful to the referee for his (or her) valuable remarks and kindness.

References

[1] BANCHOFF & POHL, "A generalization of isoperimetric inequality", J. Differ. Geom., 6, (1971), 175-192.

[2] BIRMAN, G. S., "Crofton's and Poincare's formulas in the Lorentzian plane", Geom. Dedicata, vol.15, #4, (1984), 399-411.

[3] ——————, "On L^2 and L^3 ", Elem. Math., 43/2, 1988.

[4] Birman, G. S. and Nomizu, K., " Trigonometry in Lorentzian geometry", Am. Math. Mon, vol.91, #9, (1984), 543-549.

[5] Birman & Nomizu, "The Gauss -Bonnet formula for 2-dimensional space-time", Mich. Math. J., 31, (1984), 77-81.

[6] Santaló, L. A., "Integral Geometry and Geometric probability ", Addison-Wesley Publ. Company, 1979.

[7] Wolf,J., ¨Space of constant curvature¨, Mc Graw Hill, New York, 1976.

Proceedings of the meeting
Lorentzian Geometry-Benalmádena 2001
Benalmádena, Málaga, Spain
Pub. de la RSME, Vol. 5 (2003), 95-108

Normal geodesics in Space–Times

Anna Maria Candela[1]

[1] *Dipartimento Interuniversitario di Matematica*
Università degli Studi di Bari
via E. Orabona 4, 70125 Bari – Italy

Abstract

Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ be a Lorentzian manifold equipped with a static metric $\gamma_L = \langle \alpha(x)\cdot, \cdot \rangle - \beta(x)dt^2$ where β has a subquadratic growth. Then, fixed P_0, P_1 submanifolds of $\mathcal{M}_0$, a suitable version of the Fermat principle and the classical Ljusternik–Schnirelman theory allow to prove that the existence of normal lightlike geodesics joining $P_0 \times \{0\}$ to $P_1 \times \mathbb{R}$ is influenced by the topology of $\mathcal{M}_0$, P_0 and P_1.

1 Introduction

In these last years an increasing interest has been turned to the study of geodesics in Lorentzian manifolds by using variational tools and topological methods.

In fact, if $(\mathcal{M}, \gamma_L)$ is a Lorentzian manifold, or more in general a semi–Riemannian manifold, it is well known that the geodesic equation

$$D_s \dot{z} = 0$$

has a variational structure; so, a smooth curve $z : [0,1] \to \mathcal{M}$ is a geodesic if and only if it is a critical point of the action functional

$$f(z) = \int_0^1 \langle \dot{z}(s), \dot{z}(s) \rangle_L \, ds$$

in a suitable manifold of curves Z which depends on the required "boundary conditions" (see Section 2).

But unlike to the Riemannian case, in the Lorentzian one the functional f is not bounded both from above and from below; moreover, its critical points have infinite Morse index. Hence, classical topological theories, as the Ljusternik–Schnirelman or the Morse one, can not be applied directly to the research of critical points of f in Z.

A way to overcome these difficulties is selecting some classes of semi–Riemannian manifolds in which a suitable new variational principle allows to introduce a new functional, bounded from below, whose critical points are related to those ones of f.

In this approach the main idea is that, when the metric γ_L admits Killing vector fields, then the negative contribution of the variational problem in the directions of the Killing vector fields can be "factored out" (see, for examples, in [3, 14] for standard stationary and static manifolds, in [5] for plane fronted waves, in [8] for Gödel type Space–Times). Moreover, if we are interested in lightlike geodesics, a variational principle similar to the Fermat principle in optics allows to define another *good enough* functional (cf. [11, 12]).

So, in order to give an idea of possible variational tools, we just point out how to solve a particular problem in a simple model of Lorentzian manifold.

Definition 1.1 *A Lorentzian manifold $(\mathcal{M}, \gamma_L)$ is static if there exists a connected finite dimensional Riemannian manifold $(\mathcal{M}_0, \gamma)$, a smooth symmetric linear strictly positive operator $\alpha(x) : T_x\mathcal{M}_0 \to T_x\mathcal{M}_0$ and a smooth strictly positive scalar field $\beta : \mathcal{M}_0 \to \mathbb{R}$ such that $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ and*

$$\gamma_L = \langle \alpha(x)\cdot, \cdot\rangle - \beta(x)\ dt^2 \tag{1.1}$$

on any tangent space $T_z\mathcal{M} \equiv T_x\mathcal{M}_0 \times \mathbb{R}$, $z = (x,t) \in \mathcal{M}$.

Definition 1.2 *Let $(\mathcal{M}, \gamma_L)$ be a Lorentzian manifold and let $\mathcal{N}_0$, $\mathcal{N}_1$ be two submanifolds of $\mathcal{M}$. A curve $z : [0,1] \to \mathcal{M}$ is a normal geodesic joining $\mathcal{N}_0$ to $\mathcal{N}_1$ if it is a geodesic in $\mathcal{M}$ such that*

$$\begin{cases} z(0) \in \mathcal{N}_0 \\ \dot z(0) \in T_{z(0)}\mathcal{N}_0^{\perp} \end{cases}, \qquad \begin{cases} z(1) \in \mathcal{N}_1 \\ \dot z(1) \in T_{z(1)}\mathcal{N}_1^{\perp} \end{cases},$$

where, for $i \in \{0,1\}$, $T_{z(i)}\mathcal{N}_i^{\perp}$ denotes the orthogonal space of $T_{z(i)}\mathcal{N}_i$ in $T_{z(i)}\mathcal{M}$ with respect to the Lorentzian metric γ_L.

From now on, let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ be a Lorentzian manifold equipped with the static metric (1.1) and consider $\mathcal{N}_0 = P_0 \times \{t_0\}$ and $\mathcal{N}_1 = P_1 \times \mathbb{R}$ with $t_0 \in \mathbb{R}$ and P_0, P_1 two submanifolds in $\mathcal{M}_0$.

Roughly speaking, we say that a curve $z : [0,1] \to \mathbb{R}$, $z = (x,t)$, joining $\mathcal{N}_0$ to $\mathcal{N}_1$ has arrival time in the future if it is $t(1) > t_0$ while in the past if it is $t(1) < t_0$.

Remark 1.3 If there exists $\bar{x} \in P_0 \cap P_1$, then the constant curve $(\bar{x}, t_0)$ is a (constant) lightlike normal geodesic joining $\mathcal{N}_0$ to $\mathcal{N}_1$.

Aim of this paper is to show that the minimum number of (non–constant) lightlike normal geodesics joining $\mathcal{N}_0$ to $\mathcal{N}_1$ depends on the topology of P_0, P_1 and $\mathcal{M}_0$; moreover, some of them have arrival time in the future while some others in the past.

More precisely, the following results can be stated (here and after, $d(\cdot,\cdot)$ is the distance induced on $\mathcal{M}_0$ by its Riemannian metric γ, see (2.1), while $\mathrm{cat}(\cdot)$ is the Ljusternik–Schnirelman category, see Definition 3.1).

Theorem 1.4 *Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ be a Lorentzian manifold equipped with the static metric γ_L defined in (1.1). Suppose that*

(H_1) *$(\mathcal{M}_0, \gamma)$ is a complete C^3 Riemannian manifold;*

(H_2) *there exist λ, $\nu > 0$, $R_1, R_2 \geq 0$, $q \in [0,2[$ and a point $x_0 \in \mathcal{M}_0$ such that*

$$\langle \alpha(x)\xi, \xi\rangle \ \geq\ \lambda\ \langle \xi, \xi\rangle \quad \text{for all } \xi \in T_x\mathcal{M}_0,\ x \in \mathcal{M}_0, \tag{1.2}$$

$$\nu \leq \beta(x) \leq R_1 + R_2\ d^q(x, x_0) \quad \text{for all } x \in \mathcal{M}_0. \tag{1.3}$$

Moreover, let $\mathcal{N}_0 = P_0 \times \{t_0\}$, $\mathcal{N}_1 = P_1 \times \mathbb{R}$ be two submanifolds of $\mathcal{M}$ such that $t_0 \in \mathbb{R}$ and

(H_3) *P_0 and P_1 are closed submanifolds of $\mathcal{M}_0$ and one of them is compact;*

(H_4) *$P_0 \cap P_1 = \emptyset$.*

Then, at least two lightlike normal geodesics joining $\mathcal{N}_0$ to $\mathcal{N}_1$ exist: one with arrival time in the future and the other one in the past.

Furthermore, the hypothesis

(H_5) *P_0 and P_1 are both contractible in $\mathcal{M}_0$*

implies the existence of at least $\mathrm{cat}(P_0 \times P_1)$ *of such geodesics with arrival time in the future and* $\mathrm{cat}(P_0 \times P_1)$ *ones with arrival time in the past.*

Theorem 1.5 *Let* $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ *be a static Lorentzian manifold such that* (H_1) *and* (H_2) *hold. Moreover, consider* P_0 *and* P_1 *which satisfy the assumptions* (H_3) *and* (H_5). *If* $\mathcal{M}_0$ *is not contractible in itself then there exists a sequence of (non–constant) lightlike normal geodesics joining* $\mathcal{N}_0$ *to* $\mathcal{N}_1$ *with arrival times diverging positively and a sequence with arrival times diverging negatively.*

If $\mathcal{M}$ is a (standard) stationary Lorentzian manifold some similar results have been already obtained in [11, 12] if both P_0 and P_1 are reduced to single points (see also [14]) and in [6] if P_0, P_1 satisfy (H_3) but the coefficient β in (1.1) is bounded.

2 Variational setting

Let $(\mathcal{M}, \gamma_L)$ be a static Lorentzian manifold with $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ and γ_L as in (1.1), where $(\mathcal{M}_0, \gamma)$ is a Riemannian manifold such that (H_1), (H_2) hold. Moreover, let P_0 and P_1 be two submanifolds of $\mathcal{M}_0$ which satisfy (H_3). For simplicity, assume $t_0 = 0$ and $I = [0,1]$. Assume $\mathcal{N}_0 = P_0 \times \{0\}$, $\mathcal{N}_1 = P_1 \times \mathbb{R}$.

Let us remark that we are interested in lightlike normal geodesics joining $\mathcal{N}_0$ to $\mathcal{N}_1$, i.e., we look for smooth curves $z : I \to \mathcal{M}$, $z = (x,t)$, such that

$$\begin{cases} D_s\dot{z}(s) = 0 & \text{for all } s \in I, \\ \langle \dot{z}(s), \dot{z}(s)\rangle_L = 0 & \text{for all } s \in I, \end{cases}$$

with boundary conditions

$$\begin{cases} x(0) \in P_0, \qquad t(0) = 0, \\ \langle \alpha(x(0))\dot{x}(0), \xi\rangle = 0 \quad \text{for all } \xi \in T_{x(0)}P_0 \end{cases}$$

$$\begin{cases} x(1) \in P_1, \qquad \dot{t}(1) = 0, \\ \langle \alpha(x(1))\dot{x}(1), \xi\rangle = 0 \quad \text{for all } \xi \in T_{x(1)}P_1. \end{cases}$$

So, in order to solve this problem with variational tools, let us introduce a suitable variational setting.

Since $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$, the infinite dimensional manifold $H^1(I, \mathcal{M})$ is diffeomorphic to the product manifold $H^1(I, \mathcal{M}_0) \times H^1(I, \mathbb{R})$ and is equipped with a structure of an infinite dimensional Riemannian manifold γ_1 by setting

$$\langle\zeta,\zeta\rangle_1 = \int_0^1 \langle\xi,\xi\rangle\, ds + \int_0^1 \langle D_s\xi, D_s\xi\rangle\, ds + \int_0^1 \tau^2 ds + \int_0^1 \dot{\tau}^2 ds,$$

for any $z = (x,t) \in H^1(I,\mathcal{M})$ and $\zeta = (\xi,\tau) \in T_zH^1(I,\mathcal{M}) \equiv T_xH^1(I,\mathcal{M}_0) \times H^1(I,\mathbb{R})$.

By the Nash Embedding Theorem we can assume that $\mathcal{M}_0$ is a submanifold of an Euclidean space $\mathbb{R}^N$ and γ is the restriction to $\mathcal{M}_0$ of the Euclidean metric of $\mathbb{R}^N$ while $d(\cdot,\cdot)$ is the corresponding distance, i.e.,

$$d(x_1,x_2) = \inf\left\{\int_0^1 \sqrt{\langle\dot\gamma,\dot\gamma\rangle} ds :\ \gamma \in A_{x_1,x_2}\right\} \tag{2.1}$$

if x_1, $x_2 \in \mathcal{M}_0$, where $\gamma \in A_{x_1,x_2}$ if $\gamma : I \to \mathcal{M}_0$ is a piecewise smooth curve such that $\gamma(0) = x_1$, $\gamma(1) = x_2$.

Hence, it can be proved that the manifold $H^1(I,\mathcal{M}_0)$ can be identified with the set of the absolutely continuous curves $x : I \to \mathbb{R}^N$ with square summable derivative such that $x(I) \subset \mathcal{M}_0$.

Furthermore, since $\mathcal{M}_0$ is a complete Riemannian manifold with respect to γ, also $H^1(I,\mathcal{M})$ is a complete Riemannian manifold equipped with the previous scalar product.

Let Z be the smooth manifold of all the $H^1(I,\mathcal{M})$–curves joining $\mathcal{N}_0$ to $\mathcal{N}_1$ while $\Omega(P_0,P_1)$ denotes the smooth submanifold of $H^1(I,\mathcal{M}_0)$ which contains all the curves joining P_0 to P_1 (cf. [13]). From the product structure of $H^1(I,\mathcal{M})$ it is

$$Z \equiv \Omega(P_0,P_1) \times W,$$

where $W = \{t \in H^1(I,\mathbb{R}) : t(0) = 0\}$ is a subspace of $H^1(I,\mathbb{R})$.

Clearly, it is

$$T_zZ \equiv T_x\Omega(P_0,P_1) \times W \quad \text{for all } z = (x,t) \in Z.$$

Proposition 2.1 *Since the hypotheses* (H_1), (H_3) *hold, then the submanifold* $\Omega(P_0,P_1)$ *is complete and* Z *is complete, too.*

By (1.1) it follows that the action integral $f : Z \to \mathbb{R}$ is defined as

$$f(z) = \int_0^1 \langle\dot z,\dot z\rangle_L\, ds = \int_0^1 (\langle\alpha(x)\dot x,\dot x\rangle - \beta(x)\dot t^2)\, ds$$

for any $z = (x,t) \in Z$. It is easy to prove that f is a C^1 functional with Fréchet differential

$$\begin{aligned} f'(z)[\zeta] &= \int_0^1 \langle \alpha'(x)[\xi]\ \dot{x}, \dot{x}\rangle\ ds + 2\int_0^1 \langle \alpha(x)\dot{x}, \dot{\xi}\rangle\ ds \\ &\quad - \int_0^1 \beta'(x)[\xi]\ \dot{t}^2\ ds - 2\int_0^1 \beta(x)\dot{t}\dot{\tau}\ ds, \end{aligned}$$

for all $z = (x,t) \in Z$, $\zeta = (\xi, \tau) \in T_z Z$, where α' and β' denote, respectively, the derivatives of α and β with respect to the Riemannian structure on $\mathcal{M}_0$.

In a quite standard way it can be proved that a curve $z : I \to \mathcal{M}$ is a lightlike normal geodesic joining $\mathcal{N}_0$ to $\mathcal{N}_1$ if and only if $z \in Z$ is a critical point of the action functional f such that $f(z) = 0$ (for more details, see, e.g., [7, Proposition 2.1]).

As already remarked in the introduction, a similar variational principle holds also for the study of normal geodesics joining two given submanifolds in a Riemannian manifold (cf. [13]), but in the Riemannian case the action functional is bounded from below while it is not more true in the Lorentzian case. But the coefficients in the metric (1.1) are time–independent so it is possible to get over such a difficulty by introducing a new functional which depends only on the Riemannian variable x.

Anyway, in this particular case there is a one more problem: the arrival time is unknown, so we can not work "directly" as in [4]. By the way, a Fermat type principle introduced in General Relativity by D. Fortunato, F. Giannoni and A. Masiello allows to overcome such a trouble (see [11]). The idea is to express the arrival time by means of a new functional whose critical points in $\Omega(P_0, P_1)$ are related to those ones of f in Z with null energy.

Thus, let $\lambda \in \mathbb{R}$ be fixed and consider the set of curves

$$Z_\lambda = \{z \in Z : z = (x,t),\ t(1) = \lambda\}.$$

If $f_\lambda = f|_{Z_\lambda}$, then $z = z(s)$ is a lightlike normal geodesic joining $\mathcal{N}_0$ to $\mathcal{N}_1$ with arrival time λ if and only if $z \in Z_\lambda$ is such that $f'_\lambda(z) = f_\lambda(z) = 0$.

The following propositions can be proved.

Proposition 2.2 *Let $\bar{z} = (\bar{x}, \bar{t}) \in Z_\lambda$. Then, $\bar{z}$ is a critical point of the action functional f_λ in Z_λ if and only if $\bar{x}$ is a critical point of the* *functional*

$$J_\lambda(x) = \int_0^1 \langle \alpha(x)\dot{x}, \dot{x}\rangle\ ds\ -\ \lambda^2\ K(\bar{x})$$

in $\Omega(P_0, P_1)$*, while it is*

$$\bar{t}(s) = \lambda K(\bar{x}) \int_0^s \frac{1}{\beta(\bar{x})}\ d\sigma, \quad \textit{with } K(\bar{x}) = \Big(\int_0^1 \frac{1}{\beta(\bar{x})}\ ds\Big)^{-1}.$$

In both these cases it is $f_\lambda(\bar{z}) = J_\lambda(\bar{x})$*.*

(For the proof, see [3, Theorem 2.1] or also [4, Proposition 2.2]).

Thus, Proposition 2.2 implies that we have to look for $x \in \Omega(P_0, P_1)$ such that

$$J'_\lambda(x) = J_\lambda(x) = 0, \tag{2.2}$$

where it is $J_\lambda(x) = 0$ if and only if

$$\lambda^2 = \int_0^1 \langle \alpha(x)\dot{x}, \dot{x}\rangle\ ds \cdot \int_0^1 \frac{1}{\beta(x)}\ ds.$$

Defined $H : (\lambda, x) \in \mathbb{R} \times \Omega(P_0, P_1) \mapsto J_\lambda(x) \in \mathbb{R}$, (2.2) is equivalent to look for $(\lambda, x) \in \mathbb{R} \times \Omega(P_0, P_1)$ such that

$$\frac{\partial H}{\partial x}(\lambda, x) = H(\lambda, x) = 0, \tag{2.3}$$

where (1.3) implies that $\frac{\partial H}{\partial \lambda}(\lambda, x) = 0$ if and only if $\lambda = 0$, i.e., x is constant.

In order to apply the Fermat principle, let us define the two functionals $F_\pm : \Omega(P_0, P_1) \to \mathbb{R}$ such that

$$F_\pm(x) = \pm\sqrt{\int_0^1 \langle \alpha(x)\dot{x}, \dot{x}\rangle\ ds \cdot \int_0^1 \frac{1}{\beta(x)}\ ds}.$$

Obviously, it is $F_- = -F_+$. Thus, let us consider $F = F_+$.

By simple calculations it is possible to prove that F is continuous but not differentiable at level zero while it is smooth everywhere else. Clearly, it is $F(x) \geq 0$ for all $x \in \Omega(P_0, P_1)$ and $F(x) = 0$ if and only if x is a constant function.

Remark 2.3 If (H_4) holds, then $\Omega(P_0, P_1)$ has no constant at all. Thus, F is always different from zero and C^1 in $\Omega(P_0, P_1)$.

Proposition 2.4 *Let $x \in \Omega(P_0, P_1)$ be such that*

$$F'(x) = 0, \qquad F(x) > 0. \tag{2.4}$$

Then, taken $\lambda = F(x)$, (λ, x) solves (2.3).

Proof. If we define the manifold

$$\mathcal{G}_+ = \{(\lambda, x) \in \mathbb{R}_+ \times \Omega(P_0, P_1) : H(\lambda, x) = 0\},$$

it is easy to verify that it is the graph of the functional F; whence, the conclusion follows by the Fermat principle (for more details, see, e.g., [14, Theorem 6.2.2]).

□

Thus, from now on, our aim is to solve (2.4); hence, to look for strictly positive critical levels of F in $\Omega(P_0, P_1)$.

3 Ljusternik–Schnirelman Theory

First of all, let us recall the main tools of the Ljusternik–Schnirelman Theory (for more details, see, e.g., [1, 14, 15]).

Definition 3.1 *Let X be a topological space. Given $A \subseteq X$, the Ljusternik–Schnirelman category of A in X, briefly $\mathrm{cat}_X(A)$, is the least number of closed and contractible subsets of X covering A. If it is not possible to cover A with a finite number of such sets, it is $\mathrm{cat}_X(A) = +\infty$.*

We denote $\mathrm{cat}(X) = \mathrm{cat}_X(X)$.

Definition 3.2 *A C^1 functional g on a Riemannian manifold Ω satisfies the Palais–Smale condition at level $a \in \mathbb{R}$, briefly $(PS)_a$, if any $(x_n)_n \subset \Omega$ such that*

$$g(x_n) \to a \quad \textit{and} \quad g'(x_n) \to 0 \qquad \textit{as } n \to +\infty$$

converges in Ω up to subsequences.

Theorem 3.3 *Let Ω be a complete Riemannian manifold and g a C^1 functional on Ω which satisfies $(PS)_a$ for all $a \in \mathbb{R}$. Taken any $k \in \mathbb{N}$, $k > 0$, let us define*

$$c_k = \inf_{A \in \Gamma_k} \sup_{x \in A} g(x) \quad \textit{with } \Gamma_k = \{A \subseteq \Omega : \mathrm{cat}_\Omega(A) \geq k\}. \tag{3.1}$$

Then, c_k is a critical value of g for each k such that $\Gamma_k \neq \emptyset$ and $c_k \in \mathbb{R}$; if, moreover, g is bounded from below then g attains its infimum and has at least $\operatorname{cat}(\Omega)$ *critical points.*

Remark 3.4 Let Ω and g be as in Theorem 3.3. If g is bounded from below, then for all $c \in \mathbb{R}$ it is

$$\operatorname{cat}_\Omega(g^c) < +\infty,$$

where $g^c = \{x \in \Omega : g(x) \leq c\}$.

Remark 3.5 Let g be a positive functional not differentiable at level zero while it is smooth elsewhere in a complete Riemannian manifold Ω. If $(PS)_a$ holds at any level $a > 0$, then it can be proved that c_k is a critical value of g for all k such that $\Gamma_k \neq \emptyset$ and $c_k > 0$.

In order to apply Theorem 3.3 to the functionals F_+ and F_- defined in the previous section, we need evaluating the Ljusternik–Schnirelman category of the manifold of curves $\Omega(P_0, P_1)$.

Proposition 3.6 *Let $(\mathcal{M}_0, \gamma)$ be a smooth complete connected finite dimensional Riemannian manifold and let P_0, P_1 be closed submanifolds both contractible in $\mathcal{M}_0$. Then*

$$\operatorname{cat}(\Omega(P_0, P_1)) \geq \operatorname{cat}(P_0 \times P_1)\ .$$

(For the proof, see [6, Theorem 3.7]).

Proposition 3.7 *Let $(\mathcal{M}_0, \gamma)$ be a smooth complete connected finite dimensional Riemannian manifold and let P_0, P_1 be two of its closed submanifolds. If $\mathcal{M}_0$ is not contractible in itself while both P_0 and P_1 are contractible in $\mathcal{M}_0$, then $\Omega(P_0, P_1)$ has infinite category and possesses compact subsets of arbitrary high category.*

(For the proof, see [9, 10]).

4 Proof of the main theorems

Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ be a static Lorentzian manifold such that (H_1) and (H_2) hold; moreover, let P_0 and P_1 be two submanifolds of $\mathcal{M}_0$ which satisfy (H_3).

As already remarked in Section 2, now we have to apply the Ljusternik–Schnirelman Theory to the study of strictly positive critical levels of F in $\Omega(P_0, P_1)$.

To this aim, first of all we need some more information on F.

Lemma 4.1 *The functional F is coercive with respect to $\|\dot{x}\|^2 = \int_0^1 \langle \dot{x}, \dot{x} \rangle ds$, i.e.,*

$$F(x) \to +\infty \qquad \text{if} \quad \|\dot{x}\| \to +\infty.$$

Proof. By (1.2) the proof is trivial if $\mathcal{M}_0$ is compact.
On the contrary, by the hypothesis (H_2), (2.1) and some comments (for example, see [4, Lemma 4.1]) it follows that there exists a constant $R^* > 0$ such that

$$F(x) \geq R^* \; \frac{\|\dot{x}\|}{1 + \|\dot{x}\|^{\frac{q}{2}}},$$

with $q < 2$. Hence, the proof is complete.

□

Proposition 4.2 *Taken any $a > 0$, the functional F satisfies $(PS)_a$.*

Proof. Taken $a > 0$, let $(x_n)_n \subset \Omega(P_0, P_1)$ be such that

$$F(x_n) \to a \quad \text{and} \quad F'(x_n) \to 0 \qquad \text{as } n \to +\infty. \tag{4.1}$$

Clearly, Lemma 4.1 and (4.1) imply that $(\|\dot{x}_n\|)_n$ is bounded; hence, by (H_3) it follows that $(x_n)_n$ is bounded in $\Omega(P_0, P_1)$.
Then, there exists $x \in H^1(I, \mathbb{R}^N)$ such that, up to subsequences,

$$x_n \rightharpoonup x \text{ weakly in } H^1(I, \mathbb{R}^N), \quad x_n \to x \text{ uniformly in } I.$$

By Proposition 2.1 it is $x \in \Omega(P_0, P_1)$; furthermore, by [2, Lemma 2.1] there exist two sequences $(\xi_n)_n$, $(\nu_n)_n \subset H^1(I, \mathbb{R}^N)$ such that

$$\begin{aligned} &\xi_n \in T_{x_n}\Omega(P_0, P_1), \quad x_n - x = \xi_n + \nu_n \qquad \text{for all } n \in \mathbb{N}, \\ &\xi_n \rightharpoonup 0 \;\text{ weakly} \quad \text{and} \quad \nu_n \to 0 \;\text{ strongly in } H^1(I, \mathbb{R}^N). \end{aligned} \tag{4.2}$$

Thus, (4.1) implies that $(F(x_n))_n$ is bounded and far from zero if n is large enough; moreover,

$$\begin{aligned} o(1) &= F'(x_n)[\xi_n] \\ &= \left(\int_0^1 (\langle \alpha'(x_n)[\xi_n]\dot{x}_n, \dot{x}_n \rangle + 2\langle \alpha(x_n)\dot{x}_n, \dot{\xi}_n \rangle) \; ds \;\cdot \int_0^1 \frac{1}{\beta(x_n)} \; ds \right. \\ &\quad \left. - \int_0^1 \langle \alpha(x_n)\dot{x}_n, \dot{x}_n \rangle \; ds \cdot \int_0^1 \frac{\beta'(x_n)[\xi_n]}{\beta^2(x_n)} \; ds \right) \; \frac{1}{2F(x_n)}; \end{aligned}$$

whence,

$$\begin{aligned} o(1) \ = \ & \int_0^1 (\langle \alpha'(x_n)[\xi_n]\dot{x}_n, \dot{x}_n\rangle + 2\langle \alpha(x_n)\dot{x}_n, \dot{\xi}_n\rangle)\ ds \cdot \int_0^1 \frac{1}{\beta(x_n)}\ ds \\ & - \int_0^1 \langle \alpha(x_n)\dot{x}_n, \dot{x}_n\rangle\ ds \cdot \int_0^1 \frac{\beta'(x_n)[\xi_n]}{\beta^2(x_n)}\ ds. \end{aligned}$$

On the other hand, $(\alpha'(x_n))_n$ and $(\beta'(x_n))_n$ are bounded while $\xi_n \to 0$ uniformly in I; hence, $(\beta(x_n))_n$ bounded and far from zero and $(\|\dot{x}_n\|)_n$ bounded imply

$$\begin{aligned} \int_0^1 \langle \alpha'(x_n)[\xi_n]\dot{x}_n, \dot{x}_n\rangle\ ds \cdot \int_0^1 \frac{1}{\beta(x_n)}\ ds = o(1), \\ \int_0^1 \langle \alpha(x_n)\dot{x}_n, \dot{x}_n\rangle\ ds \cdot \int_0^1 \frac{\beta'(x_n)[\xi_n]}{\beta^2(x_n)}\ ds = o(1). \end{aligned}$$

So, it is

$$\int_0^1 \langle \alpha(x_n)\dot{x}_n, \dot{\xi}_n\rangle\ ds \ = o(1),$$

with $x_n = x + \xi_n + \nu_n$; hence, by (4.2) it follows

$$\int_0^1 \langle \alpha(x_n)\dot{\xi}_n, \dot{\xi}_n\rangle ds = o(1)$$

which implies $\xi_n \to 0$ strongly in $H^1(I, \mathbb{R}^N)$.

□

Lemma 4.3 *For all $c \in \mathbb{R}$ it is*

$$\mathrm{cat}_{\Omega(P_0,P_1)}(F^c) < +\infty, \tag{4.3}$$

where $F^c = \{x \in \Omega(P_0, P_1) : F(x) \le c\}$.

Proof. If $P_0 \cap P_1 = \emptyset$, then the result follows by Remarks 2.3 and 3.4. By the way, more in general, F is not differentiable at level zero but working as in the proof of Lemma 4.1 it follows that there exists a constant $R > 0$ such that

$$F(x) \ge \ R\ \|\dot{x}\|^{1-\frac{q}{2}}, \tag{4.4}$$

with $q < 2$ as in (1.3).

Thus, let us consider the functional

$$g : x \in \Omega(P_0, P_1) \longmapsto \int_0^1 \langle \dot{x}, \dot{x} \rangle \; ds \;\in \mathbb{R}.$$

It can be proved that g is a smooth positive function which satisfies the Palais–Smale condition at any real level on the manifold of curves $\Omega(P_0, P_1)$; hence, by Remark 3.4 it follows that

$$\mathrm{cat}_{\Omega(P_0,P_1)}(g^{\ c}) < +\infty, \quad \text{for all} \;\; c \in \mathbb{R}. \tag{4.5}$$

On the other hand, (4.4) implies that for all $c \in \mathbb{R}$ there exists $c \in \mathbb{R}$ such that $F^c \subset g^{\ c}$; whence, by (4.5) it follows (4.3).

□

Proof of Theorem 1.4. By (H_4), Remark 2.3, Proposition 4.2 and Theorem 3.3 imply that F has at least $\mathrm{cat}(\Omega(P_0, P_1))$ critical points whose critical levels are the arrival times of the corresponding lightlike normal geodesics (see Proposition 2.4). On the other hand, the same arguments apply to $F_- = -F$ so by Theorem 3.6 the proof is complete.

□

Proof of Theorem 1.5. First of all let us remark that, in general, $\Omega(P_0, P_1)$ can contain some constants; hence, F attains the value zero. By the way, in the given hypotheses Proposition 3.7 implies that $\Gamma_k \neq \emptyset$ for all $k \in \mathbb{N}$ (with Γ_k as in (3.1) with $\Omega = \Omega(P_0, P_1)$); so, in order to apply the result in Remark 3.5, it is enough to prove that a sequence $(k_i)_i \subset \mathbb{N}$ exists such that

$$0 < c_{k_i} < c_{k_{i+1}} \qquad \text{with} \;\; c_{k_i} = \inf_{A \in \Gamma_{k_i}} \sup_{x \in A} F(x) \quad \text{if } i \in \mathbb{N}.$$

Clearly, this is true if we prove that fixed any $a > 0$ there exists $\bar{k} \in \mathbb{N}$ such that

$$B \in \Gamma_{\bar{k}} \quad \Longrightarrow \quad B \cap F_a \neq \emptyset \;, \tag{4.6}$$

where $F_a = \{x \in \Omega(P_0, P_1) : F(x) > a\}$. In fact, (4.6) implies $c_{\bar{k}} \geq a$ and the result follows by the arbitrariness of $a > 0$.
In order to prove (4.6), suppose that it does not hold for some $a > 0$. Then, there exists a sequence $(B_n)_n$ of subsets of $\Omega(P_0, P_1)$ such that

$$\mathrm{cat}_{\Omega(P_0,P_1)}(B_n) \geq n \quad \text{and} \quad B_n \subset F^a \qquad \text{for all } n \in \mathbb{N}$$

which implies $\mathrm{cat}_{\Omega(P_0,P_1)}(F^a) = +\infty$ in contradiction with Lemma 4.3.

□

Acknowledgments

This work has been supported by M.I.U.R. (research funds ex 40 % and 60%).

References

[1] A. Ambrosetti, *Critical points and nonlinear variational problems*, Mém. Soc. Math. France (N.S.) **49** (1992).

[2] V. Benci and D. Fortunato, *On the existence of infinitely many geodesics on space–time manifolds*, Adv. Math., **105**, 1-25 (1994).

[3] V. Benci, D. Fortunato and F. Giannoni, *On the existence of multiple geodesics in static space–times*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **8**, 79-102 (1991).

[4] A.M. Candela, *Topological methods applied to General Relativity.* Preprint.

[5] A.M. Candela, J.L. Flores and M. Sánchez, *On General Plane Fronted Waves. Geodesics.* General Relativity and Grav. (to appear).

[6] A.M. Candela and A. Salvatore, *Light rays joining two submanifolds in Space–Times*, J. Geom. Phys., **22**, 281-297 (1997).

[7] A.M. Candela and A. Salvatore, *Normal geodesics in stationary Lorentzian manifolds with unbounded coefficients.* Preprint.

[8] A.M. Candela and M. Sánchez, *Geodesic connectedness in Gödel type Space–Times*, Differential Geom. Appl., **12**, 105-120 (2000).

[9] A.M. Canino, *On p–convex sets and geodesics*, J. Differential Equations, **75**, 118-157 (1988).

[10] E. Fadell and S. Husseini, *Category of loop spaces of open subsets in Euclidean space*, Nonlinear Anal., **17**, 1153-1161 (1991).

[11] D. Fortunato, F. Giannoni and A. Masiello, *A Fermat principle for stationary space–times and applications to light rays*, J. Geom. Phys., **15**, 159-188 (1995).

[12] D. FORTUNATO AND A. MASIELLO, *Fermat principles in General Relativity and the existence of light rays on Lorentzian manifolds*, in: Proc. "Variational and local methods in the study of Hamiltonian systems" I.C.T.P. 1994 (A. Ambrosetti and G.F. Dell'Antonio Eds.), World Scientific, Singapore (1995), 34-64.

[13] W. KLINGENBERG, *Riemannian Geometry*, de Gruyter, Berlino (1982).

[14] A. MASIELLO, *Variational methods in Lorentzian geometry*, Pitman Res. Notes Math. Ser. **309**, Longman Sci. Tech., Harlow (1994).

[15] R.S. PALAIS, *Lusternik–Schnirelman Theory on Banach manifolds*, Topology, **5**, 115-132 (1966).

Proceedings of the meeting
Lorentzian Geometry-Benalmádena 2001
Benalmádena, Málaga, Spain
Pub. de la RSME, Vol. 5 (2003), 109-124

Characterizing Certain Product semi-Riemannian Structures by Differential Equations

M. Fernández-López[1], E. García-Río[1] and D. N. Kupeli[2]

[1] *Department of Geometry and Topology, University of Santiago de Compostela, 15782Santiago de Compostela, Spain.*
(manufl@usc.es, xtedugr@usc.es)

[2] *Bosna-Hersek Caddesi, No:35/B-10, Emek Mahallesi, 06510 Ankara, Turkey.*
(dnkupeli@superonline.com)

Abstract

Some characterizations of certain semi-Riemannian product, warped product and twisted product structures of semi-Riemannian manifolds by the existence of nontrivial solutions to certain partial differential equations on semi-Riemannian manifolds are surveyed.

1 Introduction

In analysis, mostly the existence of a nontrivial solution to a differential equation on a certain domain is argued. But in geometry, one can also argue the existence of a manifold structure for a differential equation to possess a nontrivial solution. This may be considered as an analytic characterization (or representation) of a manifold structure by a differential equation if this manifold structure serves as a unique domain structure for this differential equation to possess a nontrivial solution in a certain class of manifolds.

In this survey, we give several analytic characterizations of semi-Riemannian product, warped product and twisted product structures of semi-Riemannian manifolds by differential equations, that is, by the existence of nontrivial solutions to some differential equations on certain semi-Riemannian manifolds. In Section 3, we survey the analytic characterizations of the above product semi-Riemannian structures of semi-Riemannian manifolds by the existence of solutions to Eikonal equation, Obata's equation and Möbius equation in each subsection, respectively. We mostly concentrate our attention to the cases of Riemannian and Lorentzian manifolds to obtain the strongest results. Unfortunately, in generic semi-Riemannian cases, the results are mostly weak when they involve the linear algebra of indefinite inner product spaces.

In fact, the purpose of this survey paper is to call some attention to characterizations (or representations) of product semi-Riemannian structures analytically by differential equations on certain classes of semi-Riemannian manifolds determined by mild geometric/topological assumptions. Although we expect that every manifold structure cannot be characterized (or represented) by a differential equation, such manifold structures that can be characterized by differential equations may be a larger class of manifold structures than semi-Riemannian product, warped product and twisted product structures of semi-Riemannian manifolds. In any case, having a better knowledge about characterizations of manifold structures by differential equations may lead us a better understanding of a possible relation between differential equations and differential geometric structures.

2 Preliminaries

Here, we briefly state the main concepts and definitions used throughout this paper.

Let (M_1, g_1) and (M_2, g_2) be semi-Riemannian manifolds of dimensions n_1 and n_2 with Levi-Civita connections $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$, respectively, and let $f\colon (M_1, g_1) \to (M_2, g_2)$ be a map. We denote the set of vector fields on M_1 by ΓTM_1 and the set of vector fields along f by $\Gamma_f TM_2$. We also denote the pullback of $\overset{2}{\nabla}$ along f by $\overset{2}{\nabla}$. Recall that the map

$$\nabla f_*\colon \Gamma TM_1 \times \Gamma TM_1 \to \Gamma_f TM_2$$

defined by

$$(\nabla f_*)(X,Y) = \overset{2}{\nabla}_X f_* Y - f_*(\overset{1}{\nabla}_X Y)$$

is called the *second fundamental form of* f. The trace $\tau(f)$ of ∇f_* with respect to g_1 is called the *tension field of* f. That is,

$$\begin{aligned} \tau(f) &= trace\, \nabla f_* \\ &= \sum_{i=1}^{n_1} g_1(X_i, X_i)(\nabla f_*)(X_i, X_i), \end{aligned}$$

where $\{X_1, \cdots, X_{n_1}\}$ is a local orthonormal frame for TM_1. If $\nabla f_* = 0$ then f is called *affine* (or *totally geodesic*), and if $\tau(f) = 0$ then f is called *harmonic*.

Let (M,g) be an n-dimensional semi-Riemannian manifold and let ∇ denote both the Levi-Civita connection and the gradient operator on (M,g). The *Hessian tensor* h_f of a function $f\colon M \to \mathbb{R}$ is defined by $h_f(X) = \nabla_X \nabla f$, where $X \in \Gamma TM$. Also the symmetric $(0,2)$-tensor field H_f defined on (M,g) by $H_f(X,Y) = g(h_f(X),Y)$ is called the Hessian form of f, where $X,Y \in \Gamma TM$. We define the *Laplacian* Δf of a function f on (M,g) by $\Delta f = div \nabla f$ $(= trace\, h_f)$, where div is the divergence. Note that, in particular, if $f = (f_1, \cdots, f_m)\colon (M,g) \to (\mathbb{R}^m, \bar{g})$, where $\bar{g}$ is a semi-Euclidean metric tensor on $\mathbb{R}^m$, then

$$\nabla f_* = \sum_{i=1}^m H_{f_i} \frac{\partial}{\partial x^i} \circ f \quad \text{and} \quad \tau(f) = \sum_{i=1}^m (\Delta f_i) \frac{\partial}{\partial x^i} \circ f,$$

where $(x^1, \cdots, x^m)$ is the usual coordinate system of $\mathbb{R}^m$.

Next we recall some notation and terminology on product structures to be used throughout this note. Let (B, g_B) and (F, g_F) be semi-Riemannian manifolds of dimensions r and s, respectively, and let, $\pi\colon B \times F \to B$ and $\sigma\colon B \times F \to F$ be the canonical projections. Also let $\lambda\colon B \times F \to (0,\infty)$ be a smooth function. Then the *twisted product* of (B, g_B) and (F, g_F) with twisting function λ is defined to be the product manifold $M = B \times F$ with metric tensor $g = g_B \oplus \lambda^2 g_F$ given by $g = \pi^* g_B + \lambda^2 \sigma^* g_F$. For brevity in notation, we denote this semi-Riemannian manifold (M,g) by $B \times_\lambda F$. In particular, if λ only depends on the points of B then $B \times_\lambda F$ is called the *warped product* of (B, g_B) and (F, g_F) with warping function λ.

A local characterization of twisted and warped products can be stated in terms of the extrinsic geometry of the foliations $\mathcal{L}_B$ and $\mathcal{L}_F$ of the product manifold $M = B \times F$ as follows. Let g be a semi-Riemannian metric tensor on the manifold $M = B \times F$ and assume that the canonical foliations $\mathcal{L}_B$ and $\mathcal{L}_F$ intersect perpendicularly everywhere. Then g is the metric tensor of (see [18])

(i) a twisted product $B \times {}_\lambda F$ if and only if $\mathcal{L}_B$ is a totally geodesic foliation and $\mathcal{L}_F$ is a totally umbilic foliation,

(ii) a warped product $B \times {}_\lambda F$ if and only if $\mathcal{L}_B$ is a totally geodesic foliation and $\mathcal{L}_F$ is a spheric foliation,

(iii) a usual product of semi-Riemannian manifolds if and only if $\mathcal{L}_B$ and $\mathcal{L}_F$ are totally geodesic foliations.

Note that the condition relating (i) and (ii) in the above is the normal parallelism of the mean curvature vector field of the totally umbilic foliation $\mathcal{L}_F$. This can be equivalently described by a curvature condition on the Ricci tensor of (M, g) as follows: a twisted product $M = B \times {}_\lambda F$ is indeed a warped product if and only if the Ricci tensor of (M, g) satisfies $Ric(X, V) = 0$ for every $X \in \Gamma TB$ and $V \in \Gamma TF$ [6].

3 Characterizations of Product Semi-Riemannian Structures

In this section, we survey the known differential equations characterizing product semi-Riemannian structures, that is, we state the known differential equations defined on certain classes of semi-Riemannian manifolds which uniquely determine the certain semi-Riemannian product, warped product and twisted product structures as their domain structures in the case of the existence of nontrivial solutions to these differential equations. In the following subsections, we discuss the characterizations by eikonal equation, Obata's equation and Möbius equation, respectively.

3.1 Characterizations by Eikonal Equation

In this subsection, we state some results related to the characterizations of certain Riemannian and Lorentzian product structures in semi-Riemannian geometry by eikonal equation. Let (M, g) be an n-dimensional semi-Riemannian manifold. *A map $f\colon (M, g) \to \mathbb{R}$ is said to satisfy an* *eikonal equation on* (M,g) *if* $g(\nabla f,\nabla f)=k_0$, where k_0 is either of $-1,0$ or 1. (See [10, Chapter 6]).

Here, from the viewpoint of this survey, we consider special cases of eikonal equation on Riemannian and Lorentzian manifolds where we obtain the strongest splitting results due to this equation.

Proposition 3.1 *Let* (M,g) *be an* $n(\geq 2)$*-dimensional Riemannian (respectively, Lorentzian) manifold and* $f\colon (M,g)\to\mathbb{R}$ *be a map. Then,* f *satisfies the equation* $g(\nabla f,\nabla f)=1$ *(respectively,* $g(\nabla f,\nabla f)=-1$*) on* (M,g) *if and only if* $f\colon (M,g)\to(\mathbb{R},dt^2)$ *(respectively,* $f\colon (M,g)\to(\mathbb{R},-dt^2)$*) is a semi-Riemannian submersion.*

Proof. See [10, Proposition 6.1.1].

□

In fact, the following property of the solutions to eikonal equations in the above proposition yields the characterization of semi-Riemannian product structures by a splitting result of Innami [12]. In the Proposition below, by the timelike nonnegative Ricci curvature of a Lorentzian manifold (M,g), we mean $Ric(z,z)\geq 0$ for all timelike $z\in TM$. This condition is also called timelike convergence condition in relativity theory.

Proposition 3.2 *Let* (M,g) *be a connected,* $n(\geq 2)$*-dimensional complete Riemannian (respectively, timelike geodesically complete Lorentzian) manifold with nonnegative (respectively, timelike nonnegative) Ricci curvature, and let* $f\colon (M,g)\to\mathbb{R}$ *be a map. If* f *satisfies the equation* $g(\nabla f,\nabla f)=1$ *(respectively,* $g(\nabla f,\nabla f)=-1$*) on* (M,g) *then* f *is an affine map on* (M,g)*, that is,* $H_f=0$.

Proof. See [10, Proposition 6.2.5].

□

Remark 3.3 In the proof of the Lorentzian case of the above theorem, the diagonalizability of the Hessian tensor of f satisfying eikonal equation $g(\nabla f,\nabla f)=-1$ plays a crucial role. In general, this is not true in generic semi-Riemannian manifolds. Indeed, solutions of the Eikonal equation with nondiagonalizable Hessian tensor are constructed in [11] for non-Lorentzian signatures.

Hence, by using the Main Theorem of [12], we obtain the following splitting result due to eikonal equation.

Theorem 3.4 *Let (M, g) be a connected, $n(\geq 2)$-dimensional complete Riemannian (respectively, timelike geodesically complete Lorentzian) manifold with nonnegative (respectively, timelike nonnegative) Ricci curvature. Then, a necessary and sufficient condition for (M, g) to be isometric with a Riemannian (respectively, Lorentzian) product $(\mathbb{R}\times N, dt^2 \oplus g_N)$ (respectively, $(\mathbb{R} \times N, -dt^2 \oplus g_N)$) is the existence of a nonconstant map $f\colon (M, g) \to \mathbb{R}$ satisfying the equation $g(\nabla f, \nabla f) = 1$ (respectively, $g(\nabla f, \nabla f) = -1$) on (M, g), where (N, g_N) is a Riemannian manifold.*

Proof. See [10, Theorem 7.2.2] and [10, Theorem 7.3.2].

□

Remark 3.5 Note here that, by the above theorem, the differential equation $g(\nabla f, \nabla f) = 1$ (respectively, $g(\nabla f, \nabla f) = -1$) on a connected, $n(\geq 2)$-dimensional complete Riemannian (respectively, timelike geodesically complete Lorentzian) manifold (M, g) with nonnegative (respectively, timelike nonnegative) Ricci curvature, has a nontrivial solution if and only if its domain manifold (M, g) is a Riemannian product $(\mathbb{R} \times N, dt^2 \oplus g_N)$ (respectively, a Lorentzian product $(\mathbb{R} \times N, -dt^2 \oplus g_N)$). Hence the differential equation $g(\nabla f, \nabla f) = 1$ (respectively, $g(\nabla f, \nabla f) = -1$) may be considered as an analytic characterization (or representative) of Riemannian products $(\mathbb{R} \times N, dt^2 \oplus g_N)$ (respectively, Lorentzian products $(\mathbb{R} \times N, -dt^2 \oplus g_N)$) among the connected, $n(\geq 2)$-dimensional complete Riemannian (respectively, timelike geodesically complete Lorentzian) manifolds (M, g) with nonnegative (respectively, timelike nonnegative) Ricci curvature.

Finally in this subsection, we state a result of Sakai [19] related to eikonal equation characterizing a specific warped product in Riemannian geometry.

Theorem 3.6 *Let (M, g) be a connected, $n(\geq 2)$-dimensional complete Riemannian manifold with $Ric(z, z) \geq -(n-1)k$, $k > 0$, for all unit $z \in TM$. Then, a necessary and sufficient condition for (M, g) to be isometric with a Riemannian warped product $(\mathbb{R} \times N, dt^2 \oplus \phi^2 g_N)$, where $\phi(t) = e^{\pm\sqrt{k}t}$ and (N, g_N) is a complete Riemannian manifold* *of nonnegative Ricci curvature, is the existence of a nonconstant function* $f\colon (M,g) \to \mathbb{R}$ *satisfying the equations* $g(\nabla f, \nabla f) = 1$ *and* $|\Delta f| = (n-1)k$ *on* (M,g).

Proof. See [19, Theorem 3.5] and [19, Remark 3.6].

□

Remark 3.7 As in Remark 3.5, by the above theorem, the simultaneous differential equations $g(\nabla f, \nabla f) = 1$ and $|\Delta f| = (n-1)k$, $k > 0$, may be considered as an analytic characterization (or representative) of Riemannian warped products $(\mathbb{R} \times N, dt^2 \oplus \phi^2 g_N)$, where $\phi(t) = e^{\pm\sqrt{k}t}$ and (N, g_N) is a complete Riemannian manifold of nonnegative Ricci curvature, among the connected, $n(\geq 2)$-dimensional complete Riemannian manifolds (M,g) with $Ric(z,z) \geq -(n-1)k$, $k > 0$, for all unit $z \in TM$.

From the viewpoint of this survey, here we only considered the strongest consequences of eikonal equation related to splitting results for Riemannian and Lorentzian manifolds. More detailed analysis of eikonal equation in semi-Riemannian geometry may be found in [10]. Also a detailed analysis of eikonal equation in Riemannian geometry may be found in [7] and [8].

Note that timelike eikonal equation has a special significance in general relativity. In fact, a spacetime (M,g) is *stably causal* if and only if there exist a real function with timelike gradient ($\|\nabla f\| < 0$), i.e., a solution of the timelike eikonal inequality. Now, it is easy to check the existence of a conformal metric on M where the timelike eikonal equation admits a solution. Indeed, f itself satisfies $g_c(\overset{c}{\nabla} f, \overset{c}{\nabla} f) = -1$, where $g_c = (-g(\nabla f, \nabla f))g$ and $\overset{c}{\nabla}$ is the g_c-gradient operator. Then, one has the following singularity versus splitting result [9]

Theorem 3.8 *Let* (M,g) *be a stably causal spacetime. Then, either* (M, g_c) *is timelike geodesically incomplete or else,* (M,g) *is conformally diffeomorphic to a parametrized Lorentzian product* $(\mathbb{R} \times N, -(dt \otimes dt) \oplus g_t)$*, where* $g_c = -g(\nabla f, \nabla f)g$ *and* f *is a time function for a synchronizable reference frame on* (M,g).

In particular, if there exists a proper time synchronizable reference frame on (M,g) *then, either* (M,g) *is timelike geodesically incomplete or else,* (M,g) *is isometric to timelike geodesically complete a parametrized Lorentzian product* $(\mathbb{R} \times N, -(dt \otimes dt) \oplus g_t)$.

Recall here the construction of a parametrized product structure. Let (a, b) be an open interval in $\mathbb{R}$ furnished with the usual negative definite metric tensor $-dt \otimes dt$, where t is the usual coordinate on (a, b). Let N be a manifold and let g_t be a smooth 1-parameter family of Riemannian metric tensors on N parametrized over (a, b). Then the *parametrized Lorentzian product of* (a, b) *and* N is defined to be the product manifold $M = (a, b) \times N$ with Lorentzian metric tensor $g = -\psi(dt \otimes dt) \oplus g_t$, where $\psi : (a, b) \to (0, \infty)$ is a smooth function.

Note that, if $(M, g) = ((a, b) \times N,\, -\psi(dt \otimes dt) \oplus g_t)$ is a parametrized Lorentzian manifold then (M, g) is a stably causal spacetime. Indeed, if $f = t \circ \pi^1 : M \to \mathbb{R}$, then $g(\nabla f, \nabla f) = -\frac{1}{\psi \circ \pi^1} < 0$. In particular, if $\psi = 1$ then $f = t \circ \pi^1$ is a solution of the timelike eikonal equation of (M, g).

Remark 3.9 Special cases of parametrized Lorentzian products are as follows

(i) If $g_t(q) = \lambda(t, q)^2 g_N(q)$, where $\lambda : (a, b) \times N \to (0, \infty)$ and g_N is a fixed Riemannian metric on N, then (M, g) is the Lorentzian twisted product of $((a, b),\, \psi(dt \otimes dt))$ and (N, g_N) with twisting function λ.

(ii) If $g_t = \lambda(t)^2 g_N$, where $\lambda : (a, b) \to (0, \infty)$ and g_N is a fixed Riemannian metric on N, then (M, g) is the Lorentzian warped product of $((a, b),\, \psi(dt \otimes dt))$ and (N, g_N) with warping function λ.

(iii) If $g_t = g_N$, where g_N is a fixed Riemannian metric tensor on N, then (M, g) is the Lorentzian product of $((a, b),\, \psi(dt \otimes dt))$ and (N, g_N).

Note here that the special cases above can be detected through the eikonal equation, under some additional conditions on the solutions (cf [9]).

3.2 Characterizations by Obata's Equation

In this subsection, we state some results related to characterizations of certain Riemannian product and warped product structures in Riemannian geometry by Obata's equation. On an $n(\geq 2)$-dimensional Riemannian manifold (M, g), *the differential equation* $H_f + kfg = 0$, $k \in \mathbb{R}$, *where* $f\colon (M, g) \to \mathbb{R}$ *is a function, is known as Obata's equation.* In fact, the existence of a nontrivial solution to the equation $H_f + kfg =$

0, $k > 0$, on a connected, $n(\geq 2)$-dimensional complete Riemannian manifold (M, g) has a very strong consequence that (M, g) is isometric with the Euclidean sphere $\mathbb{S}^n(k)$ of sectional curvature k. (See [36, Theorem A], and also [1] for a survey of known results related to the characterizations of certain rank-one symmetric Riemannian manifolds by differential equations). However the differential equation $H_f + kfg = 0$, $k \leq 0$, is not as deterministic in characterizing its domain Riemannian manifold (M, g) as in the case of $k > 0$. Yet the results obtained by Kanai [13] for the case $k \leq 0$ of Obata's equation are of interest from viewpoint of this survey in characterizing certain Riemannian product and warped product structures in Riemannian geometry. First we consider the case $k = 0$ in Obata's equation. In fact, the theorem below is also a consequence of the Main Theorem of [12].

Theorem 3.10 *Let (M, g) be a connected, $n(\geq 2)$-dimensional complete Riemannian manifold. Then, a necessary and sufficient condition for (M, g) to be isometric with a Riemannian product $(\mathbb{R} \times N, dt^2 \oplus g_N))$ is the existence of a nonconstant map $f\colon (M, g) \to \mathbb{R}$ satisfying the equation $H_f = 0$ on (M, g), where (N, g_N) is a Riemannian manifold.*

Proof. See [13, Theorem B] or [20, Theorem 2].

□

Remark 3.11 As in Remark 3.5, by the above theorem, the equation $H_f = 0$ may be considered as an analytic characterization (or representative) of Riemannian products $(\mathbb{R} \times N, dt^2 \oplus g_N)$ among the connected, $n(\geq 2)$-dimensional complete Riemannian manifolds (M, g).

Concerning the case $k < 0$ of Obata's equation, Kanai showed in [13] (also see [20]) that, a necessary and sufficient condition for a connected, $n(\geq 2)$-dimensional complete Riemannian manifold (M, g) to be isometric with a connected component $\mathbb{H}^n_\pm(k)$ of the real hyperbolic space $\mathbb{H}^n(k)$ of sectional curvature $k(< 0)$ (respectively, with a warped product of the Euclidean line and a complete non-Euclidean Riemannian manifold, where the warping function $\phi\colon \mathbb{R} \to \mathbb{R}^+$ satisfies the equation $\phi'' + k\phi = 0$, $k < 0$), is the existence of a nonconstant function $f\colon M \to \mathbb{R}$ with a critical point (respectively, without critical points) satisfying the equation $H_f + kfg = 0$, $k < 0$, on (M, g). Note here that, a connected component $\mathbb{H}^n_\pm(k)$ of the real hyperbolic space $\mathbb{H}^n(k)$ of sectional curvature $k(< 0)$, is the warped product of the Euclidean line and the Euclidean space with warping function $\phi(t) = e^{\pm\sqrt{-k}t}$ on $\mathbb{R}$. Thus, we combine the above two results of Kanai to give a characterization of certain warped products in Riemannian geometry by differential equations.

Theorem 3.12 *Let (M, g) be a connected, $n(\geq 2)$-dimensional complete Riemannian manifold. Then, a necessary and sufficient condition for (M, g) to be isometric with a Riemannian warped product $(\mathbb{R} \times N, dt^2 \oplus \phi^2 g_N)$, where (N, g_N) is a complete Riemannian manifold and the warping function ϕ satisfies the equation $\phi'' + k\phi = 0$, $k < 0$, is the existence of a nonconstant function $f\colon M \to \mathbb{R}$ satisfying the equation $H_f + kfg = 0$, $k < 0$, on (M, g).*

Proof. See [13, Corollary E].

□

Remark 3.13 As in Remark 3.5, by the above theorem, the equation $H_f + kfg = 0$, $k < 0$, may be considered as an analytic characterization (or representative) of Riemannian warped products $(\mathbb{R} \times N, dt^2 \oplus \phi^2 g_N)$, where (N, g_N) is a complete Riemannian manifold and warping function ϕ satisfies the equation $\phi'' + k\phi = 0$, $k < 0$, among the connected, $n(\geq 2)$-dimensional complete Riemannian manifolds (M, g).

Remark 3.14 Note that, Theorem 3.6 has a similar prediction (in fact, stronger) as of Theorem 3.12. Indeed, it can be shown by using the Bochner identity $\frac{1}{2}\Delta g(\nabla f, \nabla f) = \|h_f\|^2 + Ric(\nabla f, \nabla f) + g(\nabla \Delta f, \nabla f)$ (see [10, pag. 79]) and the fact that $\|h_f\|^2 \geq \frac{(\Delta f)^2}{n-1}$ for a function f satisfying $g(\nabla f, \nabla f) = 1$ on an $n(\geq 2)$-dimensional Riemannian manifold (M, g) that, the function f in the statement of Theorem 3.6 also satisfies the equation $H_f - kfg = 0$, $(k > 0)$, on the Riemannian manifold (M, g) in the statement of Theorem 3.6.

3.3 Characterizations by Möbius Equation

In this subsection, we state some results related to the characterizations of certain semi-Riemannian twisted product and warped product structures in semi-Riemannian geometry by Möbius equation. Let (M_1, g_1) and (M_2, g_2) be semi-Riemannian manifolds of dimensions $dim\, M_1 = n_1 > n_2 = dim\, M_2 \geq 1$. A submersion $f\colon (M_1, g_1) \to (M_2, g_2)$ is called nondegenerate if the fibres of f are semi-Riemannian submanifolds of

(M_1, g_1). (Note that, if (M_1, g_1) is connected then these fibres are all semi-Riemannian submanifolds of the same metric index).

Definition 3.15 *Let (M_1, g_1) and (M_2, g_2) be semi-Riemannian manifolds of dimensions $n_1 > n_2 \geq 1$. A nondegenerate submersion $f\colon (M_1, g_1) \to (M_2, g_2)$ is said to satisfy the local Möbius equation if*

$$\begin{aligned} (\nabla f_*)(X, Y) &= \frac{\tau(f)}{n_1 - n_2} g_1(X, Y) \\ (\nabla f_*)(X, U) &= 0 \end{aligned}$$

for all $X, Y \in \Gamma ker f_*$ and $U \in \Gamma(ker f_*)^\perp$.

Now we state a local characterization of semi-Riemannian twisted products by the local Möbius equation.

Theorem 3.16 *Let (M_1, g_1) and (M_2, g_2) be semi-Riemannian manifolds of dimensions $n_1 > n_2 \geq 1$. If there exists a nondegenerate submersion $f\colon (M_1, g_1) \to (M_2, g_2)$ satisfing the local Möbius equation then (M_1, g_1) is locally a semi-Riemannian twisted product $(M_1^1 \times M_1^2, g_1^1 \oplus \lambda^2 g_1^2)$, where (M_1^1, g_1^1) and (M_1^2, g_1^2) are semi-Riemannian manifolds and $f\colon M_1^1 \times M_1^2 \to M_1^1$ is the projection.*

Conversely, if $(M_1, g_1) = (M_1^1 \times M_1^2, g_1^1 \oplus \lambda^2 g_1^2)$ is a semi-Riemannian twisted product, where (M_1^1, g_1^1) and (M_1^2, g_1^2) are semi-Riemannian manifolds, then the projection map $\pi_1\colon M_1^1 \times M_1^2 \to M_1^1$ satisfies the Möbius equation.

Proof. See [3, Theorem 1] and [3, Theorem 2].

□

Remark 3.17 Note that, by the above theorem, the local Möbius equation may be considered as an analytic characterization (or representative) of locally twisted product semi-Riemannian manifolds.

Recall that a map $f\colon (M_1, g_1) \to (M_2, g_2)$ between semi-Riemannian manifolds (M_1, g_1) and (M_2, g_2) is called harmonic if f satisfies the differential equation $\tau(f) = 0$. Also, with this additional differential equation to the local Möbius equation, we can characterize locally product semi-Riemannian manifolds.

Theorem 3.18 *Let* (M_1, g_1) *and* (M_2, g_2) *be semi-Riemannian manifolds of dimensions* $n_1 > n_2 \geq 1$. *If there exists a nondegenerate submersion* $f\colon (M_1, g_1) \to (M_2, g_2)$ *satisfying the local Möbius equation with* $\tau(f) = 0$ *then* (M_1, g_1) *is locally a semi-Riemannian product* $(M_1^1 \times M_1^2, g_1^1 \oplus g_1^2)$, *where* (M_1^1, g_1^1) *and* (M_1^2, g_1^2) *are semi-Riemannian manifolds and* $f\colon M_1^1 \times M_1^2 \to M_1^1$ *is the projection.*

Conversely, if $(M_1, g_1) = (M_1^1 \times M_1^2, g_1^1 \oplus g_1^2)$ *is a semi-Riemannian product, where* (M_1^1, g_1^1) *and* (M_1^2, g_1^2) *are semi-Riemannian manifolds, then the projection map* $\pi_1\colon M_1^1 \times M_1^2 \to M_1^1$ *satisfies the Möbius equation with* $\tau(\pi_1) = 0$.

Proof. See [3, Corollary 1].

□

Remark 3.19 Note that, by the above theorem, the local Möbius equation together with the equation $\tau(f) = 0$, may be considered as an analytic characterization (or representative) of locally product semi-Riemannian manifolds. Note also that, if $(M_1, g_1) = (M, g)$ is a Riemannian manifold and $(M_2, g_2) = (\mathbb{R}, dt^2)$ in the above theorem, then we obtain a local version of Theorem 3.10.

Also, with an additional differential equation to the local Möbius equation, we can characterize locally warped product semi-Riemannian manifolds.

Theorem 3.20 *Let* (M_1, g_1) *and* (M_2, g_2) *be semi-Riemannian manifolds of dimensions* $n_1 > n_2 \geq 1$. *If there exists a nondegenerate submersion* $f\colon (M_1, g_1) \to (M_2, g_2)$ *satisfying the local Möbius equation with* $(\nabla\nabla f_*)(X, Y, Z) = 0$ *for all* $X, Y, Z \in \Gamma ker f_*$ *then* (M_1, g_1) *is locally a semi-Riemannian warped product* $(M_1^1 \times M_1^2, g_1^1 \oplus \lambda^2 g_1^2)$, *where* (M_1^1, g_1^1) *and* (M_1^2, g_1^2) *are semi-Riemannian manifolds and* $f\colon M_1^1 \times M_1^2 \to M_1^1$ *is the projection.*

Conversely, if $(M_1, g_1) = (M_1^1 \times M_1^2, g_1^1 \oplus \lambda^2 g_1^2)$ *is a semi-Riemannian warped product, where* (M_1^1, g_1^1) *and* (M_1^2, g_1^2) *are semi-Riemannian manifolds then the projection map* $\pi_1\colon M_1^1 \times M_1^2 \to M_1^1$ *satisfies the Möbius equation with* $(\nabla\nabla\pi_*)(X, Y, Z) = 0$ *for all* X, Y, Z *tangent to the copies of* M_1^2 *in* $M_1^1 \times M_1^2$.

Proof. See [3, Theorem 4] and [3, Theorem 5].

□

Remark 3.21 Note that, by the above theorem, the local Möbius equation together with the equation $(\nabla\nabla f_*)(X,Y,Z) = 0$ for all $X, Y, Z \in \Gamma ker f_*$, may be considered as an analytic characterization (or representative) of locally warped product semi-Riemannian manifolds.

It is shown in [6] that, a semi-Riemannian twisted product $(M,g) = (N_1 \times N_2, g_{N_1} \oplus \lambda^2 g_{N_2})$ can be written as a warped product if and only if (M,g) is mixed Ricci-flat, that is, $Ric(X,U) = 0$ for all X and U tangent to the copies of N_1 and N_2 in $N_1 \times N_2$, respectively. (See [6, Theorem 1]). Thus, we also have the following version of Theorem 3.20.

Theorem 3.22 *Let (M_1, g_1) and (M_2, g_2) be semi-Riemannian manifolds of dimensions $n_1 > n_2 \geq 1$. If $f\colon (M_1,g_1) \to (M_2,g_2)$ is a nondegenerate submersion satisfying the local Möbius equation and $Ric(X,U) = 0$ for all $X \in \Gamma ker f_*$ and $U \in \Gamma(ker f_*)^\perp$ then (M_1,g_1) is locally a semi-Riemannian warped product $(M_1^1 \times M_1^2, g_1^1 \oplus \lambda^2 g_1^2)$, where (M_1^1, g_1^1) and (M_1^2, g_1^2) are semi-Riemannian manifolds and $f\colon M_1^1 \times M_1^2 \to M_1^1$ is the projection.*

Conversely, if $(M_1,g_1) = (M_1^1 \times M_1^2, g_1^1 \oplus \lambda^2 g_1^2)$ is a semi-Riemannian warped product, where (M_1^1, g_1^1) and (M_1^2, g_1^2) are semi-Riemannian manifolds, then the projection map $\pi_1\colon M_1^1 \times M_1^2 \to M_1^1$ satisfies the Möbius equation and $Ric(X,U) = 0$ for all X and U tangent to the copies of M_1^1 and M_1^2 in $M_1^1 \times M_1^2$, respectively.

Proof. Immediate from *Theorem* 3.16 and [6, Theorem 1].

□

In literature, a function f on an $n(\geq 2)$-dimensional semi-Riemannian manifold (M,g) is said to satisfy the *Möbius equation* if

$$H_f - df \otimes df - \frac{1}{n}[\Delta f - g(\nabla f, \nabla f)]g = 0$$

on (M,g). (See [17]). Also, by making the transformation $t = e^{-f}$, the Möbius equation becomes the

$$H_t = \frac{\Delta t}{n} g$$

on (M,g), which we call the localized Möbius equation. (Indeed, if t is a solution of the equation $H_t = \frac{\Delta t}{n} g$ then $f = -log\,|t|$ satisfies the Möbius equation $H_f - df \otimes df - \frac{1}{n}[\Delta f - g(\nabla f, \nabla f)]g = 0$ on the subset of M where

$t \neq 0$). The localized Möbius equation has been analyzed in the literature in detail. (See for example, [14] and [15] and the references therein). One of the results related to localized Möbius equation which is of interest from viewpoint of this survey is the following: Let (M, g) be an $n(\geq 2)$-dimensional semi-Riemannian manifold. If a nondegenerate submersion $f\colon (M, g) \to \mathbb{R}$ satisfies the equation $H_f = \frac{\Delta f}{n} g$ on (M, g) then (M, g) is locally a semi-Riemannian warped product of a semi-Euclidean open interval and a semi-Riemannian manifold. (See [15, Lemma 2.3]).

Meanwhile note that, if such an f satisfies the local Möbius equation then, by Theorem 3.16, (M, g) is locally a semi-Riemannian twisted product of a semi-Euclidean open interval and a semi-Riemannian manifold. The reason is, in the latter case, f satisfies the equations $H_f(X, Y) = \frac{\Delta f}{n-1} g(X, Y)$ and $H_f(X, U) = 0$ for all $X, Y \in \Gamma ker f*$ and $U \in \Gamma(ker f_*)^{\perp}$. But this yields loss of information which is used to show that the totally umbilic semi-Riemannian fibres of f are spherical. (See [14, Lemma 11]). In fact, this lack of information is recovered in Theorem 3.20 and Theorem 3.22 with additional assumptions. Nevertheless, the local Möbius equation is of interest in relativity theory. Although the time functions of physically realistic spacetimes do not satisfy localized Möbius equation, they satisfy local Möbius equation, hence yield splitting results for these spacetimes. For example, see [4] and [5] for the applications of local Möbius equation in relativity theory.

Acknowledgments

This work has been partially supported by a project BFM2001-3778-C03-01 (Spain).

References

[1] F. Erkekoglu, E. García-Río, D. N. Kupeli and B. Ünal, *Characterizing Specific Riemannian Manifolds by Differential Equations*, preprint.

[2] M. Fernández-López, *Resultados de descomposición asociados á ecuación de Möbius*, Publ. Depto. Geom. Top. **96**, Univ. Santiago de Compostela, 2002.

[3] M. Fernández-López, E. García-Río and D. N. Kupeli, *The Local Möbius Equation and Decomposition Theorems in Riemannian Geometry*, Canadian Math. Bull., to appear.

[4] M. Fernández-López, E. García-Río and D. N. Kupeli, *A Characterization of Cosmological Time Functions*, Ann. Glob. Anal. Geom., to appear.

[5] M. Fernández-López, E. García-Río and D. N. Kupeli, *A Local Analytic Characterization of Schwarzschild Metrics*, J. Geom. Phys., to appear.

[6] M. Fernández-López, E. García-Río, D. N. Kupeli and B. Ünal, *A Curvature Condition for a Twisted Product to be a Warped Product*, Manuscripta Math., **106(2)**, (2001), 213-217.

[7] A. E. Fischer, *Riemannian Maps Between Riemannian Manifolds*, Contemp. Math., **132**, (1992), 331-366.

[8] A. E. Fischer, *Riemannian Submersions and the Regular Interval Theorem of Morse Theory*, Ann. Glob. Anal. Geom., **14**, (1996), 263-300.

[9] E. García-Río and D. N. Kupeli, *Singularity versus splitting theorems for stably causal spacetimes*, Ann. Glob. Anal. Geom., **14**, (1996), 301-312.

[10] E. García-Río and D. N. Kupeli, *Semi-Riemannian Maps and Their Applications*, Kluwer Academic Publishers, Dordrecht, 1999.

[11] B. Z. Gürel, *On Solutions of Semi-Euclidean Eikonal Equations*, M.S. Thesis, Math. Dept., METU, August 1999.

[12] N. Innami, *Splitting Theorems of Riemannian Manifolds*, Compositio Math., **47(3)**, (1982), 237-247.

[13] M. Kanai, *On a Differential Equation Characterizing a Riemannian Structure of a Manifold*, Tokyo J. Math., **6(1)**, (1983), 143-151.

[14] W. Kühnel, *Conformal Transformations Between Einstein Spaces*, in Conformal Geometry, R. Kulkarni and U. Pinkall (Eds.), 105-146. Aspects of Mathematics, **E12**, Vieweg & Sons, Brownschweig, 1988.

[15] W. KÜHNEL, *Conformal Vector Fields on Pseudo-Riemannian Spaces*, Diff. Geom. Appl., **7**, (1997), 237-250.

[16] M. OBATA, *Certain Conditions For a Riemannian Manifold to be Isometric with a Sphere*, J. Math. Soc. Japan, **14(3)**, (1962), 333-340.

[17] B. OSGOOD AND D. STOVE, *The Schwarzian Derivative and Conformal Mapping of Riemannian Manifolds*, Duke Math. J., **67(1)**, (1992), 57-99.

[18] R. PONGE AND H. RECKZIEGEL, *Twisted Products in Pseudo-Riemannian Geometry*, Geom. Dedicata **48** (1993), 15–25.

[19] T. SAKAI, *On Riemannian Manifolds Admitting a Function Whose Gradient is of Constant Norm*, Kodai Math. J., **19**, (1996), 39-51.

[20] Y. TASHIRO, *Complete Riemannian Manifolds and Some Vector Fields*, Trans. Amer. Math. Soc., **117**, (1965), 251-275.

Proceedings of the meeting
Lorentzian Geometry-Benalmádena 2001
Benalmádena, Málaga, Spain
Pub. de la RSME, Vol. 5 (2003), 125-139

Geometry of lightlike submanifolds in Lorentzian space forms

Angel Ferrández[1], Angel Giménez[1] and Pascual Lucas[1]

[1] *Departamento de Matemáticas, Universidad de Murcia*
Campus de Espinardo, E-30100 Espinardo Murcia
e-mails: *aferr@um.es, agpastor@um.es, plucas@um.es*

Abstract

In a Lorentzian space (or more generally in a pseudo-Riemannian space) appears a class of submanifolds where the induced metric is degenerate; they are called lightlike submanifolds. This work tries to give some relations between geometric objects of a lightlike submanifold and those ones of a Riemannian submanifold in a Lorentzian space. These relations allow us to obtain some characterization results for totally geodesic submanifolds in Lorentzian space forms.

2000 Mathematics Subject Classification: 53B30, 53C50, 53A04.

1 Introduction

It is well-known that in a Lorentzian manifold we can find three causal types of submanifolds: spacelike (or Riemannian), timelike (or Lorentzian) and lightlike (degenerate or null), depending on the character of the induced metric on the tangent space. The growing importance of lightlike submanifolds in global Lorentzian geometry, and their use in general relativity, motivated the study of degenerate submanifolds in a semi-Riemannian manifold. Due to the degeneracy of the metric, basic differences occur between the study of lightlike submanifolds and the classical theory of Riemannian as well as semi-Riemannian submanifolds (see, for example, [1], [3], [5], [7], [11], and [14]).

Lightlike submanifolds (in particular, lightlike hypersurfaces) appear in many papers in physics. For instance, the lightlike submanifolds are very interesting in general relativity, since they produce models of different types of horizons (event horizons, Cauchy's horizons, Kruskal's horizons). The idea that the Universe we live in can be represented as a 4-dimensional submanifold embedded in a $(4+d)$-dimensional space-time manifold has attracted the attention of many physicists. Higher dimensional semi-Euclidean spaces should provide a theoretical framework in which the fundamental laws of physics may appear to be unified, as in the Kaluza-Klein scheme. Lightlike hypersurfaces are also studied in the theory of electromagnetism (see, for example, [4], [15] and [16]).

In the following section we introduce the necessary geometric objects to study the geometry of lightlike submanifolds. The aim of this work is to relate the geometry of a lightlike submanifold in a Lorentzian space with the geometry of a Riemannian submanifold in the same ambient space. Making use of the well known results in the non-degenerate geometry we obtain some interesting results for lightlike submanifolds. In the last section, and as an application of the results obtained before, we completely describe the totally geodesic lightlike submanifolds in a Lorentzian space of constant curvature.

2 Preliminaries

Let $(M_1^{n+1}, \mathcal{N}^\circ)$ be an $(n+1)$-dimensional Lorentzian manifold endowed with a metric g (also denoted by $\langle,\rangle$). An m-dimensional submanifold P^m in M_1^{n+1} is said to be a *lightlike submanifold* if the induced metric on P^m is degenerate. In the sequel, and for simplicity of notation, we will write M and P instead of M_1^{n+1} and P^m, respectively. Also, we will suppose that $m \geq 3$.

The notation and basic facts on lightlike submanifolds are taken from [5]. The tangent vector bundle on M restricted to P can be, not uniquely, decomposed as

$$TM|_P = TP \oplus tr(TP) = S(TP) \perp (K \oplus \widetilde{K}) \perp S(TP^\perp), \tag{2.1}$$

where K is the 1-dimensional *radical distribution* of TP, that is, $K = TP \cap TP^\perp$, K is called a *lightlike transversal vector bundle*, and $S(TP)$

and $S(TP^{\perp})$ are called a *screen distribution* and a *screen transversal vector bundle* of P, respectively. They satisfy

$$TP = S(TP) \perp K \qquad \text{and} \qquad TP^{\perp} = S(TP^{\perp}) \perp K.$$

Bearing in mind the decomposition

$$TM|_P = TP \oplus tr(TP),$$

we obtain the Gauss formula for lightlike submanifolds,

$$\mathcal{N}^{\circ}_X Y = \nabla_X Y + \theta(X, Y), \quad \text{for all } X, Y \in \Gamma TP,$$

where $\nabla_X Y = (\mathcal{N}^{\circ}_X Y)^{\top}$ and $\theta(X, Y) = (\mathcal{N}^{\circ}_X Y)^t$ stand for the tangential and transversal parts, respectively. ∇ is called the *induced connection* on P and θ the *lightlike second fundamental form* of $P \subset M$, which is bilinear and symmetric. If U and X are normal and tangent sections, respectively, we can consider

$$A_U(X) = -S\left(\mathcal{N}^{\circ}_X U\right),$$

where $S : \Gamma TP \longrightarrow \Gamma S(TP)$ denotes the projection map on the screen distribution. The operator A_U is called the *lightlike shape operator* with respect to the section U. The properties of these operators can be found in [5]. Now we will restrict our attention on a special type of lightlike submanifolds defined by Kupeli, [11].

Definition 2.1 *A lightlike submanifold P of M is said to be* irrotational *if $\mathcal{N}^{\circ}_X \xi \in \Gamma TP$ for all tangent section X of P, where ξ is a section of the radical distribution K.*

It is easy to check that this definition is independent of the choice of ξ, and equivalent to the condition $\theta(X, \xi) = 0$ for any decomposition.

A submanifold P is called *geodesic* or *totally geodesic* if it contains the geodesics of M which are somewhere tangent to it. In other words, if $q \in P$ and $v \in T_qP$, then the geodesic γ in M with initial conditions $\gamma(0) = q$ and $\gamma'(0) = v$ lies in P. This is equivalent to saying that the vector fields on P are invariant by covariant derivation (this equivalence is valid for all torsion free connections, [9]). Although there is no way to induce connections on arbitrary lightlike submanifolds, the totally geodesic ones have such a connection, which is compatible with the degenerate metric but not derived from it, as degenerate metrics do not have Levi-Civita connections.

Trivial examples of irrotational submanifolds are the totally geodesic lightlike submanifolds, since $\mathcal{N}^{\circ}_X Y = \nabla_X Y$, or equivalently, $\theta = 0$. Another important example are the lightlike hypersurfaces, since $\langle \mathcal{N}^{\circ}_X \xi, \xi \rangle = 0$ and so the transversal part vanishes.

Definition 2.2 *Let P be an m-dimensional lightlike submanifold of an $(n+1)$-dimensional Lorentzian manifold M. We say that P is a* totally umbilical lightlike submanifold *if for all $U \in \Gamma TP^{\perp}$, there exist a differentiable function λ_U verifying*

$$A_U(X) = \lambda_U SX, \quad \textit{for all } X \in \Gamma TP. \tag{2.2}$$

This definition is independent of the choice of the distribution $S(TP)$.

If P is an irrotational lightlike submanifold, for $\xi \in \Gamma K$, $U \in \Gamma TP^{\perp}$ and $W \in \Gamma S(TP)$, we have

$$\left\langle \mathcal{N}^{\circ}_\xi U, W \right\rangle = -\left\langle \mathcal{N}^{\circ}_\xi W, U \right\rangle = 0,$$

then $S(\mathcal{N}^{\circ}_\xi U) = 0$, which implies

$$A_U(X) = A_U(SX), \quad \text{for all } U \in \Gamma TP^{\perp}, X \in \Gamma TP. \tag{2.3}$$

From now on, $\{E_1, \ldots, E_{m-1}, \xi, \eta, N_1, \ldots, N_{n-m}\}$ will denote a pseudo-orthonormal basis of $TM|_P$ adapted to the decomposition (2.1), where $E_i \in \Gamma S(TP)$, $N_j \in \Gamma S(TP^{\perp})$, $\xi \in \Gamma K$, $\eta \in \Gamma \widetilde{K}$, satisfying

$$\langle E_i, E_i \rangle = \langle N_j, N_j \rangle = \langle \xi, \eta \rangle = 1, \quad \langle \xi, \xi \rangle = \langle \eta, \eta \rangle = 0.$$

The following proposition can be found in [11, page 80]. Here we present a different point of view.

Proposition 2.3 *Let P be an m-dimensional irrotational lightlike submanifold of a Lorentzian manifold M. The following statements are equivalent:*

(i) There exist a transversal section H satisfying that

$$\theta(X, Y) = \langle X, Y \rangle H, \quad \textit{for all } X, Y \in \Gamma TP.$$

(ii) P is totally umbilical.

Proof. Let $\{E_1, \ldots, E_{m-1}, \xi, \eta, N_1, \ldots, N_{n-m}\}$ be a pseudo-orthonormal basis adapted to $TM|_P$. We claim that $\theta(X,Y) = \theta(SX, SY)$ for $X, Y \in \Gamma TP$. The proof is a consequence of the following computation

$$\begin{aligned}
\theta(SX,SY) &= \langle \mathcal{N}^\circ_{SX} SY, \xi\rangle\, \eta + \sum_{j=1}^{n-m} \langle \mathcal{N}^\circ_{SX} SY, N_j\rangle\, N_j \\
&= -\langle \mathcal{N}^\circ_{SX}\xi, SY\rangle\, \eta - \sum_{j=1}^{n-m} \langle \mathcal{N}^\circ_{SX} N_j, SY\rangle\, N_j \qquad (2.4)\\
&= \langle A_\xi(SX), SY\rangle\, \eta + \sum_{j=1}^{n-m} \left\langle A_{N_j}(SX), SY\right\rangle N_j.
\end{aligned}$$

Let us assume (i). Then $\theta(SX,SY) = \langle SX, SY\rangle\, H$, H being

$$H = \lambda\eta + \sum_{j=1}^{n-m} \lambda_j N_j, \qquad (2.5)$$

and so, combining this equality with (2.4), we obtain

$$\langle A_\xi(SX), SY\rangle = \lambda \langle SX, SY\rangle, \quad \left\langle A_{N_j}(SX), SY\right\rangle = \lambda_j \langle SX, SY\rangle.$$

These equations are equivalent to

$$\langle A_\xi(SX) - \lambda SX, SY\rangle = 0, \quad \left\langle A_{N_j}(SX) - \lambda_j SX, SY\right\rangle = 0$$

$\forall X, Y \in \Gamma TP$, which imply

$$A_\xi(X) = A_\xi(SX) = \lambda SX, \quad A_{N_j}(X) = A_{N_j}(SX) = \lambda_j SX$$

$\forall X, Y \in \Gamma TP$. Taking into account that $\{\xi, N_1, \ldots, N_{n-m}\}$ is a basis of $TP^\perp$, we obtain (ii).

Conversely, if P is totally umbilical, then equation (2.2) implies that there exist differentiable functions λ, λ_j satisfying $A_\xi(X) = \lambda SX$ and $A_{N_j}(X) = \lambda_j SX$ for all $X \in \Gamma TP$. Substituting these equalities in (2.4) we complete the proof.

$\square$

If P is irrotational, we know that $S(\mathcal{N}^\circ_\xi \xi) = 0$, and so there exist a differentiable function ρ such that $\mathcal{N}^\circ_\xi \xi = -\rho\xi$. As a consequence of this computation we have the following result.

Proposition 2.4 *Let P be an irrotational lightlike submanifold of a Lorentzian manifold M. Then the integral curves of $\xi \in \Gamma K$ are null pregeodesics of M.*

Proof. Let h be a function such that $\xi(\log h) = \rho$, and take $\xi = h\xi$. It is easy to check that $\mathcal{N}^\circ_\xi\, \xi = 0$, hence the integral curves of ξ are geodesics and, in consequence, the integral curves of ξ are pregeodesics.

□

The above statement is true for all lightlike hypersurfaces. Furthermore, if M have constant curvature, it is well know that the null geodesics are null lines, which implies that P is swept out by null lines.

3 Lightlike submanifolds and submersions

Let P be an m-dimensional lightlike submanifold immersed in an $(n+1)$-dimensional Lorentzian manifold M by $\psi : P \longrightarrow M$. Let $\pi : M \longrightarrow B$ be a Lorentzian submersion of codimension one (that is, dim(B)=1), and set $\pi = \pi \circ \psi$. Let us assume that $\pi : P \longrightarrow B = \pi(P)$ is a submersion, or equivalently, P is not contained in any fiber of π.

Let us denote by $\mathcal{F}_t$ and Σ_t the fibers of π and π, respectively. It is clear that Σ_t is a Riemannian submanifold immersed in $\mathcal{F}_t$. The following diagram illustrates the situation:

$$\begin{tikzcd}
(\mathcal{F}_t, \widehat{\mathcal{N}}^\circ_t) \arrow[r, "i_t"] & (M, \mathcal{N}^\circ) \arrow[r, "\pi"] & B \\
(\Sigma_t, \widehat{\mathcal{N}}_t) \arrow[u, "\psi_t"] \arrow[r, "j_t"] & (P, \nabla) \arrow[u, "\psi"'] \arrow[r, "\widetilde{\pi}"] & B = \pi(P) \arrow[u, "i"']
\end{tikzcd} \tag{3.1}$$

The Levi-Civita connections $\widehat{\mathcal{N}}^\circ_t$ and $\widehat{\mathcal{N}}_t$ on $\mathcal{F}_t$ and Σ_t, respectively, can be extended to connections $\widehat{\mathcal{N}}^\circ$ and $\widehat{\mathcal{N}}$ on the vertical distributions $\mathcal{V}(TM)$ and $\mathcal{V}(TP)$, respectively.

Our aim now is to define geometric objects with respect to these submersions.

Definition 3.1 *Let P be a lightlike submanifold of a Lorentzian manifold M and $\pi : M \longrightarrow B$ a submersion as before. The screen vector bundle $S(TP) = \mathcal{V}(TP)$ on P is called the* canonical screen distribution *associated to the submersion π.*

Bearing in mind the above diagram, definitions and notations, we can split the tangent vector bundle $TM|_P$ in a different way from (2.1), as follows,

$$\begin{aligned} TM|_P &= \mathcal{V}(TM)|_P \perp \mathcal{H}(TM)|_P \\ &= \left(\mathcal{V}(TP) \perp \mathcal{V}(TP)^{\perp} \right) \perp \mathcal{H}(TM)|_P \\ &= S(TP) \oplus \left(\mathcal{V}(TP)^{\perp} \perp \mathcal{H}(TP) \right), \end{aligned} \tag{3.2}$$

where $\mathcal{V}(TP)^{\perp}$ denotes the orthogonal of $\mathcal{V}(TP)$ in $\mathcal{V}(TM)|_P$. Comparing the decompositions (2.1) and (3.2) we deduce

$$(K \oplus K) \perp S(TP^{\perp}) = \mathcal{V}(TP)^{\perp} \perp \mathcal{H}(TP).$$

Let $\chi \in \mathcal{H}(TM)$ be a unit local basic vector field with respect to π and write $\chi = \chi|_P \in \mathcal{H}(TP)$. Since P is not contained in any fiber of π, then $\langle \xi, \chi \rangle \neq 0$ for $\xi \in \Gamma K$, so that $K \oplus \mathcal{H}(TP)$ is a hyperbolic plane. Choose K such that $\Pi = K \oplus K = K \oplus H(TP)$. We can construct local frames $\{\xi, \eta\}$ and $\{N, \chi\}$, with $N \in \mathcal{V}(TP)^{\perp}$ and $\eta \in K$, satisfying

$$\varepsilon = \langle N, N \rangle = -\langle \chi, \chi \rangle, \quad \xi = \frac{1}{\sqrt{2}}(N + \chi), \quad \eta = \frac{\varepsilon}{\sqrt{2}}(N - \chi), \tag{3.3}$$

where $\varepsilon = \pm 1$. In this case $S(TP^{\perp})$ is necessarily the orthonormal complementary of span$\{N\}$ in $\mathcal{V}(TP)^{\perp}$.

Definition 3.2 *The section ξ and the vector bundle $S(TP^{\perp})$ defined above are called the* canonical radical section *and the* canonical screen transversal vector bundle *associated to the submersion π.*

If M is time-oriented, we can choose ξ and η pointing out to the future. In this case they are completely determined by the submersion π. Under these conditions, if $\{N_0 = N, N_1, \ldots, N_{n-m}\}$ is a basis of $\mathcal{V}(TP)^{\perp}$, where $\{N_1, \ldots, N_{n-m}\}$ expands the canonical screen transversal vector bundle, we consider the operators

$$\begin{array}{rcl} A_{N_j} : \mathcal{V}(TP) & \longrightarrow & \mathcal{V}(TP) \\ W & \rightsquigarrow & -\mathcal{V}(\widehat{\mathcal{N}}^{\circ}_W N_j) \end{array} \qquad \begin{array}{rcl} \sigma : \mathcal{V}(TP) \times \mathcal{V}(TP) & \longrightarrow & \mathcal{V}(TP)^{\perp} \\ (W_1, W_2) & \rightsquigarrow & \left(\widehat{\mathcal{N}}^{\circ}_{W_1} W_2 \right)^{\perp} \end{array}$$

where $0 \leq j \leq n-m$. Moreover, these operators restricted to each fiber are the shape operator respect to N_j and the second fundamental form of the immersion $\psi_t : \Sigma_t \longrightarrow \mathcal{F}_t$, respectively.

On the other hand, bearing in mind the diagram (3.1), we can consider the operators defined by

$$\begin{array}{rcl} \hat{A}^{\circ} : \mathcal{V}(TM) & \longrightarrow & \mathcal{V}(TM) \\ V & \rightsquigarrow & -\mathcal{N}^{\circ}_{V}\chi \end{array} \qquad \begin{array}{rcl} \hat{\sigma}^{\circ} : \mathcal{V}(TM)\times\mathcal{V}(TM) & \longrightarrow & \mathcal{H}(TM) \\ (V_1,V_2) & \rightsquigarrow & \mathcal{H}\left(\mathcal{N}^{\circ}_{V_1}V_2\right) \end{array}$$

These operators, restricted to each fiber $\mathcal{F}_t$, are the shape operator and the second fundamental form of the immersion $i_t : \mathcal{F}_t \to M$. We will consider both operators acting on $\mathcal{V}(TM)|_P$.

We can write the following equations relating the above geometric objects,

$$\begin{aligned} \mathcal{N}^{\circ}_{V_1}V_2 &= \widehat{\mathcal{N}}^{\circ}_{V_1}V_2 + \hat{\sigma}^{\circ}(V_1,V_2) = \widehat{\mathcal{N}}^{\circ}_{V_1}V_2 - \varepsilon\left\langle \hat{A}^{\circ}(V_1),V_2\right\rangle\ \chi, \\ \widehat{\mathcal{N}}^{\circ}_{W_1}W_2 &= \widehat{\mathcal{N}}_{W_1}W_2 + \ \sigma(W_1,W_2) \\ &= \widehat{\mathcal{N}}_{W_1}W_2 + \varepsilon\left\langle\ A_N(W_1),W_2\right\rangle N + \sum_{j=1}^{n-m}\left\langle\ A_{N_j}(W_1),W_2\right\rangle N_j, \end{aligned} \tag{3.4}$$

where W_1, W_2 are sections on $\mathcal{V}(TP) = S(TP)$ and V_1, V_2 are sections on $\mathcal{V}(TM)|_P$. These equations restricted to each fiber represent the Gauss equations of both immersions $\mathcal{F}_t \subset M$ and $\Sigma_t \subset \mathcal{F}_t$, respectively.

We are going to state some results relating the different geometric objects defined above. From these relationships we will obtain interesting applications for particular cases.

Proposition 3.3 *Let P be an m-dimensional irrotational lightlike submanifold of an $n+1$-dimensional Lorentzian manifold M, and let $\pi : M \longrightarrow B$ be a totally umbilical semi-Riemannian (Riemannian or Lorentzian) submersion. Let $S(TP)$ be the canonical screen distribution, ξ the canonical radical section associated to π and $\{N, N_1, \dots, N_{n-m}\}$ an orthonormal basis of $\mathcal{V}(TP)^{\perp}$. Then the following statements hold:*

(i) $\mathcal{N}^{\circ}_{W}\xi = A_{\xi}(W)$, *for all* $W \in \Gamma S(TP)$.

(ii) $A_{N_j} =\ A_{N_j}$ *for* $1 \le i \le n-m$, *and* $A_{\xi} = \frac{1}{\sqrt{2}}(\ A_N + \mu\mathrm{Id})$, *where μ is the differentiable function satisfying the equation $\hat{A}^{\circ}(V) = \mu V$.*

Proof. *(i)* By hypothesis, $\mathcal{N}^{\circ}_{X}\xi$ is a section of TP. The proof follows by showing that $\langle \mathcal{N}^\circ_W \xi, \eta \rangle = 0$. Indeed,

$$\begin{aligned}
\langle \mathcal{N}^\circ_W \xi, \eta \rangle &= \left\langle \frac{1}{\sqrt{2}} \mathcal{N}^\circ_W (N + \ \chi), \frac{\varepsilon}{\sqrt{2}} (N - \ \chi) \right\rangle \\
&= \frac{\varepsilon}{2} \left(- \langle \mathcal{N}^\circ_W N, \ \chi \rangle + \langle \mathcal{N}^\circ_W \ \chi, N \rangle \right) \\
&= \varepsilon \, \langle \mathcal{N}^\circ_W \ \chi, N \rangle \\
&= \varepsilon \left\langle -\hat{A}^\circ(W), N \right\rangle = 0.
\end{aligned}$$

(ii) Clearly, $A_{N_j} = \ A_{N_j}$ since $S(TP) = \ \mathcal{V}(TP)$. Bearing in mind the above statement and that the fibers of π are totally umbilical, that is, $\hat{A}^\circ(W) = \mu W$, we obtain

$$\begin{aligned}
A_\xi(W) &= -\frac{1}{\sqrt{2}} \mathcal{N}^\circ_W (N + \ \chi) \\
&= -\frac{1}{\sqrt{2}} \left(\mathcal{N}^\circ_W N + \mathcal{N}^\circ_W \ \chi \right) \\
&= -\frac{1}{\sqrt{2}} \left(\widehat{\mathcal{N}}^\circ_W N - \hat{A}^\circ(W) \right) \\
&= \frac{1}{\sqrt{2}} \left(\ A_N(W) + \mu W \right).
\end{aligned}$$

□

Proposition 3.4 *Let P be an irrotational lightlike submanifold of a Lorentzian manifold M and $\pi : M \longrightarrow B$ a totally umbilical submersion with semi-Riemannian fibers $\mathcal{F}_t$. Let $\ \pi$ be the submersion induced by π on P with fibers Σ_t. Then P is totally umbilical if and only if Σ_t is totally umbilical in $\mathcal{F}_t$ for all $t \in \ \pi(P)$.*

The proof is a direct consequence of Proposition 3.3. In particular, for totally geodesic lightlike submanifolds we have $\ A_{N_j} = 0$, $1 \leq j \leq n - m$, and $\ A_N = -\mu \mathrm{Id}$. Then if the fibers $\mathcal{F}_t$ are totally geodesics ($\mu = 0$), then the immersions $\Sigma_t \subset \mathcal{F}_t$ are totally geodesics.

4 Applications to Lorentzian space forms

This section contains some applications to Lorentzian manifolds of constant curvature $M_1^{n+1}(c)$. In particular we describe the totally geodesic lightlike submanifolds in $M_1^{n+1}(c)$. We study separately the ambient spaces $\mathbb{R}_1^{n+1}$, $\mathbb{S}_1^{n+1}$ and $\mathbb{H}_1^{n+1}$.

Lightlike submanifolds in $\mathbb{R}_1^{n+1}$

As it is well known, an irrotational lightlike submanifold is swept out by null geodesics, and these ones can be naturally extended to complete geodesics. The problem is the appearance of singular points, but we can work locally.

Fix a vector $a \in \mathbb{R}_1^{n+1}$ such that $\langle a, a\rangle = -1$ and consider $\pi_a : \mathbb{R}_1^{n+1} \longrightarrow \mathbb{R}$ the map defined by $\pi_a(x) = \langle x, a\rangle$. It is easy to prove that π is a totally geodesic submersion with Riemannian fibers, and π is a submersion, since P can not be contained in any fiber (they are Riemannian). In particular, if we choose $a = (1, 0, \ldots, 0)$ we obtain a submersion where the fibers are $\{t\} \times \mathbb{R}^n$.

We have the following situation:

$$\begin{tikzcd} \{t\} \times \mathbb{R}^n \arrow[r, "i_t"] & \mathbb{R}_1^{n+1} \arrow[r, "\pi_a"] & \mathbb{R}^- \\ \Sigma_t \arrow[u, "\psi_t"] \arrow[r, "j_t"] & P \arrow[u, "\psi"'] \arrow[r, "\pi_a"] & \pi_a(P) \arrow[u, "i"'] \end{tikzcd} \tag{4.1}$$

Remark 4.1 *Whenever P is a lightlike hypersurface of the $(n+1)$- dimensional Lorentz-Minkowski space and π_a is as above, with $a = (1, 0, \ldots, 0)$, the lightlike transversal vector bundle expanded by the vector field η, given by* (3.3)*, agrees with the canonical lightlike transversal vector bundle introduced in [2] (up to the orientation). In particular, if $n = 3$, the Gauss map N^t of the immersion $\Sigma_t \subset \mathcal{F}_t$ defined by*

$$\begin{aligned} N^t : \Sigma_t &\longrightarrow \mathbb{S}^2 \\ p \quad &\rightsquigarrow \quad N|_{\Sigma_t}(p) \end{aligned}$$

where the N is given by (3.3)*, agrees with the Gauss map associated to a lightlike hypersurface P with base Σ_t introduced by Kossowski in [10].*

It is well-known that the only totally geodesic submanifolds of $\mathbb{R}^n$ are pieces of r-planes, with $2 \leq r < n$. Moreover, the only non geodesic totally umbilical hypersurfaces of $\mathbb{R}^n$ are pieces of spheres. Then from this fact and by using Proposition 3.4 we deduce the following results already known.

Proposition 4.2 *The only totally geodesic lightlike submanifods in the Lorentz-Minkowski space $\mathbb{R}_1^{n+1}$ are pieces of null m-planes.*

Proof. We know that P is sweot out by null lines, and the sections Σ_t provided by the submersion π are $(m-1)$-planes. We only have to prove that ξ is a parallel section with respect to $\mathcal{V}(TP) = S(TP)$. Using Proposition 3.3 we have

$$\mathcal{N}^\circ_W \xi = A_\xi(W) = 0, \qquad \text{for all } W \in \Gamma S(TP),$$

which is the desired conclusion.

□

The following result, already proved by Akivis and Goldberg in [1], can also be easily deduced.

Proposition 4.3 *The only totally umbilical lightlike hypersurfaces in* $\mathbb{R}^{n+1}_1$ *are the lightlike cones.*

Proof. Let P be a totally umbilical lightlike hypersurface, then the fibers Σ_t are spherical. It is easy to see that there are only two lightlike hypersurfaces that contain the fiber Σ_t, but we can construct two lightlike cones containing Σ_t. This concludes the proof.

□

Lightlike submanifolds in the De-Sitter space $\mathbb{S}^{n+1}_1$

Fix a vector $a \in \mathbb{R}^{n+2}_1$ such that $\langle a, a\rangle = -1$ and consider as before $\bar\pi_a : \mathbb{R}^{n+2}_1 \longrightarrow \mathbb{R}$ the map defined by $\bar\pi_a(x) = \langle x, a\rangle$. It is not difficult to prove that $\pi_a = \bar\pi_a|_{\mathbb{S}^{n+1}_1}$ is a totally umbilical submersion with Riemannian fibers. To simplify the computations, choose $a = (1, 0, \ldots, 0)$. Then, for each $t \in \mathbb{R}$, the fiber $\mathcal{F}_t = \pi_a^{-1}(t)$ is a totally umbilical hypersurface with shape operator $\hat{A}^\circ = \mu I$, where $\mu = t/\sqrt{t^2+1}$. Therefore $\mathcal{F}_t$ have positive constant curvature $1/(t^2+1)$, so that $\mathcal{F}_t$ is a sphere of radius $\sqrt{t^2+1}$. Note that the fiber $\mathcal{F}_0$ is totally geodesic ($\mu|_{\mathcal{F}_0} = 0$). We have the following situation:

$$\begin{array}{ccccc}
\{t\}\times\mathbb{R}^{n+1} & \xrightarrow{\iota_t} & \mathbb{R}^{n+2}_1 & \xrightarrow{\bar\pi} & \mathbb{R}^- \\
{\scriptstyle\bar\psi_t}\uparrow & & \uparrow{\scriptstyle\bar\psi} & & \uparrow{\scriptstyle i} \\
\{t\}\times\mathbb{S}^n(1/(1+t^2)) & \xrightarrow{i_t} & \mathbb{S}^{n+1}_1 & \xrightarrow{\pi_a} & \mathbb{R}^- \\
{\scriptstyle\psi_t}\uparrow & & \uparrow{\scriptstyle\psi} & & \uparrow{\scriptstyle i} \\
\Sigma_t & \xrightarrow{j_t} & P & \xrightarrow{\pi_a} & \pi_a(P)
\end{array} \tag{4.2}$$

Proposition 4.4 *The m-dimensional totally geodesic lightlike submanifolds P in the De-Sitter space $\mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2}$ are exactly the intersections of null $(m+1)$-planes in $\mathbb{R}_1^{n+2}$ through the origin with $\mathbb{S}_1^{n+1}$.*

Proof. Consider $\pi_a : \mathbb{S}_1^{n+1} \longrightarrow \mathbb{R}$ with $a = (1, 0, \ldots, 0)$, defined as above. Then the fibers $\mathcal{F}_t$ are exactly $\mathbb{S}^n(1/(1+t^2))$. Since totally geodesic lightlike submanifolds are swept out by null lines, then it is very important to study the immersion $\Sigma_0 \subset \mathcal{F}_0 = \mathbb{S}^n(1)$. We work in $\mathcal{F}_0$ because it is the unique totally geodesic fiber in $\mathbb{S}_1^{n+1}$.

By using Proposition 3.3, for a suitable basis $\{N, N_1, \ldots, N_{n-m}\}$ of $\mathcal{V}(TP)^\perp$, we have

$$A_\xi = \frac{1}{\sqrt{2}}(\ A_N + \mu \mathrm{Id}), \quad A_{N_j} = \ A_{N_j}, \quad 1 \le j \le n-m.$$

Then P is totally geodesic if and only if $\ A_{N_j} = 0$ and $\ A_N = -\mu\mathrm{Id}$. In particular, if we restrict these operators to the fiber Σ_0, then $\ A_N = 0$ and consequently the immersion $\Sigma_0 \subset \mathbb{S}^n$ is totally geodesic. It is well-known that the totally geodesics submanifolds of $\mathbb{S}^n$ are $(m-1)$-dimensional spheres of maximum radius (that is, intersections of m-planes through the origin with $\mathbb{S}^n$). Until now, we have proved that $\Sigma_0 = \Pi^m \cap \mathbb{S}^n$ where Π^m is an m-plane contained in $\mathcal{F}_0 = \mathbb{R}^{n+1}$. We are going to prove that ξ is a parallel section on Σ_0 in $\mathbb{R}_1^{n+2}$.

We denote by ν the normal vector field of the immersion $\mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2}$, so that the shape operator of $\mathbb{S}_1^{n+1}$ is given by $A_\nu = -\mathrm{Id}$. If $W \in S(TP) = \ \mathcal{V}(TP)$ and ξ is the canonical radical section, then

$$\bar{\nabla}^\circ_W \xi = \mathcal{N}^\circ_W \xi + \langle A_\nu(W), \xi \rangle\, \nu = A_\xi(W) = 0.$$

Let $\bar{\Pi}$ be the null $(m+1)$-plane given by $\bar{\Pi} = \Pi^m \perp \mathrm{span}\,\{v\}$, where v have the same direction of $\xi|_{\Sigma_0}$, then it is easy to check that $P = \bar{\Pi} \cap \mathbb{S}_1^{n+1}$.

$\square$

Lightlike submanifolds of the Anti-De Sitter space $\mathbb{H}_1^{n+1}$

We can obtain similar results as in the De Sitter space. Fix a vector $a \in \mathbb{R}_2^{n+2}$ such that $\langle a, a \rangle = 1$ and let us consider $\bar{\pi}_a : \mathbb{R}_2^{n+2} \longrightarrow \mathbb{R}$ the map defined by $\bar{\pi}_a(x) = \langle x, a \rangle$. Consider $\pi_a = \bar{\pi}_a|_{\mathbb{H}_1^{n+1}}$. It can be proved that π_a is a totally umbilical submersion with Lorentzian fibers. The fibers $\mathcal{F}_t = \pi_a^{-1}(t)$ are totally umbilical hypersurfaces with shape operator $\hat{A}^\circ = (-t/\sqrt{t^2+1})I$. Then they are of negative constant curvature $-1/(1+t^2)$

and therefore they are pseudo-hyperbolic spaces $\mathbb{H}^n_1(-1/(1+t^2))$. Note that the fiber $\mathcal{F}_0$ is totally geodesic in $\mathbb{H}^{n+1}_1$. The induced map π is not in general a submersion, but in this case, since the fibers are again anti De-Sitter spaces, we can suppose that P is not contained in any fiber and then π is a submersion. Choosing the point $a=(0,\dots,0,1)$, we are in the following situation.

$$\begin{array}{ccccc} \{t\}\times\mathbb{R}^{n+1}_2 & \xrightarrow{\iota_t} & \mathbb{R}^{n+2}_2 & \xrightarrow{\bar\pi} & \mathbb{R}^+ \\ {\scriptstyle\bar\psi_t}\uparrow & & \uparrow{\scriptstyle\bar\psi} & & \uparrow{\scriptstyle i} \\ \{t\}\times\mathbb{H}^n_1(-1/(1+t^2)) & \xrightarrow{i_t} & \mathbb{H}^{n+1}_1 & \xrightarrow{\pi_a} & \mathbb{R}^+ \\ {\scriptstyle\psi_t}\uparrow & & \uparrow{\scriptstyle\psi} & & \uparrow{\scriptstyle i} \\ \Sigma_t & \xrightarrow{j_t} & P & \xrightarrow{\pi_a} & \pi_a(P) \end{array} \tag{4.3}$$

Denote by $\bar\Pi^{m+1}_{i,r}$ an $(m+1)$-plane through the origin of $\mathbb{R}^{n+2}_2$, where i and r denote the index and the dimension of the radical distribution, respectively. In the semi-Euclidean space $\mathbb{R}^{n+2}_2$ there exist six different types of $(m+1)$-planes, they are: $\bar\Pi^{m+1}_{0,0}$, $\bar\Pi^{m+1}_{1,0}$, $\bar\Pi^{m+1}_{2,0}$, $\bar\Pi^{m+1}_{1,1}$, $\bar\Pi^{m+1}_{0,1}$ and $\bar\Pi^{m+1}_{0,2}$.

The intersection of $(m+1)$-planes of index 0 with the Anti De-Sitter space is empty. On the other hand, $\bar\Pi^{m+1}_{1,0}\cap\mathbb{H}^{n+1}_1$ is a hyperbolic space $\mathbb{H}^m$ and $\bar\Pi^{m+1}_{2,0}\cap\mathbb{H}^{n+1}_1$ is an Anti De-Sitter space $\mathbb{H}^m_1$. The following proposition describes the intersection $\bar\Pi^{m+1}_{1,1}\cap\mathbb{H}^{n+1}_1$.

Proposition 4.5 *The m-dimensional totally geodesic lightlike submanifolds P of the Anti De-Sitter space $\mathbb{H}^{n+1}_1\subset\mathbb{R}^{n+2}_2$ are exactly the intersections of the $(m+1)$-planes $\bar\Pi^{m+1}_{1,1}$ in $\mathbb{R}^{n+2}_2$ through the origin with $\mathbb{H}^{n+1}_1$.*

Proof. Consider the submersion $\pi_a:\mathbb{H}^{n+1}_1\longrightarrow\mathbb{R}$ defined above and let π be the submersion induced by π. Similar considerations as in the De-Sitter space apply to this case, and prove that P is of the form $\Sigma_0\times\ell$ where ℓ stands for a constant null direction and Σ_0 is an $(m-1)-$dimensional totally geodesic Riemannian submanifold of $\mathcal{F}_0=\mathbb{H}^n_1$, that is, $\Sigma_0=\mathbb{H}^{m-1}$. Actually, we can write $\Sigma_0=\Pi^m\cap\mathbb{H}^n$, where Π^m is a Lorentzian plane of dimension m. Take $\bar\Pi^{m+1}_{1,1}=\Pi^m\perp\operatorname{span}\{v\}$, where v have the same direction of ℓ, then it is easy to show that $P=\bar\Pi^{m+1}_{1,1}\cap\mathbb{H}^{n+1}_1$.

□

Acknowledgments

This research has been partially supported by Dirección General de Investigación (MCYT) grant BFM2001-2871 and Fundación Séneca grant PI-3/00854/FS/01.
The second author is supported by a FPI Grant, Program PG, Ministerio de Educación y Ciencia.

References

[1] M.A. AKIVIS AND V.V. GOLDBERG , *On some methods of construction of invariant normalizations of lightlike hypersurfaces*, Differential Geom. Appl, **12**, 121-143 (2000).

[2] A. BEJANCU, A. FERRÁNDEZ AND P. LUCAS, *A new viewpoint on geometry of a lightlike hypersurface in semi-Euclidean space*, Saitama Mathematical Journal, **16**, 31-38 (1998).

[3] W.B. BONNOR, *Null hypersurfaces in Minkowski spacetime*, Tensor, N. S., **24**, 329-345 (1972).

[4] G.C. DEBNEY AND J.D. ZUND, *A note on the classification of electromagnetic fields,* Tensor N.S. **22**, 333–340 (1971).

[5] K.L. DUGGAL AND A. BEJANCU, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Academic Publishers, Amsterdam (1996).

[6] P.E. EHRLICH, J.K. BEEM AND K.L. EASLEY, *Global Lorentzian Geometry*, Marcel Dekker, Inc., New York (1996).

[7] A. FRIEDMAN, *Isotropic embedding of Riemannian manifolds into Euclidean spaces,* Rev. of Modern Phys. **37**, 201–203 (1965).

[8] G.J. GALLOWAY, *Maximum principles for null hypersurfaces and null splitting theorems*, Ann. Henri Poincaré, **1**, 543-567, (2000).

[9] S. KOBAYASHI AND K. NOMIZU, *Foundations of Differential Geometry,* Wiley (interscience), New York, Vol. I (1963).

[10] M. KOSSOWSKI, *The intrinsic conformal structure and Gauss map of a lightlike hypersurface in Minkowski space*, Transactions of the American Mathematical Society, **316**, 369-383, (1989).

[11] D.N. KUPELI, *Singular semi-Riemannian geometry*, Kluwer Academic Publishers, Dordrecht (1996).

[12] B. O'NEILL, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York (1983).

[13] R. ROSCA, *On null hypersurfaces of a Lorentzian manifold*, Tensor, N. S., **23**, 66-74 (1972).

[14] J. ROSEN, *Embedding of various relativistic Riemannian spaces in pseudo-Riemannian spaces,* Rev. of Modern Phys. **37**, 204–214 (1965).

[15] H.S. RUSE, *On the geometry of the electromagnetic field in general relativity,* Proc. London Math. Soc. **41**, 302 (1936).

[16] J.L. SYNGE, *Principal Null Directions Defined in Spacetime by an Electromagnetic Field,* Univ. of Toronto Studies Appl. Math., Ser. 1, 1935.

Proceedings of the meeting
Lorentzian Geometry-Benalmádena 2001
Benalmádena, Málaga, Spain
Pub. de la RSME, Vol. 5 (2003), 141-154

A topological technique for the geodesic connectedness in some Lorentzian manifolds

J. L. Flores[1] and M. Sánchez[1]

[1] *Depto. Geometría y Topología, Fac. Ciencias, Univ. Granada, Avda. Fuentenueva s/n, 18071-Granada, Spain*

Abstract

In some Lorentzian manifolds the problem of the geodesic connectedness can be reduced to a topological problem related to Brouwer's degree. This approach allows us to obtain the geodesic connectedness of some important spacetimes as multiwarped ones or the exterior region of slow Kerr spacetime. Other related properties of interest can be also studied.

1 Introduction

This talk is based on the results obtained in the articles [11], [12], [13] and [14]. Our main aim is to describe briefly a new technique for the study of the geodesic connectedness in Lorentzian Geometry, that is, the problem of finding a geodesic joining two given points in a connected Lorentzian manifold (M, g). This problem is not easy (even if M is compact, the geodesic connectedness may fail), and different techniques have been developed to solve it (see [23] for a survey). Recall that these techniques are related to methods of groups theory [9], geometrical methods (introduced in [8] and studied in the book [2]), variational methods (introduced in [3]; see for example [18] or [23]) and methods based on a direct integration of the geodesic equations (for example [22]). Other techniques allow the study of the existence of geodesics with a defined causal character between two given points. Among the results for these geodesics, we emphasize the classical Avez-Seifert one: *in a globally hyperbolic spacetime, two points can be joined by a causal geodesic if and only if they can be joined by a causal curve.*

In Section 3 we introduce our topological technique. Given a splitting manifold, we give an overview of the steps to obtain conditions which imply the geodesic connectedness of the manifold.

After a brief description of *multiwarped spacetimes*, in Section 4 we stablish the results on geodesic connectedness obtained when the topological methods are applied to this class of spacetimes. In the particular case of *Generalized Robertson-Walker (GRW) spacetimes*, the results on existence are more accurate and we also give results on multiplicity, conjugate points and Morse type relations.

Finally, in Section 5 we apply the technique to *Kerr spacetime*. Here we obtain results not only on connectedness but also on non-connectedness of different natural regions of this spacetime.

From now on, we will assume for the manifolds to be connected. Given a Lorentzian manifold (M, g), a tangent vector v will be called *timelike* (resp. *lightlike*; *causal*; *spacelike*) if $g(v, v) < 0$ (resp. $= 0$ and $v \neq 0$; ≤ 0 and $v \neq 0$; > 0 or $v = 0$).

2 A topological technique

First, let us explain intuitively the relation between geodesic connectedness and topological arguments. Consider two points $p_0 \neq p_1$ of a splitting manifold M^{n+1}, and fix a topological sphere of the tangent space to p_0, $S \subset T_{p_0}M$, such that the vector 0 is included in the interior of S. Consider now the subset $\exp_{p_0} sS$, for each $s \in \mathbb{R}$, yielded by the geodesics emanating at p_0 ($\exp_{p_0}$ is the exponential map at p_0). Initially, for small s, p_1 is outside $\exp_{p_0} sS$, but for some bigger s, p_1 may lie inside $\exp_{p_0} sS$. This topological change (from being outside to being inside the exponential of a sphere) reflects that p_0 and p_1 can be connected by a geodesic, and suggest to use a topological argument.

Even though this is quite intuitive, the mathematical formalization of these ideas is rather long. More precise mathematically, consider the following steps:

– Step 1. Assume that M is an open subset of $\mathbb{R}^{n+1}$, and consider the function

$$\mathcal{F} : \mathcal{D} \subset T_{p_0}M \equiv \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, \quad \mathcal{F}(v) = \gamma_v(1) - p_1, \qquad (2.1)$$

where γ_v is the unique geodesic starting at p_0 satisfying $\gamma_v'(0) = v$, for any $v \in T_{p_0}M$, and γ_v is defined at 1 for all v in the domain $\mathcal{D}$. Now, the zeroes of the function $\mathcal{F}$ correspond with geodesics connecting p_0 and p_1. (If M is not included in $\mathbb{R}^{n+1}$, assume that suitable coordinates can be chosen. Otherwise, the structure of the spacetime may also make useful the idea underlying in (2.1), as in multiwarped spacetimes).

– Step 2. Suppress the degree of freedom associated to the reparametrization of the geodesics and reduce the problem to find zeroes of a new function $\overline{\mathcal{F}} : \overline{\mathcal{D}} \to \mathbb{R}^n$, where the new domain $\overline{\mathcal{D}}$ is also included in $\mathbb{R}^n$.

– Step 3. Assume that, as in most classical spacetimes, a partial integration of the geodesics equations can be done (by Noether Theorem this is possible if there exist Killing vector fields on M). Then, when possible, rewrite $\overline{\mathcal{F}}$ and $\overline{\mathcal{D}}$ in terms of the constants of motion associated to the partial integration or Killing vector fields (typically, $\overline{\mathcal{D}}$ may be chosen as a compact n-rectangle).

– Step 4. If $\overline{\mathcal{F}}$ satisfies certain conditions at the boundary of $\overline{\mathcal{D}}$ then Brouwer's topological arguments may imply the existence of a zero. In dimension $n = 1$ these conditions will be quite trivial: if $[a, b] \subset \overline{\mathcal{D}}$ and $\overline{\mathcal{F}}(a) \cdot \overline{\mathcal{F}}(b) < 0$ then $\overline{\mathcal{F}}$ will have a zero. For dimension $n = 2$ and, say, $[a, b] \times [a', b'] \subset \overline{\mathcal{D}}$, $\overline{\mathcal{F}} = (\overline{\mathcal{F}}^1(x, y), \overline{\mathcal{F}}^2(x, y))$, if $\overline{\mathcal{F}}^1(a, y) \cdot \overline{\mathcal{F}}^1(b, y) < 0, \forall y \in [a', b']$, $\overline{\mathcal{F}}^2(x, a') \cdot \overline{\mathcal{F}}^2(x, b') < 0, \forall x \in [a, b]$, then the degree of $\overline{\mathcal{F}}$ will be $\neq 0$, and $\overline{\mathcal{F}}$ will have a zero; natural extensions of these conditions will be needed for $n \geq 3$.

More exactly, we will need some variations of previous arguments. For example under, say, the condition for $\overline{\mathcal{F}}^1$, $\overline{\mathcal{F}}^1(a, y) \cdot \overline{\mathcal{F}}^1(b, y) < 0, \forall y \in [a', b']$, a connected set $\mathcal{C}$ of zeroes of $\overline{\mathcal{F}}^1$ which joins the horizontal lines $y = a', y = b'$ can be found. Then, we will look for a zero of $\overline{\mathcal{F}}^2$ in $\mathcal{C}$. Some other subtleties will be taken into account for specific applications.

3 Application to multiwarped spacetimes

The first class of Lorentzian manifolds where our technique was applied, is the class of *multiwarped spacetimes*, which have a remarkable role in General Relativity.

Consider n Riemannian manifolds (F_i, g_i), an open interval $I = (a, b) \subseteq \mathbb{R}$ with opposite metric to the usual $-d\tau^2$, and n differentiable functions $f_i > 0 \quad i = 1, \dots, n$ on I. A *multiwarped spacetime* with base $(I, -d\tau^2)$, fibers (F_i, g_i) $i = 1, \dots, n$ and warping functions $f_i > 0$, $i = 1, \dots, n$ is the product manifold $M = I \times F_1 \times \cdots \times F_n$ endowed with the Lorentz metric:

$$g = -\pi_I^* d\tau^2 + \sum_{i=1}^{n} (f_i \circ \pi_I)^2 \pi_i^* g_i \equiv -d\tau^2 + \sum_{i=1}^{n} f_i^2 g_i \ ,$$

where π_I and $\pi_i \quad i = 1, \dots, n$ are the natural projections of $I \times F_1 \times \cdots \times F_n$ onto I and $F_1, \dots, F_n$, respectively.

Examples of multiwarped spacetimes are the following:

(1) For $n = 1$ fibers these manifolds correspond with *Generalized Robertson-Walker (GRW) spacetimes* $(I \times F, -d\tau^2 + f^2 g_F)$, which are natural generalizations of Friedmann-Lemaitre-Robertson-Walker cosmological spacetimes [1], [22].

(2) Two classical spacetimes can be seen as particular cases when $n = 2$ [21]:

Consider the metric on $(r_-, r_+) \times \mathbb{R} \times S^2$

$$-\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

being e, m constants (m positive), $e^2 < m^2$, $\quad r \in (r_-, r_+)$, $\quad r_- = m - (m^2 - e^2)^{1/2}$, $\quad r_+ = m + (m^2 - e^2)^{1/2}, t \in \mathbb{R}$ and θ, φ are the spherical coordinates on S^2. [Notice that the new variable $\tau \in (0, \tau_+)$ obtained from

$$d\tau = \left(-1 + \frac{2m}{r} - \frac{e^2}{r^2}\right)^{-1/2} dr$$

provides the interval I and the warping functions in our definition of multiwarped spacetime]. When $e = 0$ this metric represents the *Schwarzschild black hole*, generated by a spherically symmetric massive object. When $e \neq 0$ this metric is the intermediate zone of usual *Reissner-Nordström spacetime*, generated by a spherically symmetric charged body.

(3) Moreover, multiwarped spacetimes include *Bianchi type IX spacetimes* as Kasner's, and may also represent relativistic spacetimes together with internal spaces attached at each point or multidimensional inflationary models (see [19] and references therein).

In order to apply the topological technique, the following subtleties must be considered in relation to the four steps of Section 3:

Step 1 and 3. Multiwarped spacetimes are not included in $\mathbb{R}^{n+1}$, but the projection of a geodesic $\gamma(s)$ on each fiber F_i is a pregeodesic $\gamma_i(s)$. Fixed $x_i, x'_i \in F_i$ we will assume that $\gamma_i(s)$ is the reparametrization of a minimizing geodesic, and the only relevant degree of freedom will be its initial velocity $c_i = (f_i^4 \circ \tau) \cdot g_i(d\gamma_i/ds, d\gamma_i/ds)$. On the other hand, the projection on the base $\tau(s)$ is then characterized by $\tau'(0)$; by technical reasons, a new parameter $K \in \mathbb{R}$ directly related to $\tau'(0)$ is introduced. So, every geodesic $\gamma(s)$ is characterized by $(K, c_1, \ldots, c_n)$ which takes values in $\mathbb{R}^{n+1}$.

Step 2. Every geodesic is reparametrized by the component τ. Thus, the points with $\tau'(s) = 0$ must be specifically studied.

Step 4. Given the two points to be connected $z_0 = (\tau_0, x), z'_0 = (\tau'_0, x') \in I \times (F_1 \times \cdots \times F_n)$, one can prove first an Avez-Seifert type result (see below) and thus, if $|\tau'_0 - \tau_0|$ is big enough the two points can be joined by a causal geodesic. For the intermediate values of $\tau'_0 \in [\hat{a}, \hat{b}]$ some functions $\overline{\mathcal{F}}^1, \ldots, \overline{\mathcal{F}}^n$ of $(K, c_1, \ldots, c_n)$ can be defined. The simultaneous zeroes of these functions correspond with geodesics joining z_0 with a point in the line $\{(\tau, x') : \tau \in (a, b)\}$. The boundary conditions of the $\overline{\mathcal{F}}^i$'s ensure the existence of a connected set of zeroes $\mathcal{C}$ which correspond with geodesics joining z_0 and (τ'_0, x') for any τ'_0 in an interval which includes $[\hat{a}, \hat{b}]$.

As consequence, the problem of the geodesic connectedness is essentially solved if we assume *weak convexity* for every fiber F_i, that is, each two points $x_i, x'_i \in F_i$ can be joined by a minimizing F_i-geodesic (the fiber is *strongly convex* if previous minimizing geodesic is the unique one). In fact, we have:

- An (Avez-Seifert type) result on geodesic connectedness [12, Th. 2]: *in a multiwarped spacetime with weakly convex fibers two points causally related can be joined by a causal geodesic.*

- A result with a natural geometric interpretation [12, Th. 1]:

Theorem 3.1 *A multiwarped spacetime* $(I \times F_1 \times \cdots \times F_n, g)$ *with weakly convex fibers is geodesically connected if the following condition holds: every point* $z = (\tau, x) \in I \times (F_1 \times \cdots \times F_n)$ *and every line* $L[x'] = \{(\tau, x') : \tau \in I\}, x' \in F_1 \times \cdots \times F_n$ *can be joined by both a future and a past directed causal curve.*

Previous condition is equivalent to the following property on the warping functions $f_1, \ldots, f_n$:

$$\int_a^c f_i^{-2}(\frac{1}{f_1^2} + \cdots + \frac{1}{f_n^2})^{-1/2} = \infty, \quad \int_c^b f_i^{-2}(\frac{1}{f_1^2} + \cdots + \frac{1}{f_n^2})^{-1/2} = \infty,$$

for all i and for $c \in (a, b)$.

This condition seems appropriate when either $I = \mathbb{R}$ or f_i goes to zero at the extremes. But, for example, a strip $I \times \mathbb{R}^n, I \neq \mathbb{R}$ in Lorentz-Minkowski spacetime $\mathbb{L}^{n+1}$ does not satisfy this condition. However, Theorem 3.1 does not cover all the possibilities of the technique, and we can obtain results stronger (although under more intrincated hypotheses on the f_i's). As a consequence, all previous results are reproven or extended; in particular, geodesic connectedness of Reissner-Nordström Intermediate spacetime is reproven [12, Section 6] by using a technique completely different to the one in [15] (and without the restriction $2r_- > r_+$ in this reference). Moreover, the accuracy of our technique is shown by proving the geodesic connectedness of Schwarzschild black hole [12, Th. 6]. The geodesic connectedness of spacetimes as Kasner's can be trivially determined by using our results.

- Finally, we remark that these results are easily extendible to the case that every fiber F_i is a manifold with boundary ∂F_i (improving results in [16]). In fact, in this case the problem is reduced to answer when the structure of ∂F_i imply that F_i is weakly convex (if $F_i \cup \partial F_i$ is a complete smooth manifold with boundary, F_i is weakly convex if and only if the second fundamental form of the boundary, with respect to the interior normal, is positive semidefinite; for more general results see [6]). Of course, the results work when we consider strips $(\hat{a}, \hat{b}) \times D_1 \times \cdots \times D_n \subseteq I \times F_1 \times \cdots \times F_n$. Moreover, further multiplicity results can be obtained if the topology of one of the fibers is not trivial (see for example [11, Th. 3])

As we commented before, we can give more accurate conditions for the geodesic connectedness when only $n = 1$ fibers are considered (Conditions (A), (B), (C) and (R) in [11, Section 3]). These conditions are somewhat cumbersome, because they yield not only sufficient but also *necessary* hypotheses for geodesic connectedness. More precisely, if the fiber is strongly convex then the GRW is geodesically connected if and only if one of the Conditions (C) or (R) in [11] holds (Condition (A) in [11] is less accurate than (C), but it is translatable to any multiwarped spacetime to obtain geodesic connectedness; Condition (B) is an intermediate condition). Nevertheless, from these conditions it is easy to obtain simple general sufficient conditions for connectedness. For example (see [11, Lemmas 3 and 9]), *if the GRW spacetime is* not *geodesically connected then f must admit a limit at some extreme of the interval $I = (a, b)$; if this extreme is b (resp. a) then f' must be strictly positive (resp. negative) in a non-empty subinterval $(\bar{b}, b) \subseteq (a, b)$ (resp. $(a, \bar{a}) \subseteq (a, b)$),* (etc.)

On the other hand, this machinery allows us to obtain a precise relation between the conjugate points $z_0 = (\tau_0, x_0), z'_0 = (\tau'_0, x'_0)$ of a geodesic $\gamma(s) = (\tau(s), \gamma_F(s))$ in the GRW and the points x_0, x'_0 of its projection $\gamma_F(s)$ on F (recall that $\gamma_F(s)$ is a pregeodesic on F). Concretely, if m is the multiplicity of conjugation along γ of z_0, z'_0 then the multiplicity of the projections x_0, x'_0 along γ_F is $m' \in \{m, m-1\}$. In particular, if z_0, z'_0 are non-conjugate then so are x_0, x'_0. Even more, if γ is a causal geodesic (or any geodesic without zeroes in $d\tau/ds$) then $m' = m$ [11, Th. 4].

From here, Morse type relations which relate the topology of the space of curves joining two non-conjugate points with the Morse indexes of the geodesics joining them can be obtained [11, Section 5]. Finally, all these results can be applied to the following two cases [11, Section 6].

– F is an interval of $\mathbb{R}$ and, thus, $-g$ is static standard. Our results improve the previous ones in [5] obtained from completely different techniques. Concretely, we obtain:

Corollary 3.2 *Given $(y_0, x_0), (y'_0, x'_0)$ in the static spacetime $(K \times J \subseteq \mathbb{R}^2, g_S = dy^2 - f^2(y)dx^2)$, these points are*

(i) spacelike related (i.e. timelike related for $-g_S$) if and only if $\int_{y_0}^{y'_0} f^{-1} > d(x_0, x'_0)$. In this case there exist a unique geodesic which joins them; this geodesic is necessarily spacelike and without conjugate points.

(ii) lightlike related (connectable with a lightlike curve) if and only if $\int_{y_0}^{y_0'} f^{-1} = d(x_0, x_0')$. In this case there exist a unique geodesic which joins them; this geodesic is necessarily lightlike and without conjugate points.

(iii) timelike related if and only if $\int_{y_0}^{y_0'} f^{-1} < d(x_0, x_0')$. All points which are timelike related can be joined by a geodesic (necessarily timelike) if and only if either Condition (C) or Condition (R) holds.

– $Ric(\partial_\tau, \partial_\tau) \geq 0$ ($\Leftrightarrow f'' \leq 0$). This hypothesis is natural from a physical point of view (it is implied by the *timelike convergence condition* and, thus, by some *energy conditions*). Concretely, we obtain:

Corollary 3.3 *An inextendible GRW spacetime (i.e. with f defined on a maximal interval I) with $Ric(\partial_\tau, \partial_\tau) \geq 0$ and weakly convex fiber satisfies*

(i) Each two causally related points can be joined with causal geodesic, which is unique if the fiber is strongly convex.

(ii) The spacetime is geodesically connected. Moreover, each strip $(\hat{a}, \hat{b}) \times F \subset I \times F$, $a < \hat{a} < \hat{b} < b$ with the restricted metric is geodesically connected if and only if $f'(\hat{a}) \geq 0$ and $f'(\hat{b}) \leq 0$ (i.e. $f'(\hat{a}) \cdot f'(\hat{b}) \leq 0$).

(iii) There exist a natural surjective map between geodesics connecting $z_0 = (\tau_0, x_0)$, $z_0' = (\tau_0', x_0') \in I \times F$ and F-geodesics connecting x_0 and x_0'. Under this map, when the geodesic connecting z_0 and z_0' is causal then the multiplicity of its conjugate points is equal to the multiplicity for the corresponding geodesic connecting x_0, x_0'.

(iv) If (F, g_F) is complete and F is not contractible in itself, then any $z_0, z_0' \in I \times F$ can be joined by means of infinitely many spacelike geodesics. If x_0, x_0' are not conjugate there are at most finitely many causal geodesics connecting them.

Other applications (for example an extension of [24, Th. 5.3] for GRW spacetimes) are possible.

4 Application to Kerr spacetime

Kerr spacetime represents the stationary axis-symmetric asymptotically flat gravitational field outside a rotating massive object. The simplest description of the Kerr metric tensor is in terms of the time coordinate t on $\mathbb{R}$ and spherical coordinates r, θ, φ on $\mathbb{R}^3$ (θ denotes colatitud and φ longitud), which are called Boyer-Lindquist coordinates (we follow the notation in [20, Section 1.1]). Let $m > 0$ and a be two constants, such that m represents the mass of the object and ma the angular momentum as measured from infinity. In previous coordinates, Kerr metric takes the form

$$ds^2 = g_{t,t}dt^2 + g_{r,r}dr^2 + g_{\theta,\theta}d\theta^2 + g_{\varphi,\varphi}d\varphi^2 + 2g_{\varphi,t}d\varphi dt$$

with

$$\begin{array}{ll} g_{r,r} = \frac{\rho(r,\theta)^2}{\triangle(r)} & g_{\varphi,\varphi} = [r^2 + a^2 + \frac{2mra^2\sin^2\theta}{\rho(r,\theta)^2}]\sin^2\theta \\ g_{\theta,\theta} = \rho(r,\theta)^2 & g_{\varphi,t} = -\frac{2mra\sin^2\theta}{\rho(r,\theta)^2} \\ g_{t,t} = -1 + \frac{2mr}{\rho(r,\theta)^2} & \end{array}$$

and being

$$\rho(r,\theta)^2 = r^2 + a^2\cos^2\theta \quad \text{and} \quad \triangle(r) = r^2 - 2mr + a^2.$$

Halting the rotation by setting $a = 0$, Kerr spacetime becomes Schwarzschild spacetime; if, then, the mass is removed ($m = 0$) only (empty) Lorentz-Minkowski spacetime remains.

Formulae above show that Kerr metric fails when either $\rho(r,\theta)^2 = 0$ or $\triangle(r) = 0$. The zeroes of the function $\rho(r,\theta)^2$ correspond with a physical singularity. On the other hand, the function $\triangle(r)$ has the zeroes $r_+ = m + \sqrt{m^2 - a^2}$ and $r_- = m - \sqrt{m^2 - a^2}$ and,

$$\triangle(r) = (r - r_-)(r - r_+).$$

However, these zeroes do not correspond with singularities in the curvature. In fact, the hypersurfaces $\mathbb{R} \times \{x \in \mathbb{R}^3 : r = r_+\}$ and $\mathbb{R} \times \{x \in \mathbb{R}^3 : r = r_-\}$ are *event horizons*. More precisely:

– If $a^2 < m^2$ (*slow* Kerr spacetime) we have two event horizons $r = r_+$ and $r = r_-$.

– If $a^2 = m^2$ (*extreme* Kerr spacetime) we only have one event horizon $r = r_+ = r_-$.

– Finally, if $a^2 > m^2$ (*fast* Kerr spacetime) we do not have event horizons.

The region in slow Kerr spacetime satisfying $r > r_+$ (resp. $r_- < r < r_+$; $r < r_-$) is called exterior (resp. intermediate; interior) Kerr spacetime. Intermediate Kerr spacetime has a strange physical behaviour: matter might disappear in finite proper time, or suddenly appear from nowhere. In interior Kerr spacetime, the ring singularity appears with its associated time machine.

In order to apply the topological technique, the following subtleties must be considered:

Step1. Even though Kerr spacetime is an open subset of $\mathbb{R}^4$, arbitrary values for coordinates θ, φ will be permitted for convenience.

Step 2. We reparametrize our geodesic by the component r. So, the points with $r'(s) = 0$ must be specifically studied.

Step 3. The partial integrations are given by the constants of motion L (*angular momentum*; ∂_φ is a Killing vector field), E (*energy*; ∂_t is a Killing vector field), q (normalization of the geodesic; *rest mass*) and K (*Carter constant*).

Step 4. In this case we actually consider a sequence of increasing rectangles $[a_m, b_m] \times [a'_m, b'_m] \times [a''_m, b''_m] \subset \overline{\mathcal{D}}$, and the function $\overline{\mathcal{F}} = (\overline{\mathcal{F}}^1, \overline{\mathcal{F}}^2, \overline{\mathcal{F}}^3)$ will have the boundary conditions $\overline{\mathcal{F}}^1(a_m, y, z) \cdot \overline{\mathcal{F}}^1(b_m, y, z) < 0 \;\; \forall (y, z) \in [a'_m, b'_m] \times [a''_m, b''_m]$, $\overline{\mathcal{F}}^2(x, a'_m, z) \cdot \overline{\mathcal{F}}^2(x, b'_m, z) < 0 \;\; \forall (x, z) \in [a_m, b_m] \times [a''_m, b''_m]$. So, a sequence of connected sets $\mathcal{C}_m$ of zeroes of $\overline{\mathcal{F}}^1$ and $\overline{\mathcal{F}}^2$ connecting the faces $z = b''_m$ and $z = a''_m$ can be found and, for m big enough, the existence of a zero in some $\mathcal{C}_m$ for $\overline{\mathcal{F}}^3$ is proved.

Then, the following result is obtained:

Theorem 4.1 *Exterior Kerr spacetime* $\mathbb{K}$ *with* $a^2 < m^2$ *is geodesically connected [14].*

In particular, this result reproves that (outer) Schwarzschild spacetime is geodesically connected (see also [4] or [13, Th. 6]).

Even more, by using exactly the same technique we obtain the geodesic connectedness of the Schwarzschild black hole ($r < r_+$, $a = 0$) [10]. Notice that this result has been obtained only by applying our technique, and can be obtained in two different ways: (1) considering Schwarzschild black hole as a multiwarped spacetime, and (2) working as in a region of Kerr spacetime.

It is worth pointing out that our technique circumvents the following difficulties:

(1) In $\mathbb{K}$ there is not a globally defined timelike Killing vector field

K (if K existed the problem would be reduced to a "Riemannian" one, where variational methods yield very precise results).

(2) Another possibility would be to consider $\mathbb{K}$ as a splitting manifold as those studied by using variational methods and Rabinowitz's saddle point Theorem in [17]. But these results are specially appropiate to study globally hyperbolic spacetimes under a splitting with complete Cauchy hypersurfaces t = constant and no clear choice of the time function t seems to be natural to apply such techniques for $\mathbb{K}$.

(3) At any case, the role of the event horizon $r = r_+$ seems to be unavoidable (under our approach, geodesics approaching to the event horizon play an essential role). As we have commented, the convexity of the boundary of a semi-Riemannian manifold sometimes yields the geodesic connectedness of the manifold, especially in the Riemannian case [23]. The boundary $r = r_+$ of $\mathbb{K}$ is singular, and the approaching hypersurfaces $r = r_+ + \nu, \nu > 0$, are not convex (regions $r > r_+ + \nu$ are not geodesically connected, see below). In the Riemannian case, there are techniques to measure if the lack of convexity goes to zero, when $\nu \to 0$, yielding geodesic connectedness, [6]. In the static case, some of these techniques are translatable [7], and geodesic connectedness of some spacetimes with singular boundary, including Schwarzschild spacetime, has been proven [4]. But none of these techniques seem appliable to a non-stationary situation.

On the other hand, we obtain the non-geodesic connectedness of some regions of the slow, extreme and fast Kerr spacetime [15, Section 4]. Firstly, we have:

Theorem 4.2 *Stationary Kerr spacetime* $M^a = \mathbb{R} \times \{x \in \mathbb{R}^3 : r > m + \sqrt{m^2 - a^2\cos^2\theta}\}$ *with* $0 < a^2 \leq m^2$ *is not geodesically connected.*

Even more, we prove that (stationary or not) regions R of exterior Kerr spacetime with $a^2 \leq m^2$ satisfying $r > r_+ + \nu$ ($\nu > 0$) are not geodesically connected. Moreover, no region $M^a_\varepsilon = \mathbb{R} \times \{x \in \mathbb{R}^3 : r > m + \sqrt{m^2 + \varepsilon - a^2\cos^2\theta}\}$ with $a^2 \leq m^2$ and $\varepsilon > 0$ is geodesically connected.

Finally, for fast Kerr spacetime we have:

Theorem 4.3 *(i) Stationary fast Kerr spacetime is not geodesically connected (if we assume* $r > 0$ *as well as if* $r \in \mathbb{R}$*).*

(ii) Regions (stationary or not) of fast Kerr spacetime determined by $r > \nu$ *for some* $\nu > 0$ *are not geodesically connected.*

(iii) The whole fast Kerr spacetime (including non-stationary regions and $r \in \mathbb{R}$*) is not geodesically connected.*

Acknowledgments

Work partially supported by a MCyT grant number BFM2001-2871-C04-01.

References

[1] L.J. ALÍAS, A. ROMERO AND M. SÁNCHEZ, *Uniqueness of complete spacelike hypersurfaces of constant mean curvature in Generalized Robertson-Walker spacetimes*, Gen. Relat. Gravit., **27**, 71-84, (1995).

[2] J.K. BEEM, P.E. EHRLICH AND K.L. EASLEY, *Global Lorentzian geometry*, Monographs Textbooks Pure Appl. Math., New York (1996).

[3] V. BENCI AND D. FORTUNATO, Proc. on Variational Methods (Editado por H. Beresticky- J.M. Coron- I. Ekeland), Paris, 413-429, (1988).

[4] V.BENCI, D.FORTUNATO AND F.GIANNONI, *On the existence of geodesics in static Lorentz manifolds with singular boundary*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., (4) **XIX**, 255-289, (1992).

[5] V. BENCI, F. GIANNONI AND A. MASIELLO, *Some properties of the spectral flow in semiriemannian geometry*, J. Geom. Phys., **27**, 267-280, (1998).

[6] R. BARTOLO, A. GERMINARIO AND M. SÁNCHEZ, *Convexity of domains of Riemannian manifolds*, Annals Global Anal. Geom., **21**, 63-84, (2002).

[7] R. BARTOLO, A. GERMINARIO AND M. SÁNCHEZ, *A note on the boundary of a static Lorentzian manifold*, Diff. Geom. Appl., to appear.

[8] J.K. BEEM AND P.E. PARKER, *Pseudoconvexity and geodesic connectedness*, Ann. Mat. Pure Appl., **155**, 137-142, (1989).

[9] E. CALABI AND L. MARKUS, *Relativistic space forms*, Ann. Math., **75**, 63-76, (1962).

[10] J.L. Flores, *Conectividad geodésica en algunos espaciotiempos: una técnica topológica*, Ph. D. Thesis, Granada (2002), in preparation.

[11] J.L. Flores and M. Sánchez, *Geodesic connectedness and conjugate points in GRW spacetimes*, J. Geom. Phys., **36**, 285-314, (2000).

[12] J.L. Flores and M. Sánchez, *Geodesic connectedness of multiwarped spacetimes*, J. Diff. Equat, to appear.

[13] J.L. Flores and M. Sánchez, *Geodesics in stationary spacetimes. Application to Kerr spacetime*, Int. J. Theor. Phys. Group Th. Nonlin. Opt., **8**, (2002), to appear.

[14] J.L. Flores and M. Sánchez, *A topological method for geodesic connectedness of spacetimes: outer Kerr spacetime*, preprint.

[15] F. Giannoni, *Geodesics on nonstatic Lorentz manifolds of Reissner–Nordström type*, Math. Ann., **291**, 383-401, (1991).

[16] F. Giannoni and A. Masiello, *Geodesics on Lorentzian manifolds with quasi-convex boundary*, Manuscripta Math., **78**, 381-396, (1993).

[17] F. Giannoni and A. Masiello, *Geodesics on product Lorentzian manifolds*, it Ann.-Inst.-H.-Poincare-Anal.-Non-Lineaire, **12** N^o. 1, 27-60, (1995).

[18] A. Masiello, *Variational methods in Lorentzian Geometry*, Pitman Research Notes in Mathematics Series **309** Longman Scientific and Technical, Harlow, Essex (1994).

[19] S. Mignemi and H.-J. Schmidt, *Classification of multidimensional inflationary models*, J. Math. Phys., **39** No. 2, 998-1010, (1998).

[20] B. O'Neill, *The Geometry of Kerr Black Holes*, A K Peters, Ltd. Wellesley (1995).

[21] M. Sánchez, *Geodesic connectedness in Generalized Reissner–Nordström type Lorentz manifolds*, Gen. Relat. Gravit., **29**, 1023-1037, (1997)

[22] M. Sánchez, *On the geometry of Generalized Robertson Walker spacetimes: geodesics*, Gen. Relat. Gravit., **30**, 915-932, (1998).

[23] M. SÁNCHEZ, *A survey on geodesic connectedness*, these proceedings.

[24] K. UHLENBECK, *A Morse Theory for geodesics on Lorentz manifolds*, Topology, **14**, 69-90, (1975).

Proceedings of the meeting
Lorentzian Geometry-Benalmádena 2001
Benalmádena, Málaga, Spain
Pub. de la RSME, Vol. 5 (2003), 155-169

Causal relations and their applications

Alfonso García-Parrado[1] and José M. M. Senovilla[1]

[1]*Departamento de Física Teórica. Universidad del País Vasco. Apartado 644, 48080 Bilbao, Spain*

Abstract

In this work we define and study the relations between Lorentzian manifolds given by the diffeomorphisms which map causal future directed vectors onto causal future directed vectors. This class of diffeomorphisms, called *proper causal relations*, contains as a subset the well-known group of conformal relations and are deeply linked to the so-called causal tensors of Ref.[1]. If two given Lorentzian manifolds are in *mutual* proper causal relation then they are said to be causally isomorphic: they are indistinguishable from the causal point of view. Finally, the concept of causal transformation for Lorentzian manifolds is introduced and its main mathematical properties briefly investigated.

1 Basics on causal relations

In this section the definitions of the basic concepts and the notation to be used throughout this contribution shall be presented. Differentiable manifolds are denoted by italic capital letters $\mathcal{V}, W, U, \dots$ and, to our purposes, all such manifolds will be connected causally orientable Lorentzian manifolds of dimension n. The signature convention is set to $(+ - \cdots -)$. $T_x(V)$ and $T^*_x(V)$ will stand respectively for the tangent and cotangent spaces at $x \in V$, and $T(\mathcal{V})$ (resp. $T^*(\mathcal{V})$) is the tangent bundle (cotangent bundle) of $\mathcal{V}$. Similarly the bundle of j-contravariant and k-covariant tensors of V is denoted $\mathcal{T}^j_k(V)$. If φ is a diffeomorphism between V and W, the push-forward and pull-back are written as φ' and φ^* respectively. The hyperbolic structure of the Lorentzian scalar product naturally splits the elements of $T_x(V)$ into timelike, spacelike, and null, and as usual we use the term *causal* for the vectors (or vector fields) which are non-spacelike. To fix the notation we introduce the sets:

$$\begin{aligned} \Theta^+(x) &= \{\vec{X} \in T_x(V) : \vec{X} \text{ is causal future directed}\}, \\ \Theta(x) &= \Theta^+(x) \cup \Theta^-(x), \qquad \Theta^+(V) = \bigcup_{x\in V} \Theta^+(x) \end{aligned}$$

with obvious definitions for $\Theta^-(x)$, $\Theta^-(V)$ and $\Theta(V)$. Before we proceed, we need to introduce a further concept taken from [1].

Definition 1.1 *A tensor* $\mathbf{T} \in \mathcal{T}_r^0(x)$ *satisfies the dominant property if for every* $\vec{k}_1, \ldots, \vec{k}_r \in \Theta^+(x)$ *we have that* $\mathbf{T}(\vec{k}_1, \ldots, \vec{k}_r) \geq 0$.

The set of all r-tensors with the dominant property at $x \in V$ will be denoted by $\mathcal{DP}_r^+(x)$ whereas $\mathcal{DP}_r^-(x)$ is the set of tensors such that $-\mathbf{T} \in \mathcal{DP}_r^+(x)$. We put $\mathcal{DP}_r(x) \equiv \mathcal{DP}_r^+(x) \cup \mathcal{DP}_r^-(x)$. All these definitions extend straightforwardly to the bundle $\mathcal{T}_r^0(V)$ and we may define the subsets $\mathcal{DP}_r^+(\mathcal{U})$, $\mathcal{DP}_r^-(\mathcal{U})$ and $\mathcal{DP}_r(\mathcal{U})$ for an open subset $\mathcal{U} \subseteq V$ as follows:

$$\mathcal{DP}_r^{\pm}(U) = \bigcup_{x\in\mathcal{U}} \mathcal{DP}_r^{\pm}(x), \;\; \mathcal{DP}_r(\mathcal{U}) = \mathcal{DP}_r^+(\mathcal{U}) \cup \mathcal{DP}_r^-(\mathcal{U}).$$

The simplest example (leaving aside $\mathbb{R}^+$) of causal tensors are the causal 1-forms ($\equiv \mathcal{DP}_1(V)$) [1][1], while a general characterization of $\mathcal{DP}_r^+ \equiv \mathcal{DP}_r^+(V)$ is the following (see [3] for a proof)[1]:

Proposition 1.2 $\mathbf{T} \in \mathcal{DP}_r^+$ *if and only if the components* $T_{i_1\ldots i_r}$ *of* $\mathbf{T}$ *in all orthonormal bases fulfill* $T_{0\ldots 0} \geq |T_{i_1\ldots i_r}|$, $\forall i_1 \ldots i_r$, *where the* 0*-index refers to the temporal component.*

We are now ready to present our main concept, which tries to capture the notion of some kind of relation between the causal structure of V and W ([2]).

Definition 1.3 *Let* $\varphi : V \to W$ *be a global diffeomorphism between two Lorentzian manifolds. We shall say that* W *is properly causally related*

[1]See also Bergqvist's and Senovilla's contributions to this volume.

with V by φ, denoted $V \prec_{\varphi} W$, if for every $\vec{X} \in \Theta^{+}(V)$ we have that $\varphi' \vec{X}$ belongs to $\Theta^{+}(W)$. W is said to be properly causally related with V, denoted simply as $V \prec W$, if $\exists \varphi$ such that $V \prec_{\varphi} W$.

Remarks

1. This definition can also be given for any set $\zeta \subseteq V$ by demanding that $(\varphi' \vec{X})_{\varphi(x)} \in \Theta^{+}(\varphi(x)) \ \ \forall \vec{X} \in \Theta^{+}(x), \forall x \in \zeta$.
2. Two diffeomorphic Lorentzian manifolds may fail to be properly causally related as we shall show later with explicit examples.

Definition 1.4 *Two Lorentzian manifolds V and W are called causally isomorphic if $V \prec W$ and $W \prec V$. This shall be written as $V \sim W$.*

We claim that if $V \sim W$ then their causal structure are somehow the same.

Let $\mathbf{g}$ and $\mathbf{g}$ be the Lorentzian metrics of V and W respectively. By using

$$\mathbf{g}(\varphi' \vec{X}, \varphi' \vec{Y}) = \varphi^{*} \ \mathbf{g}(\vec{X}, \vec{Y}), \tag{1.1}$$

we immediately realize that $V \prec_{\varphi} W$ implies that $\varphi^{*} \ \mathbf{g} \in \mathcal{DP}_2^{+}(V)$. Conversely, if $\varphi^{*} \ \mathbf{g} \in \mathcal{DP}_2^{+}(V)$ then for every $\vec{X} \in \Theta^{+}(V)$ we have that $(\varphi^{*} \ \mathbf{g})(\vec{X}, \vec{X}) = \ \mathbf{g}(\varphi' \vec{X}, \varphi' \vec{X}) \geq 0$ and hence $\varphi' \vec{X} \in \Theta(W)$. However, it can happen that $\Theta^{+}(V)$ is actually mapped to $\Theta^{-}(W)$, and $\Theta^{-}(V)$ to $\Theta^{+}(W)$. This only means that W *with the time-reversed orientation* is properly causally related with V. Keeping this in mind, the assertion $\varphi^{*} \ \mathbf{g} \in \mathcal{DP}_2^{+}(V)$ will be henceforth taken as equivalent to $V \prec_{\varphi} W$.

2 Mathematical properties

Let us present some mathematical properties of proper causal relations.

Proposition 2.1 *If $V \prec_{\varphi} W$ then:*

1. *$\vec{X} \in \Theta^{+}(V)$ is timelike $\Longrightarrow \varphi' \vec{X} \in \Theta^{+}(W)$ is timelike.*
2. *$\vec{X} \in \Theta^{+}(V)$ and $\varphi' \vec{X} \in \Theta^{+}(W)$ is null $\Longrightarrow \vec{X}$ is null.*

Proof. For the first implication, if $\vec{X} \in \Theta^+(V)$ is timelike we have, according to equation (1.1), that $\varphi^* \mathbf{g}(\vec{X}, \vec{X}) = \mathbf{g}(\varphi'\vec{X}, \varphi'\vec{X})$ which must be a strictly positive quantity as $\varphi^* \mathbf{g} \in \mathcal{DP}_2^+(W)$ [1]. For the second implication, equation (1.1) implies $0 = \varphi^* \mathbf{g}(\vec{X}, \vec{X})$ which is only possible if $\vec{X}$ is null since $\varphi^* \mathbf{g} \in \mathcal{DP}_2^+(V)$ and $\vec{X} \in \Theta^+(V)$ (see again [1]).

□

Proposition 2.2 $V \prec_\varphi W \iff \varphi'\vec{X} \in \Theta^+(W)$ *for all null* $\vec{X} \in \Theta^+(V)$.

Proof. For the non-trivial implication, making again use of (1.1) we can write:

$$\varphi'\vec{X} \in \Theta^+(W)\, \forall \vec{X} \text{ null in } \Theta^+(V) \Leftrightarrow \varphi^* \mathbf{g}(\vec{X}, \vec{Y}) \geq 0 \;\forall\, \vec{X}, \vec{Y} \text{ null in } \Theta^+(V)$$

which happens if and only if $\varphi^* \mathbf{g}$ is in $\mathcal{DP}_2^+(V)$ (see [1] property 2.4).

□

Proposition 2.3 (Transitivity of the proper causal relation)
If $V \prec_\varphi W$ *and* $W \prec_\psi U$ *then* $V \prec_{\psi\circ\varphi} U$

Proof. Consider any $\vec{X} \in \Theta^+(V)$. Since $V \prec_\varphi W$, $\varphi'\vec{X} \in \Theta^+(W)$ and since $W \prec_\psi U$ we get $\psi'[\varphi'\vec{X}] \in \Theta^+(U)$ so that $(\psi \circ \varphi)'\vec{X} \in \Theta^+(U)$ from what we conclude that $V \prec_{\psi\circ\varphi} U$.

□

Therefore, we see that the relation $\prec$ is a preorder. Notice that if $V \sim W$ (that is $V \prec W$ and $W \prec V$) this does not imply that $V = W$. Nevertheless, one can always define a partial order for the corresponding classes of equivalence.

Next, we identify the part of the boundary of the null cone which is preserved under a proper causal relation. A lemma is needed first. Recall that $\vec{X}$ is called an "eigenvector" of a 2-covariant tensor $\mathbf{T}$ if $\mathbf{T}(\cdot, \vec{X}) = \lambda \mathbf{g}(\cdot, \vec{X})$ and λ is then the corresponding eigenvalue.

Lemma 2.4 *If* $\mathbf{T} \in \mathcal{DP}_2^+(x)$ *and* $\vec{X} \in \Theta^+(x)$ *then* $\mathbf{T}(\vec{X}, \vec{X}) = 0 \iff \vec{X}$ *is a null eigenvector of* $\mathbf{T}$.

Proof. Let $\vec{X} \in \Theta^+(x)$ and assume $0 = \mathbf{T}(\vec{X}, \vec{X}) = T_{ab}X^aX^b$. Then since $T_{ab}X^b \in \mathcal{DP}_1^+(x)$ [1] we can conclude that X_a and $T_{ab}X^b$ must be proportional which results in X^a being a null eigenvector of T_{ab}. The converse is straightforward.

□

Proposition 2.5 *Assume that $V \prec_\varphi W$ and $\vec{X} \in \Theta^+(x)$, $x \in V$. Then $\varphi' \vec{X}$ is null at $\varphi(x) \in W$ if and only if $\vec{X}$ is a null eigenvector of $\varphi^* \ \mathbf{g}(x)$.*

Proof. Let $\vec{X}$ be in $\Theta^+(x)$ and suppose $\varphi' \vec{X}$ is null at $\varphi(x)$. Then, according to proposition 2.1, $\vec{X}$ is also null at x. On the other hand we have $0 = \mathbf{g}(\varphi' \vec{X}, \varphi' \vec{X}) = \varphi^* \ \mathbf{g}(\vec{X}, \vec{X})$ and since $\varphi^* \ \mathbf{g}|_x \in \mathcal{DP}_2^+(x)$, lemma 2.4 implies that $\vec{X}$ is a null eigenvector of $\varphi^* \ \mathbf{g}$ at x.

□

The vectors which remain null under the causal relation φ are called its *canonical null directions.* On the other hand, the null eigenvectors of $\mathbf{T} \in \mathcal{DP}_2^+$ can be used to classify this tensor, as proved in [1]. As a result we have

Proposition 2.6 *If the relation $V \prec_\varphi W$ has n linearly independent canonical null directions then $\varphi^* \ \mathbf{g} = \lambda \mathbf{g}$.*

Proof. If there exist n independent canonical null directions, then $\varphi^* \ \mathbf{g}$ has n independent null eigenvectors which is only possible if $\varphi^* \ \mathbf{g}$ is proportional to the metric tensor $\mathbf{g}$ ([1, 3].)

□

Proposition 2.6 has an interesting application in the following theorem

Theorem 2.7 *Suppose that $V \prec_\varphi W$ and $W \prec_{\varphi^{-1}} V$. Then $\varphi^* \ \mathbf{g} = \lambda \mathbf{g}$ and $(\varphi^{-1})^* \mathbf{g} = \frac{1}{(\varphi^{-1})^* \lambda} \ \mathbf{g}$ for some positive function λ defined on V.*

Proof. Under these hypotheses, using proposition 2.1, we get the following intermediate results

$$\varphi' \vec{X} \in \Theta^+(W) \text{ null and } \vec{X} \in \Theta^+(V) \Longrightarrow \vec{X} \text{ is null},$$
$$(\varphi^{-1})' \vec{Y} \in \Theta^+(V) \text{ null and } \vec{Y} \in \Theta^+(W) \Longrightarrow \vec{Y} \text{ is null}.$$

Now, let $\vec{X} \in \Theta^+(V)$ be null and consider the unique $\vec{Y} \in T(V)$ such that $\vec{X} = (\varphi^{-1})'\vec{Y}$. Then $\vec{Y} = \varphi'\vec{X}$ and $\vec{Y} \in \Theta^+(W)$ as φ sets a proper causal relation and $\vec{X}$ is in $\Theta^+(V)$. Hence, according to the second result above $\vec{Y}$ must be null and we conclude that every null $\vec{X} \in \Theta^+(V)$ is push-forwarded to a null vector of $\Theta^+(W)$ which in turn implies that $\varphi^*\ \mathbf{g} = \lambda\mathbf{g}$. In a similar fashion, we can prove that $(\varphi^{-1})^*\mathbf{g} = \mu\ \mathbf{g}$ and hence $(\varphi^{-1})^*\lambda = 1/\mu$.

$\square$

Corollary 2.8 *$V \prec_\varphi W$ and $W \prec_{\varphi^{-1}} V \iff \varphi$ is a conformal relation.*

3 Applications to causality theory

In this section we will perform a detailed study of how two Lorentzian manifolds V and W such that $V \prec_\varphi W$ share common causal features. To begin with, we must recall the basic sets used in causality theory, namely $I^\pm(p)$ and $J^\pm(p)$ for any point $p \in V$ (these definitions can also be given for sets). One has $q \in J^+(p)$ (respectively $q \in I^+(p)$) if there exists a continuous future directed causal (resp. timelike) curve joining p and q. Recall also the Cauchy developments $D^\pm(\zeta)$ for any set $\zeta \subseteq V$ [4, 5, 6]. Another important concept is that of future set: $\mathrm{A} \subset V$ is said to be a future set if $I^+(\mathrm{A}) \subseteq \mathrm{A}$. For example $I^+(\zeta)$ is a future set for any ζ. All these concepts are standard in causality theory and are defined in many references, see for instance [4, 5, 6].

Proposition 3.1 *If $V \prec_\varphi W$ then, for every set $\zeta \subseteq V$, we have $\varphi(I^\pm(\zeta)) \subseteq I^\pm(\varphi(\zeta))$ and $\varphi(J^\pm(\zeta)) \subseteq J^\pm(\varphi(\zeta))$.*

Proof. It is enough to prove it for a single point $p \in V$ and then getting the result for every ζ by considering it as the union of its points. For the first relation, let y be in $\varphi(I^+(p))$ arbitrary and take $x \in I^+(p)$ such that $\varphi(x) = y$. Choose a future-directed timelike curve γ joining p and x. From proposition 2.1, $\varphi(\gamma)$ is then a future-directed timelike curve joining $\varphi(p)$ and y, so that $y \in I^+(\varphi(p))$. The second assertion is proven in a similar way using again proposition 2.1. The proof for the past sets is analogous.

$\square$

The converse of this proposition does not hold in general unless we impose further causality conditions over our spacetime.

Definition 3.2 *A Lorentzian manifold V is said to be distinguishing if for every neighbourhood U_p of $p \in V$ there exists another neighbourhood $B_p \subset U_p$ containing p which meets every causal curve starting at p in a connected set.*

We need some concepts of standard causality theory. For any $p \in V$ one can introduce normal coordinates in a neighbourhood $\mathcal{N}_p$ of p (see, e.g. [6]). Then the exponential map provides a diffeomorphism exp : $\mathcal{O} \subset T_p(V) \to \mathcal{N}_p$ where $\mathcal{O}$ is an open neighbourhood of $\vec{0} \in T_p(V)$. The interior of the future (past) light cone of p is defined by $C_p^{\pm} = \exp(\mathrm{int}(\Theta^{\pm}(p)) \cap \mathcal{O})$, and obviously $C_p^{\pm} \subseteq I^{\pm}(p)$ [6]. Other important issue deals with the chronology relation $\ll$ between two points. We have $p \ll q$ if there exist a future timelike curve joining p and q. See [7] for an axiomatic study of this relation.

Proposition 3.3 *Let γ be a continuous curve of the Lorentzian manifold $(V, \mathbf{g})$ and assume that γ is a total set with respect to the relation $\ll$ (that is to say every pair of elements of the curve is comparable by $\ll$.) Then γ is timelike iff V is distinguishing.*

Proof. Clearly if γ is timelike then γ must be a total set for the relation $\ll$ (this is true for every spacetime). For the converse consider a curve γ which is total with respect to $\ll$ and let $q \in \gamma$ be an arbitrary point of the curve. If we take a normal neighbourhood of q, $\mathcal{N}_q$ then we may find a neighbourhood U_q of q which is intersected in a connected set by every causal curve meeting q. Now, if we pick up a point $z \in \gamma \cap U_q$ we have that either $q \ll z$ or $z \ll q$. Assuming the former we deduce that there exists a timelike curve γ joining q and z which implies that $\gamma \cap U_q$ is a connected set. This property together with the distinguishability of V implies that γ must be a subset of U_q and hence $\gamma \subset \mathcal{N}_q$ from what we conclude that $\gamma \subset C_p$ ([6]) and hence $z \in C_p$ $\forall z \in \gamma \cap U_q$ which is only possible if $\gamma \cap U_q$ timelike. By covering γ with sets of the form $\gamma \cap U_q$, $q \in \gamma$ we arrive at the desired result.

$\square$

Proposition 3.4 *Let $\varphi : V \to W$ be a diffeomorphism with the property $\varphi(I^+(p)) \subseteq I^+(\varphi(p))$ $\forall p \in V$. Then if W is distinguishing, φ is a proper causal relation. A similar result holds replacing I^+ by I^-.*

Proof. From the statement of this proposition is clear that $\forall\ p, q$ of V such that $p \ll q$ then $\varphi(p) \ll \varphi(q)$. Therefore every timelike curve γ of V is mapped onto a continuous curve in W total with respect to $\ll$ and hence timelike due to the indistiguishability of W. Furthermore if the curve γ is future directed then $\varphi(\gamma)$ must be also future directed since $\ll$ is preserved which is only possible if every timelike future-pointing vector is mapped onto a future-pointing timelike vector. As a consequence, if $\vec{k}$ is a null vector, $\varphi'\vec{k}$ must be a causal vector (to see it just construct a sequence of timelike future directed vectors converging to $\vec{k}$) which proves that φ is a proper causal relation.

□

The results for the Cauchy developments are the following:

Proposition 3.5 *If* $V \prec_\varphi W$ *then* $D^\pm(\varphi(\zeta)) \subseteq \varphi(D^\pm(\zeta)),\ \forall \zeta \subseteq V$.

Proof. It is enough to prove the future case. Let $y \in D^+(\varphi(\zeta))$ arbitrary and consider any causal past directed curve $\gamma^-_{\varphi^{-1}(y)} \subset V$ containing $\varphi^{-1}(y)$. Since the image curve by φ of $\gamma^-_{\varphi^{-1}(y)}$ is a causal curve passing through y, ergo meeting $\varphi(\zeta)$, we have that $\gamma^-_{\varphi^{-1}(y)}$ must meet ζ from what we conclude that $y \in \varphi(D^+(\zeta))$ due to the arbitrariness of $\gamma^-_{\varphi^{-1}(y)}$.

□

Corollary 3.6 *If* $\mathbb{S} \subset W$ *is a Cauchy hypersurface then* $\varphi^{-1}(\mathbb{S})$ *is also a Cauchy hypersurface of* V.

Proof. If $\mathbb{S}$ is a Cauchy hypersurface then $D(\mathbb{S}) = W$, and from proposition 3.5 $D(\mathbb{S}) \subseteq \varphi(D(\varphi^{-1}(\mathbb{S})))$. Since φ is a diffeomorphism the result follows.

□

One can prove the impossibility of the existence of proper causal relations sometimes. For instance, from the previous corollary we deduce that $V \prec W$ is impossible if W is globally hyperbolic but V is not. Other impossibilities arise as follows. Let us recall that, for any inextendible causal curve γ, the boundaries $\partial I^\pm(\gamma)$ of its chronological future and past are usually called its future and past event horizons, sometimes also called particle horizons [4, 5, 6]. Of course these sets can be empty (then one says that γ has no horizon).

Proposition 3.7 *Suppose that every inextendible causal future directed curve in W has a non-empty $\partial I^-(\gamma)$ $(\partial I^+(\gamma))$. Then any V such that $V \prec W$ cannot have inextendible causal curves without past (future) event horizons.*

Proof. If there were a future-directed curve γ in V with $\partial I^-(\gamma) = \emptyset$, $I^-(\gamma)$ would be the whole of V. But according to proposition 3.1 $\varphi(I^-(\gamma)) \subseteq I^-(\varphi(\gamma))$ from what we would conclude that $I^-(\varphi(\gamma)) = W$ against the assumption.

□

The class of future (or past) sets characterize the proper causal relations for distinguishing spacetimes as it is going to be shown next (every statement for future objects has a counterpart for the past).

Lemma 3.8 *If A is a future set then $p \in \overline{A} \Longleftrightarrow I^+(p) \subseteq A$.*

Proof. Suppose $I^+(p) \subseteq A$. Then since $C^+_p \subseteq I^+(p)$ and $p \in \overline{C^+_p}$ we have that $U_p \cap C^+_p \neq \emptyset$ for every neighbourhood U_p of p which in turn implies that $U_p \cap \mathrm{A} \neq \emptyset$ and hence $p \in \overline{\mathrm{A}}$. Conversely, let p be any point of $\overline{\mathrm{A}}$ then $I^+(p) \subseteq I^+(\overline{A}) = I^+(A) \subseteq A$.

□

Theorem 3.9 *Suppose that $(W,\ \mathbf{g})$ is a distinguishing spacetime. Then a diffeomorphism $\varphi : (V, \mathbf{g}) \to (W,\ \mathbf{g})$ is a proper causal relation if and only if $\varphi^{-1}(A)$ is a future set for every future set. $A \subseteq W$.*

Proof. Suppose $\mathrm{A} \subseteq W$ is a future set, $V \prec_\varphi W$ and take $\varphi^{-1}(\mathrm{A}) \subseteq V$. Proposition 3.1 implies $\varphi(I^+(\varphi^{-1}(\mathrm{A}))) \subseteq I^+(\varphi(\varphi^{-1}(\mathrm{A}))) = I^+(\mathrm{A}) \subseteq \mathrm{A}$ which shows that $I^+(\varphi^{-1}(\mathrm{A})) \subseteq \varphi^{-1}(\mathrm{A})$. Conversely, for any $p \in V$ take the future set $I^+(\varphi(p))$ and consider the future set $\varphi^{-1}(I^+(\varphi(p)))$. As $\varphi(p) \in \overline{I^+(\varphi(p))}$ then $p \in \overline{\varphi^{-1}(I^+(\varphi(p)))}$ and according to lemma 3.8 $I^+(p) \subseteq \varphi^{-1}(I^+(\varphi(p)))$ so that $\varphi(I^+(p)) \subseteq I^+(\varphi(p))$. Since this holds for every $p \in V$ and W is distinguishing, proposition 3.4 ensures that φ is a proper causal relation.

□

This theorem has important consequences.

Proposition 3.10 *If $V \sim W$ then there is a one-to-one correspondence between the future (and past) sets of V and W.*

Proof. If $V \sim W$ then $V \prec_{\varphi} W$ and $W_{\prec_{\Psi}}V$ for some diffeomorphisms φ and Ψ. By denoting with $\mathcal{F}_V$ and $\mathcal{F}_W$ the set of future sets of V and W respectively, we have that $\varphi^{-1}(\mathcal{F}_W) \subseteq \mathcal{F}_V$ and $\Psi^{-1}(\mathcal{F}_V) \subseteq \mathcal{F}_W$, due to theorem 3.9. Since both φ and Ψ are bijective maps we conclude that $\mathcal{F}_V$ is in one-to-one correspondence with a subset of $\mathcal{F}_W$ and vice versa which, according to the equivalence theorem of Bernstein [2], implies that $\mathcal{F}_V$ is in one-to-one correspondence with $\mathcal{F}_W$.

□

4 Causal transformations

In this section we will see how the concepts above generalize, in a natural way, the group of conformal transformations in a Lorentzian manifold V.

Definition 4.1 *A transformation $\varphi : V \longrightarrow V$ is called* causal *if $V \prec_{\varphi} V$.*

The set of causal transformations of V will be denoted by $\mathcal{C}(V)$. This is a subset of the group of transformations of V which is closed under the composition of diffeomorphisms, due to proposition 3.9, and contains the identity map. This algebraic structure is well-known, see e.g. [9], and called subsemigroup with identity or submonoid. Thus, $\mathcal{C}(V)$ is a *submonoid* of the group of diffeomorphisms of V. Nonetheless, $\mathcal{C}(V)$ usually fails to be a group. In fact we have,

Proposition 4.2 *Every subgroup of causal transformations is a group of conformal transformations.*

Proof. Let $G \subseteq \mathcal{C}(V)$ be a subgroup of causal transformations and consider any $\varphi \in G$, so that both φ and φ^{-1} are causal transformations. Then φ is necessarily a conformal transformation as follows from Theorem 2.7.

□

[2]See e.g. [8].

From standard results, see [9], we know that $\mathcal{C}(V)\cap\mathcal{C}(V)^{-1}$ is just the group of conformal transformations of V and there is no other subgroup of $\mathcal{C}(V)$ which contains $\mathcal{C}(V)\cap\mathcal{C}(V)^{-1}$. The causal transformations which are not conformal transformations are called proper causal transformations.

It is now a natural question whether one can define infinitesimal generators of one-parameter families of causal transformations which generalize the "conformal Killing vectors", and in which sense. Notice, however, that if $\{\varphi_s\}_{s\in\mathbb{R}}$ is a one-parameter group of causal transformations, from the previous results the only possibility is that $\{\varphi_s\}$ be in fact a group of conformal motions. On the other hand, things are more subtle if there are no conformal transformations in the family $\{\varphi_s\}$ other than the identity, in which case it is easy to see that the 'best' one can accomplish is that either $G^+\equiv\{\varphi_s\}_{s\in\mathbb{R}^+}$ or $G^-\equiv\{\varphi_s\}_{s\in\mathbb{R}^-}$ is in $\mathcal{C}(V)$. If this happens one talks about maximal one-parameter submonoids of proper causal transformations. Of course, it is also possible to define local one-parameter submonoids of causal transformations $\{\varphi_s\}_{s\in I}$ for some interval $I=(-\varepsilon,\varepsilon)$ of the real line assuming that $\{\varphi_s\}_{s\in(0,\varepsilon)}$ consists of proper causal transformations. In any of these cases, we can define the infinitesimal generator of $\{\varphi_s\}$ as the vector field $\vec{\xi}=d\varphi_s/ds|_{s=0}$. Given that $\varphi_s^*\mathbf{g}\in\mathcal{DP}_2$ for all $s\geq 0$ (or for all non-positive s), one can somehow control the Lie derivative of $\mathbf{g}$ with respect to $\vec{\xi}$. For instance, it is easy to prove that $\pounds_{\vec{\xi}}\mathbf{g}(\vec{k},\vec{k})\geq 0$ (or ≤ 0) for all null $\vec{k}$, clearly generalizing the case of conformal Killing vectors. An explicit example of this will be shown in the next section.

5 Examples

Example 5.1 Einstein static universe and de Sitter spacetime. *Let us take V as the Einstein static universe [4] and $W=\mathcal{SS}$ as de Sitter spacetime. In both cases the manifold is $\mathbb{R}\times S^3$ and hence they are diffeomorphic. By proposition 3.7 we know that $V\not\prec W$ because every causal curve in de Sitter spacetime possesses event horizons. However, the proper causal relation in the opposite way does hold as can be shown by constructing it explicitly. The line element of each spacetime is (with the notation $d\Omega^2=d\theta^2+\sin^2\theta d\phi^2$):*

$$\begin{aligned} V: &\quad ds^2=dt^2-a^2(d\chi^2+\sin^2\chi d\Omega^2)\\ W: &\quad d\ s^2=d\bar{t}^2-\alpha^2\cosh^2(\bar{t}/\alpha)(d\bar{\chi}^2+\sin^2\bar{\chi}d\bar{\Omega}^2)\ , \end{aligned}$$

where χ, θ, ϕ (and their barred versions) are standard coordinates in S^3 and a, α are constants. The diffeomorphism $\Psi : W \to V$ is chosen as $\{t = b\bar{t}, \chi = \bar{\chi}, \theta = \bar{\theta}, \phi = \bar{\phi}\}$ for a constant b. One can readily get $\Psi^*\mathbf{g}$

$$(\Psi^*\mathbf{g})_{ab}dx^a dx^b = b^2 d\bar{t}^2 - a^2(d\bar{\chi}^2 + \sin^2\bar{\chi}d\bar{\Omega}^2)$$

which on using proposition 1.2 shows that $\Psi^*\mathbf{g} \in \mathcal{DP}_2^+(W)$ if $b^2 \geq a^2/\alpha^2$ and therefore Ψ is proper causal relation for those b.

Example 5.2 *Consider the following spacetimes: $\mathbb{L}_a$ is the region of Lorentz-Minkowski spacetime with $R > a > 0$ in spherical coordinates $\{T, R, \Theta, \Phi\}$; W_c is the outer region of Schwarzschild spacetime with $r > c \geq 2M$ in Schwarzschild coordinates $\{t, r, \theta, \phi\}$. Define the diffeomorphism $\varphi : \mathbb{L}_a \to W_c$ given by $\{t = bT, r = R-a+c, \theta = \Theta, \phi = \Phi\}$ for an appropriate positive constant b, so that we have*

$$(\varphi^*\,\mathbf{g})_{ab}dx^a dx^b = b^2\left(1 - \frac{2M}{R-a+c}\right)dT^2 - \frac{dR^2}{1-\frac{2M}{R-a+c}} - (R-a+c)^2 d\Omega^2.$$

By choosing b and a one can achieve $\varphi^*\,\mathbf{g} \in \mathcal{DP}_2^+(V_a)$ whenever *$c > 2M$, while for $c = 2M$ φ fails to be a proper causal relation. Actually $\mathbb{L}_a \not\prec W_{2M}$ due to corollary 3.6 as W_{2M} is globally hyperbolic but $\mathbb{L}_a$ is not.*

Take now the diffeomorphism $\Psi : W_c \to \mathbb{L}_a$ defined by $\{T = t, R = r, \Theta = \theta, \Phi = \phi\}$, so that $\Psi^*\mathbf{g}$ reads $(\Psi^*\mathbf{g})_{ab}dx^a dx^b = dt^2 - dr^2 - r^2 d\Omega^2$ from where we immediately deduce that $\Psi^*\mathbf{g} \in \mathcal{DP}_2^+(V)$ for every $c \geq 2M$ as long as $a \geq 2M$. We have thus proved that $W_c \sim \mathbb{L}_c$ if $c > 2M$, but not for $c = 2M$. This is quite interesting and clearly related to the null character of the event horizon $r = 2M$ in Schwarzschild's spacetime.

Example 5.3 (Friedman cosmological models with $p = \gamma\rho$.) *Let us take as $(W, \mathbf{g})$ the flat Friedman-Robertson-Walker (FRW) spacetimes in standard FRW coordinates $\{t, \chi, \theta, \phi\}$ with line element given by*

$$ds^2 = dt^2 - a^2(t)(d\chi^2 + \chi^2 d\Omega^2)$$

and assume that the source of Einstein's equations is a perfect fluid with equation of state given by $p = \gamma\rho$ (p = pressure, ρ = density, $\gamma \in (-1, 1)$ constant). Then the scale factor is $a(t) = Ct^{\frac{2}{3(1+\gamma)}}$ with constant

C. By straightforward calculations, it can be proven the following causal equivalences:

$$W \sim \mathbb{L}_0 \text{ for } \gamma = -1/3 \text{ where } \mathbb{L}_0 \text{ is the whole Minkowski spacetime}$$
$$W \sim V \text{ for } \gamma \neq -1/3 \text{ where } (V, \mathbf{g}) \text{ is the steady state part of } \mathcal{SS}, [4].$$

These causal equivalences are rather intuitive if we have a look at the Penrose diagram of each spacetime (figure 1).

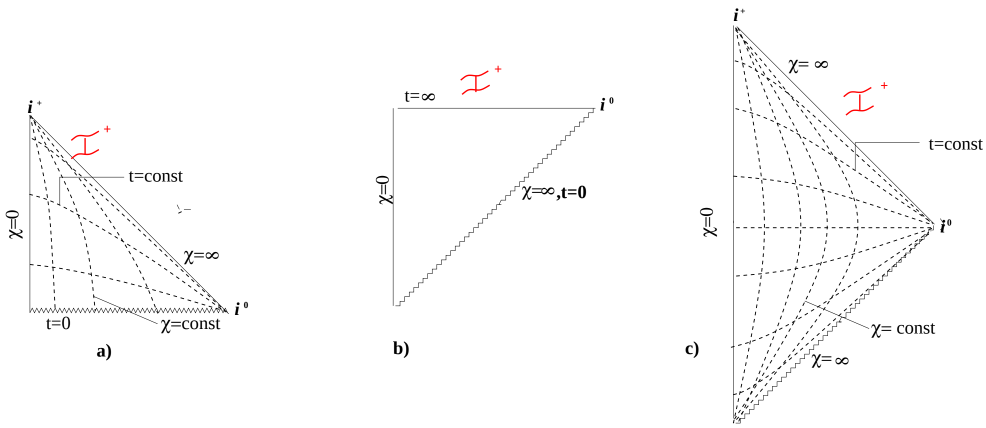


Figure 1: Penrose's diagrams of FRW spacetimes for a) $-1/3 < \gamma < 1$, b) $-1 < \gamma < -1/3$ and c) $\gamma = 1/3$. Notice the similar shape of diagram c) with that of $\mathbb{L}$, and of the steady state part of $\mathcal{SS}$ with a) and b) [4].

Example 5.4 (Vaidya's Spacetime.) *Let us show finally an example of a submonoid of causal transformations. Consider the Vaidya spacetime whose line element is [10]*

$$ds^2 = \left(1 - \frac{2M(t)}{r}\right) dt^2 - 2dtdr - r^2 d\Omega^2, \quad -\infty < t < \infty, \; 0 < r < \infty$$

where t is a null coordinate (that is, dt is a null 1-form), and $M(t)$ is a non-increasing function of t interpreted as the mass. Take the diffeomorphisms $\varphi_s : t \mapsto t + s$. Then $\varphi_s^* \mathbf{g}$ can be cast in the form

$$\varphi_s^* \mathbf{g} = \mathbf{g} - \frac{2}{r}(M(t+s) - M(t)) dt \otimes dt.$$

Hence, $\varphi_s^*\mathbf{g} \in \mathcal{DP}_2^+(V)$ *iff* $M(t+s) - M(t) \leq 0$*, which implies that* $\{\varphi_s\}_{s\geq 0}$ *are causal transformations, so that* $\{\varphi_s\}_{s\in\mathbb{R}}$ *is a maximal submonoid of causal transformations. The differential equation for the infinitesimal generator* $\vec{\xi} = \partial/\partial t$ *of this submonoid is easily calculated and reads*

$$\pounds(\vec{\xi})\mathbf{g} = -\frac{2}{r}\dot{M}(t)dt \otimes dt\,.$$

This is a particular case of a proper Kerr-Schild vector field, recently studied in [11]. Notice that Schwarzschild spacetime is included for the case M *=const., in which case* $\vec{\xi}$ *is a Killing vector. This may lead to a natural generalization of symmetries.*

6 Conclusions

In this work a new relation between Lorentzian manifolds which keeps the causal character of causal vectors has been put forward. With the aid of this relation, we have introduced the concepts of causal relation and causal isomorphism of Lorentzian manifolds which allow us to establish rigorously when two given Lorentzian manifolds are causally indistinguishable regardless their metric properties. This tools could be also useful in order to find out the global causal structure of a given spacetime by just putting it in causal equivalence with another known spacetime.

Finally a new transformation for Lorentzian manifolds, called causal transformation has been defined. These transformations are a natural generalization of the group of conformal transformations and their actual relevance is one of our main lines of future research.

7 Acknowledgements

This research has been carried out under the research project UPV 172.310-G02/99 of the University of the Basque Country.

References

[1] G. BERGQVIST AND J. M. M. SENOVILLA *Null cone preserving maps, causal tensors and algebraic Rainich theory.* Class. Quantum Grav.**18**, 5299-5326, (2001)

[2] A. García-Parrado and J. M. M. Senovilla *Causal relationship: a new tool for the causal characterization of Lorentzian manifolds.* Class. Quantum Grav. **20**, 625-664, (2003)

[3] J. M. M. Senovilla *Super-Energy tensors.* Class. Quantum Grav. **17**, 2799-2842, (2000).

[4] S. W. Hawking and G. F. R. Ellis *The large scale structure of spacetime.* Cambridge University Press, Cambridge, (1973).

[5] R. M. Wald *General Relativity.* The University of Chicago Press, (1984).

[6] J. M. M. Senovilla *Singularity Theorems and their Consequences.* Gen. Rel. Grav. **30**, 701-848, (1998).

[7] E. H. Kronheimer and R. Penrose Proc. Camb. Phil. Soc. **63** 481-501 (1967)

[8] F. Hausdorff *Set Theory* Chelsea N.Y. 1978.

[9] J. Hilgert, K. H. Hofmann and J. D. Lawson *Lie groups, Convex Cones and Semigroups.* Oxford Sciencie Publications, (1989).

[10] P.C. Vaidya *Proc. Indian Acad. Sci.* **A**33, 264 (1951).

[11] B. Coll, S. R. Hildebrandt and J. M. M. Senovilla. *Kerr Schild Symmetries.* Gen. Rel. Grav. **33**, 649-670, (2001).

Proceedings of the meeting
Lorentzian Geometry-Benalmádena 2001
Benalmádena, Málaga, Spain
Pub. de la RSME, Vol. 5 (2003), 171-181

A New Approach to the Study of Conjugate Points along Null Geodesics on Certain Compact Lorentzian Manifolds

Manuel Gutiérrez[1], Francisco J. Palomo[1] and Alfonso Romero[2]

[1]*Departamento de Álgebra, Geometría y Topología, Universidad de Málaga*

[2]*Departamento de Geometría y Topología, Universidad de Granada*

E-mail: `mgl@agt.cie.uma.es`, `fjpalomo@eresmas.com`, `aromero@ugr.es`

Abstract

An integral inequality on a compact Lorentzian manifold admitting a timelike conformal vector field is shown under some assumption on its conjugate points along null geodesics. The inequality relates the behaviour of these conjugate points to global geometrical results. As an application, several properties of the null geodesics of a natural Lorentzian metric on each odd dimensional sphere are obtained.

1 Introduction

In [5], [6] the authors have introduced a new integral inequality on a remarkable family of compact Lorentzian manifolds. It reproves a classical result of M. Berger and L.W. Green in Riemannian geometry [4, Theors. 4.2, 5.3], and, in a suitable way, extends it to the Lorentzian setting.

The main aim of this note is to show the use of that integral inequality to the study of conjugate points along null geodesics on Lorentzian odd dimensional spheres. In fact, each odd dimensional sphere may be endowed with a natural Lorentzian metric (section 3). Our method permits us to study null conjugate points without using the Jacobi equation and related techniques. Moreover, as far as we know, there are not many examples of compact Lorentzian manifolds where the behaviour of their null geodesics, null conjugate points and null conjugate loci have been described.

Compact Lorentzian manifolds have been historically neglected because of both physical and mathematical reasons. Recall that they have closed timelike curves, and therefore they are acausal (in particular, they cannot be isometrically immersed in a Lorentz-Minkowski space of any dimension) and not physically admissible. On the other hand, a compact Lorentzian manifold may be geodesically incomplete (this fact is well known) and the elliptic model of Lorentzian space form is not compact (contrary to the Riemannian case). However, it has been recently argued [20] that the study of field theory on compact spacetimes could be interesting for Physics and it could give valuable information about the underlying manifold, complementary to the one obtained from the Riemannian theory. From a mathematical point of view, the lack of completeness in the compact case gave rise to the obtention of extra conditions which joint to compactness would imply completeness of the Lorentzian manifold. For instance, in [10] it has been proved that every compact Lorentzian manifold with constant sectional curvature is geodesically complete (the flat case was previously shown in [2]); in [16] that every compact Lorentzian manifold which admits a timelike conformal vector field is geodesically complete (see also [14] for a wide information on completeness of Lorentzian manifolds). Physicists are familiarized with the study of conformal vector fields, in fact the assumption of their existence on spacetime is a way to impose some symmetry useful, for instance, to study the Einstein equations (see, for example [3]). Finally, recall the outstanding role of timelike conformal vector fields in the introduction of Bochner's technique in Lorentzian manifolds [17], [18], [13].

The content of this note is organized as follows. Section 2 is first devoted to recall the notion and main properties of the null congruence associated to a timelike conformal vector field on a Lorentzian manifold. In the compact case, an integral inequality is shown, Theorem 2.1, and, using a well known result of H. Karcher, it is analyzed when the equality holds. Moreover, we also show that Theorem 2.1 provides information on the manifold from the nonexistence of null conjugate points.

Finally, in section 3 we consider a natural Lorentzian metric g on each sphere $\mathbb{S}^{2n+1}$ (it was called *canonical* in [20]) which is introduced from three different procedures. It is shown that g has a large group of isometries, an isotropic property for null tangent directions and that it is homogeneous, Proposition 3.2. Its null geodesics are studied, showing that no null geodesic is closed. Null conjugate points and null conjugate loci are analyzed, Proposition 3.3. In fact, it is shown that all past (or future) null geodesics starting from a point p meet at the second conjugate point of p, and that the null conjugate locus at every point is an imbedded $(2n-1)-$dimensional sphere $\mathbb{S}^{2n-1}$.

2 Preliminaries

Let (M,g) be an $n(\geq 2)-$dimensional Lorentzian manifold; that is a (connected) smooth manifold M endowed with a non-degenerate metric g with index 1, i.e. with signature $(-,+,...,+)$. As usual, T_pM denotes the tangent space at $p \in M$, TM the tangent bundle of M, and $\pi : TM \longrightarrow M$ the natural projection. We shall write ∇ for the Levi-Civita connection of g, R for the Riemannian curvature tensor (our convention on the curvature tensor is $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z)$, Ric for the Ricci tensor, $\widetilde{\mathrm{Ric}}$ for the corresponding quadratic form, S for the scalar curvature and $d\mu_g$ for the canonical measure induced from g.

The causal character of a tangent vector $v \in T_pM$ is *timelike* (resp. *null*, *spacelike*) if $g(v,v) < 0$ (resp. $g(v,v) = 0$ and $v \neq 0$, $g(v,v) > 0$ or $v = 0$). If $v \in T_pM$ then, γ_v will denote the unique geodesic such that $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. It is well-known that the causal character of the velocities $\gamma'(t)$, for any geodesic γ of (M,g), does not depend on the parameter t. In particular, a null geodesic γ of (M,g) is a geodesic such that $\gamma'(t)$ is a null vector. A vector field $K \in \mathfrak{X}(M)$ is said to be timelike if K_p is timelike for all $p \in M$. A timelike or null tangent vector $v \in T_pM$ is said to be future (resp. past) with respect to K if $g(v,K_p) < 0$ (resp. $g(v,K_p) > 0$). We will write $U = hK$ where $h = [-g(K,K)^{-\frac{1}{2}}]$ and so $g(U,U) = -1$ holds on all M.

Let $\widehat{g}$ be the *Sasaki metric* on TM induced from the Lorentzian metric g. We point out that it may be introduced in a similar way to the Riemannian case. But now $\widehat{g}$ is semi-Riemannian with index 2, and the fact that the natural projection $\pi : (TM, \widehat{g}) \longrightarrow (M, g)$ is a semi-Riemannian submersion remains true.

Throughout the remainder of this paper, (M, g) will denote a Lorentzian manifold with dimension $n \geq 3$, time oriented by a timelike vector field K. Recall [8], [12] that the *null congruence associated to* K is defined as follows:

$$C_K M = \left\{ v \in TM \,:\, g(v, v) = 0,\, g(v, K_{\pi(v)}) = 1 \right\}. \tag{2.1}$$

This subset of TM has the following nice properties [5], [6]:

(i) For each null tangent vector v, there exists a unique $t \in \mathbb{R}$ such that $tv \in C_K M$, and the map $v \longmapsto [v]$ is a diffeomorphism from $C_K M$ to the manifold $\mathcal{N} = \{[v] \in PM : g(v, v) = 0\}$ of the null directions of M (here PM denotes the projective fiber bundle associated to TM).

(ii) It is an orientable imbedded submanifold of TM with dimension $2(n-1)$. Moreover $(C_K M, \pi, M)$ is a fiber bundle with fibre type $\mathbb{S}^{n-2}$, and so $C_K M$ will be compact if M is assumed to be compact.

(iii) The induced metric on $C_K M$ from the Sasaki metric of TM, which we agree also to represent by $\widehat{g}$, is Lorentzian. Moreover, the restriction of π to $C_K M$ is a semi-Riemannian submersion with spacelike fibers.

Sectional curvature of a Lorentzian metric can be defined for non-degenerate tangent planes but it cannot be stated for null planes (i.e. degenerate planes). If v is a null tangent vector and σ a null plane containing it, the *null sectional curvature* with respect to v of the plane σ is defined to be $\mathcal{K}_v(\sigma) = g(R(u, v)v, u)/g(u, u)$, where $\{u, v\}$ is a basis of σ [7], [8], [1, Def. A.6]. Note that $\mathcal{K}_v(\sigma)$ does not depend on the choice of the non-zero spacelike vector u, but it does quadratically on v.

From now on let us suppose that a null congruence associated with a timelike vector field K has been fixed. Then, we may choice, for every null plane σ, the unique null vector $v \in C_K M \cap \sigma$, thus the null sectional curvature can be thought as a function on null tangent planes. In this note we always use such convention, and we will call it the $K-$*normalized null sectional curvature*.

Until now, no extra hypothesis on the timelike vector field K has been assumed. Recall that a vector field X is called *conformal* (resp. *Killing*) if each of its (local) fluxes consists of (local) conformal (resp. isometric) transformations. It is well known that X is conformal if and only if the Lie derivative of g with respect to X satisfies $\mathcal{L}_X g = \rho g$, where $\rho : M \to \mathbb{R}$ (Killing when $\rho = 0$). If K is assumed to be conformal, then every null geodesic γ_v of (M, g) with $v \in C_K M$, provides us with the null geodesic γ_v' of $(C_K M, \widehat{g})$. Furthermore, each null geodesic β of (M, g) may be reparametrized to obtain a null geodesic α which satisfies $\alpha'(t) \in C_K M$. In fact, consider the real number $a = g(\beta', K_\beta)$, which satisfies $a \neq 0$. If we put $\alpha(t) = \beta(\frac{t}{a})$, then $g(\alpha', K_\alpha) = 1$ holds for all t. Null geodesics will be considered to be parametrized by this $K-$*affine parameter*.

We will next assume that (M, g) is a compact Lorentzian manifold and K a timelike conformal vector field. Recall that in this case (M, g) is geodesically complete [16]. The following integral inequality is the key tool to relate null conjugate points to global geometric properties:

Theorem 2.1 [5, Theor. 3.5] *Let (M, g) be a compact Lorentzian manifold which admits a timelike conformal vector field K. If there exists $a \in (0, +\infty)$ such that every null geodesic $\gamma_v : [0, a] \longrightarrow M$, with $v \in C_K M$, has no conjugate point of $\gamma_v(0)$ in $[0, a)$, then*

$$\int_M h^{n-2} d\mu_g \geq \frac{a^2}{\pi^2 (n-1)(n-2)} \int_M \left[n\widetilde{\mathrm{Ric}}(U) + S \right] h^n d\mu_g. \qquad (2.2)$$

Moreover, equality holds if and only if (M, g) has $U-$normalized null sectional curvature $\frac{\pi^2}{a^2 h^2}$.

Observe that if equality holds in (2.2) then the $U-$normalized null sectional curvature of (M, g) is an everywhere non-zero point function. On the other hand, it was proven by H. Karcher, [9] the following result:

Let U be a unit timelike vector field on an $n(\geq 4)-$ dimensional Lorentzian manifold (M, g). The $U-$normalized null sectional curvature is an everywhere non-zero point function if and only if the following conditions hold:

1. The distribution $U^\perp$ is integrable.

2. The integral manifolds of $U^\perp$ are totally umbilic and have constant sectional curvature.

3. (M, g) is locally conformal to a flat Lorentzian manifold.

Combining Theorem 2.1 and Karcher's result we can give a characterization of the equality in (2.2) in terms of the distribution $U^\perp (= K^\perp)$ and the locally conformal flatness of (M, g).

Moreover, Theorem 2.1 provides also information of (M, g) from the nonexistence of null conjugate points. In fact, if it is assumed that every null geodesic does not contain a pair of mutually conjugate points, then (2.2) is valid for any positive real number a. Therefore it must happen

$$\int_M \left[n\widetilde{\mathrm{Ric}}(U) + S \right] h^n d\mu_g \leq 0. \tag{2.3}$$

In the next section we will use the integral inequality (2.2) to get a bound of a for a relevant family of compact Lorentzian manifolds which admit a unit timelike Killing vector field.

3 Lorentzian Odd Dimensional Spheres

We consider $\mathbb{R}^{2n+2}$ identified to $\mathbb{C}^{n+1}$ as usual: $(x_1, ..., x_{2n+2}) = (z_1, ..., z_{n+1})$, with $z_j = x_j + ix_{n+1+j}$. So that, the unit sphere of $\mathbb{R}^{2n+2}$ is written

$$\mathbb{S}^{2n+1} = \Big\{ z = (z_1, ..., z_{n+1}) \in \mathbb{C}^{n+1} : \sum_{j=1}^{n+1} |z_j|^2 = 1 \Big\}.$$

Let $U \in \mathfrak{X}(\mathbb{S}^{2n+1})$ be given by $U_z = iz$ at any $z \in \mathbb{S}^{2n+1}$. For the canonical Riemannian metric g_{can} of $\mathbb{S}^{2n+1}$, U is Killing and satisfies $g_{can}(U, U) = 1$. Therefore, $\nabla_U U = 0$, where ∇ is the Levi-Civita connection of g_{can}; that is, the integral curves of U are geodesics of g_{can}.

Let ω be the $1-$form metrically equivalent to U with respect to g_{can}. A Lorentzian metric on $\mathbb{S}^{2n+1}$ can be defined by

$$g \,=\, g_{can} \,-\, 2\,\omega \otimes \omega. \tag{3.1}$$

This construction of g from g_{can} is standard, but the Lorentzian metric g deserves of making stand out among all the Lorentzian metrics of $\mathbb{S}^{2n+1}$. In fact, it has previously considered [20]. It is not difficult to show that the Levi-Civita connection $\widetilde{\nabla}$ of g satisfies:

$$\widetilde{\nabla}_X Y = \nabla_X Y - 2\,\omega(X)\nabla_Y U - 2\,\omega(Y)\nabla_X U, \tag{3.2}$$

where $X, Y \in \mathfrak{X}(\mathbb{S}^{2n+1})$. Moreover, the vector field U satisfies $g(U, U) = -1$, it is Killing for g and $\widetilde{\nabla}_U U = 0$; so that, its integral curves are unit timelike geodesics of g. On the other hand, observe that the inclusion

$\mathbb{S}^{2n+1} \hookrightarrow \mathbb{S}^{2m+1}$, $n < m$, $(z_1, .., z_{n+1}) \mapsto (z_1, .., z_{n+1}, 0, .., 0)$ is a totally geodesic Lorentzian submanifold, when both spheres are endowed with the corresponding Lorentzian metrics (3.1).

Recall now the classical Hopf fibration $\Pi : (\mathbb{S}^{2n+1}, g_{can}) \to (\mathbb{C}P^n, g_{FS})$, $z \mapsto [z]$, where $\mathbb{C}P^n$ is the complex projective space endowed with its Fubini-Study Kähler metric g_{FS} of constant holomorphic sectional curvature 4 [11, p. 273]. Recall that Π permits to consider $\mathbb{S}^{2n+1}$ as a principal fiber bundle over $\mathbb{C}P^n$ with structural group $\mathbb{S}^1$. Moreover, Π is a Riemannian submersion with totally geodesic fibres. If the Riemannian metric g_{can} is replaced by the Lorentzian metric g, then Π becomes a semi-Riemannian submersion from $(\mathbb{S}^{2n+1}, g)$ to $(\mathbb{C}P^n, g_{FS})$ with timelike totally geodesics fibres. Let us remark that g may be considered as a particular case of a Kaluza-Klein metric. In fact, if we put $\mathfrak{s}^1 = i\mathbb{R}$ for the Lie algebra of $\mathbb{S}^1$ then $i\omega$ is a connection form on $\mathbb{S}^{2n+1}$, and $g = \Pi^*(g_{FS}) - \omega \otimes \omega$, [15].

As a third description of the Lorentzian metric g, note that it can be characterized from the properties:

$$g_{|\mathcal{V}} = -g_{can|\mathcal{V}}, \quad g_{|\mathcal{H}} = g_{can|\mathcal{H}}, \quad g(\mathcal{V}, \mathcal{H}) = 0, \tag{3.3}$$

where $\mathcal{V}$ and $\mathcal{H}$ are respectively the vertical and the horizontal distributions for the canonical connection of the Hopf fibration.

Now recall that if U is a unit timelike vector field on a Lorentzian manifold (M, g), and $p \in M$, (M, g) is said to be *spatially isotropic* with respect to U at p if for every two unit vectors $u_1, u_2 \in U_p^{\perp}$ there exists an isometry $\phi : M \longrightarrow M$ such that $\phi(p) = p$, $d\phi_p(U_p) = U_p$ and $d\phi_p(u_1) = u_2$. (M, g) is said to be *spatially isotropic* with respect to U if it is *spatially isotropic* with respect to U at every point, [19, p. 47]. The following results are easy to show:

Lemma 3.1 *Let (M, g) be a Lorentzian manifold which admits a unit timelike vector field U. Then (M, g) is spatially isotropic with respect to U if and only if for every $p \in M$ and for every $u, v \in (C_U M)_p$ there exists an isometry $\phi : M \longrightarrow M$ such that $\phi(p) = p$, $d\phi_p(U_p) = U_p$ and $d\phi_p(u) = v$.*

Proposition 3.2 [5, Prop. 4.2] *$(\mathbb{S}^{2n+1}, g)$ is spatially isotropic with respect to U and the unitary group $U(n+1)$ acts transitively by $g-$isometries on $\mathbb{S}^{2n+1}$.*

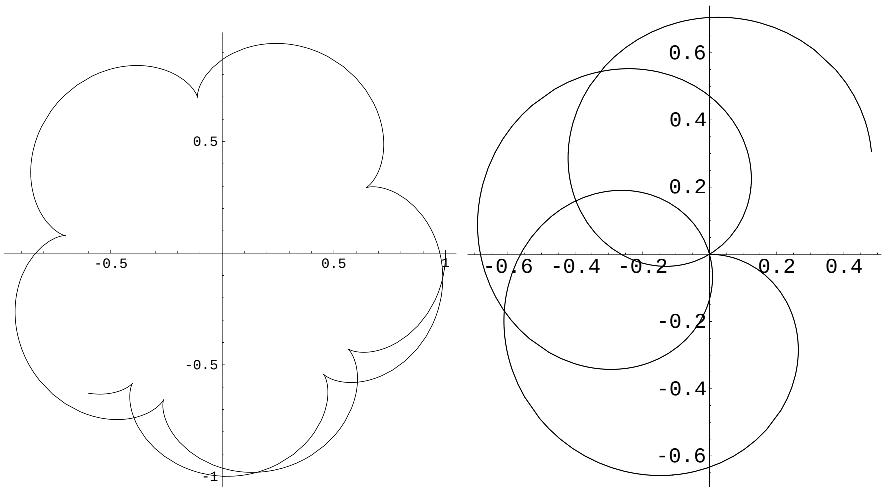
Now note that in order to analyze the behaviour of the null geodesics of the Lorentzian odd dimensional spheres, it suffices to consider the ones starting from the specific point $p = (1, ..., 0) \in \mathbb{S}^{2n+1}$. Observe that $v \in (C_U\mathbb{S}^{2n+1})_p$ if and only if $v = (-i, v_2, ..., v_{n+1})$ with $\sum_{j=2}^{n+1} \mid v_j \mid^2 = 1$.

If we agree to represent $\gamma_v(t) = (\Theta_1^v(t), ..., \Theta_{n+1}^v(t))$, with $\Theta_k^v : \mathbb{R} \to \mathbb{C}$, $1 \leq k \leq n+1$, then we get:

$$\Theta_1^v(t) = \frac{2-\sqrt{2}}{4}\, e^{(-2-\sqrt{2})it} + \frac{2+\sqrt{2}}{4}\, e^{(-2+\sqrt{2})it}$$

$$\Theta_j^v(t) = \frac{\sqrt{2} i v_j}{4}\left[e^{(-2-\sqrt{2})it} - e^{(-2+\sqrt{2})it}\right],\ j \geq 2,$$

The following figures show each kind of components of a lightlike geodesic.



From the previous equations the following facts directly follow:

(1) *There is no closed null geodesic in* $(\mathbb{S}^{2n+1}, g)$,

(2) *For every* $v, u \in (C_U\mathbb{S}^{2n+1})_p$, $v \neq u$, $\gamma_v(t) = \gamma_u(t)$ *holds if and only if* $t = \frac{k\pi}{\sqrt{2}}$ *for some* $k \in \mathbb{Z}$.

Now we pay attention to curvature properties of $(\mathbb{S}^{2n+1}, g)$. Its scalar curvature S can be computed to obtain $S = 2n(2n+3)$. On the other hand, we get $\widetilde{\mathrm{Ric}}(U) = 2n$ and the $U-$normalized null sectional curvature of $(\mathbb{S}^{2n+1}, g)$ is a point function if and only if $n = 1$, with $\mathcal{K}_v(v^\perp) = 8$ for any $v \in C_U\mathbb{S}^3$, (see [5] for details). So, it should be pointed out that the first conclusion in Karcher's theorem does not remain true if it is assumed $\dim M = 3$, because of non integrability of the distribution $U^{\perp}$.

We end this note with an application of our integral inequality (2.2) to the study of the behaviour of conjugate points along null geodesics in Lorentzian odd dimensional spheres.

Proposition 3.3 [5, Prop. 4.4] *For every null geodesic γ_v of $(\mathbb{S}^{2n+1}, g)$ with $v \in C_U\mathbb{S}^{2n+1}$, the points $\gamma_v(0)$ and $\gamma_v(\frac{\pi}{2\sqrt{2}})$ are conjugate and there is no conjugate point to $\gamma_v(0)$ on $[0, \frac{\pi}{2\sqrt{2}})$. Moreover the past null conjugate locus of each point $p \in \mathbb{S}^{2n+1}$ is a $(2n-1)-$dimensional imbedded sphere.*

Observe that previous result may be dualized to analyze the future null conjugate locus.

Remark 3.4 A conjugate point $\gamma(a)$ of $\gamma(0) = p$ along a null geodesic γ can be interpreted as an "almost-meeting point" of null geodesics starting from p. In our case, the first conjugate point along any null geodesic is exactly at the middle of the path to the first "meeting point" which is the second null conjugate point. Thus, null geodesics of $\mathbb{S}^{2n+1}$ have an "Auf wiedersehensflächen" type property as in Riemannian case, but in contrast to that, in the Lorentzian setting the first "meeting point" is not the first conjugate point.

Acknowledgments

The first author has been partially supported by MCYT-FEDER Grant BFM2001-1825 and CECJA Grant FQM-213, the second author has been partially supported by CECJA Grant FQM-213, and the third author has been partially supported by MCYT-FEDER Grant BFM2001-2871-C04-01.

References

[1] J.K. Beem, P.E. Ehrlich and K.L. Easley, *Global Lorentzian Geometry*, Second ed., Pure and Applied Math. **202**, Marcel Dekker, 1996.

[2] Y. Carrière, *Autour de la conjeture de L. Markus sur les variétés affines*, Inven. Math. **95** (1989), 615–628.

[3] A.A. Coley and B.O.J. Tupper, *Special conformal Killing vector space-times and symmetry inheritance*, J. Math. Phys. **30** (1989), 2616–2625.

[4] L.W. Green, *Auf Wiedersehensflächen*, Ann. of Math. **78** (1963), 289–299.

[5] M. Gutiérrez, F.J. Palomo and A. Romero, *A Berger-Green type inequality for Compact Lorentzian Manifolds*, Trans. Amer. Math. Soc. **354** (2002), 4505-4523. (Erratum, **355** (2003), 5119-5120).

[6] M. Gutiérrez, F.J. Palomo and A. Romero, *Conjugate points along null geodesics on Lorentzian manifolds with symmetries*, Proc. X Fall Workshop on Geometry and Physics, Miraflores de la Sierra (Madrid), Pub. R.S.M.E., (to appear).

[7] S.G. Harris, *A triangle comparison theorem for Lorentz manifolds*, Indiana Univ. Math. J. **31** (1982), 289–308.

[8] S.G. Harris, *A characterization of Robertson-Walker Spaces by null sectional curvature*, Gen. Relativity Gravitation **17** (1985), 493–498.

[9] H. Karcher, *Infinitesimale Charakterisierung von Friedmann-Universen*, Arch. Math. **38** (1982), 58–64.

[10] B. Klinger, *Completude des varietes lorentziennes a courbure constante*, Math. Ann. **306** (1996), 353–370.

[11] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* Vol. **2**, Wiley Inters. Publ., New York, 1969.

[12] L. Koch-Sen, *Infinitesimal null isotropy and Robertson-Walker metrics*, J. Math. Phys. **26** (1985), 407–410.

[13] A. Romero, *The introduction of Bochner's technique on Lorentzian manifolds*, Nonlinear Anal. **47** (2001), 3047–3059.

[14] A. Romero and M. Sánchez, *On completeness of compact Lorentzian manifolds*, Geometry and Topology of Submanifolds **VI**, World Scientific, (1994), 171–182.

[15] A. Romero and M. Sánchez, *On Completeness of Certain Families of Semi-Riemannian Manifolds*, Geom. Dedicata **53** (1994), 103–117.

[16] A. Romero and M. Sánchez, *Completeness of compact Lorentz manifolds admiting a timelike conformal-Killing vector field*, Proc. Amer. Math. Soc. **123** (1995), 2831–2833.

[17] A. Romero and M. Sánchez, *An integral inequality on compact Lorentz manifolds and its applications*, Bull. London Math. Soc. **28** (1996), 509–513.

[18] A. Romero and M. Sánchez, *Bochner's technique on Lorentz manifolds and infinitesimal conformal symmetries*, Pacific J. Math. **186** (1998), 141–148.

[19] R. Sachs and H. Wu, *General Relativity for Mathematicians*, Springer-Verlag, 1977.

[20] U. Yurtsever, *Test fields on compact space-times*, J. Math. Phys. **31** (1990), 3064–3077.

Proceedings of the meeting
Lorentzian Geometry-Benalmádena 2001
Benalmádena, Málaga, Spain
Pub. de la RSME, Vol. 5 (2003), 183-203

Geodesics on Lorentzian manifolds: a variational approach

Antonio Masiello[1]

[1] *Dipartimento Interuniversitario di Matematica, Politecnico di Bari Via E. Orabona 4, 70125 Bari, Italy*

Abstract

The variational properties of the action integral on a semiriemannian manifold are studied. In particular we state the Morse Inequalities for the geodesics joining two nonconjugate points on a Lorentzian manifold.

1 Introduction

We consider a semiriemannian manifold $(\mathcal{M}, g)$, where $\mathcal{M}$ is a smooth, connected, finite dimensional differentiable manifold and g is a metric tensor on $\mathcal{M}$. For any $z \in \mathcal{M}$, the tensor g defines a bilinear form $g(z)$ on the tangent space $T_z\mathcal{M}$ at z to $\mathcal{M}$ such that $g(z)$ is symmetric and nondegenerate. The number of the negative eigenvalues of the bilinear form $g(z)$ does not depend on z and such number is called the *index* of the metric g and it is denoted by $\nu(g)$. The semiriemannian manifold $(\mathcal{M}, g)$ is called *Riemannian* if $\nu(g) = 0$ and it is called *Lorentzian* if $\nu(g) = 1$. We refer to the books [4, 29] for the basic properties of semiriemannian manifolds.

A smooth curve $\gamma :]a, b[\longrightarrow \mathcal{M}$ is said *geodesic* if

$$D_s\dot{\gamma} = 0, \tag{1.1}$$

where D_s denotes the covariant derivative along γ induced by the Levi-Civita connection of g and $\dot{\gamma}$ is the tangent vector field along γ. It is well known that the geodesics joining two points on a semiriemannian manifold satisfy a variational principle. Indeed the geodesics joining two fixed points p and q on $\mathcal{M}$ are the extremals of the action integral

$$f(z) = \int_0^1 g(z(s))[\dot{z}(s), \dot{z}(s)]ds \tag{1.2}$$

defined on the infinite dimensional Sobolev manifold of $\Omega^{1,2}(p, q; \mathcal{M})$ of the curves $z(s) : [0, 1] \longrightarrow \mathcal{M}$ such that $z(0) = p$, $z(1) = q$, z is continuous and its derivative $\dot{z}$ is square integrable. It is well known that the space $\Omega^{1,2}(p, q; \mathcal{M})$ is equipped of a structure of infinite dimensional manifold modelled on the Sobolev–Hilbert space $H^{1,2}([0, 1], \mathbb{R}^n)$ of the absolutely continuous curves on $\mathbb{R}^n$, $n = \dim \mathcal{M}$, having square integrable derivative. If $z \in \Omega^{1,2}(p, q; \mathcal{M})$, the tangent space $T_z\Omega^{1,2}(p, q; \mathcal{M})$ at z is given by

$$T_z\Omega^{1,2}(p, q; \mathcal{M}) = \{\zeta \in \Omega^{1,2}((p, 0), (q, 0); T\mathcal{M}) : \pi \circ \zeta = z\},$$

where $T\mathcal{M}$ denotes the tangent bundle of $\mathcal{M}$ and $\pi\colon T\mathcal{M} \to \mathcal{M}$ is the bundle projection. In other words $T_z\Omega^{1,2}(p, q; \mathcal{M})$ consists of the vector fields ζ along z of class $H^{1,2}$ and having null boundary conditions.

The study of the existence and the multiplicity of geodesics joining two points on a Riemannian manifold and the relations between the set of such geodesics and the topology of the manifold $\mathcal{M}$ have played a central role in the XX century in the development of what is called now the *Calculus of Variations in the Large* and in particular in the *Critical Point Theory*, the study of the critical points of a functional which are not only global or local minima (or maxima) as in the classical Calculus of Variations, but also saddle points. This kind of studies take their origin essentially by the seminal work of Henri Poincaré on celestial mechanics and dynamical systems. After some first results by G. Birkhoff on closed geodesics on compact surfaces, the major impetus on Critical Point Theory came with the fundamental work by M. Morse in the U.S.A and by Ljusternik and Schnirelmann in the U.S.S.R. In particular Morse developed an exhaustive theory for geodesics on Riemannian manifolds, while Ljusternik, Schnirelmann obtained many results about another classical and difficult problem in Differential Geometry as the existence of closed geodesics on a compact Riemannian manifold, which has been completely solved only few years ago, see the book of Klingenberg [25].

The extension of the results of Morse, Ljusternik and Schnirelmann to semiriemannian manifolds of positive index presents immediately many difficulties with respect to the Riemannian case. If the semiriemannian manifold $(\mathcal{M}, g)$ has index $\nu(g) > 0$, the action functional (1.2) is *strongly indefinite*, i.e. now $f\colon \Omega^{1,2}(p,q;\mathcal{M}) \longrightarrow \mathbb{R}$ is unbounded both from below and from above, so the direct methods of the Calculus of Variations can not be directly applied. Moreover any critical point of f (i.e. any geodesic joining the points p and q) has its Morse index equal to $+\infty$. This means that any geodesic is an infinite dimensional saddle point for the functional f. For these reasons the classical Morse Theory and the classical Ljusternik–Schnirelmann Theory do not allow to obtain results for semiriemannian manifolds with positive index, as for instance for Lorentzian manifolds.

Critical Point Theory for strongly indefinite functionals has been the object of several deep studies in the last twenty-five years and it has many applications in the study of Hamiltonian systems, nonlinear hyperbolic equations and in symplectic geometry, related to Floer homology and the resolution of Arnold conjectures. We refer to a recent book by Alberto Abbondandolo [1] on strongly indefinite functionals and applications to Hamiltonian systems.

In this paper we present some recent results obtained in collaboration with Alberto Abbondandolo, Vieri Benci and Dino Fortunato [2]. We shall state the Morse inequalities for the geodesics joining two non-conjugate points on two classes of Lorentzian manifolds, the *stationary* and the *orthogonal splitting* Lorentzian manifolds. The Morse inequalities are obtained by applying an abstract Morse Theory for a class of strongly indefinite functionals developed in [2].

In order to study the variational properties of geodesics as critical points of the action integral, is fundamental to evaluate the second derivative $f''(z)$ at a geodesic z. Let $(\mathcal{M}, g)$ be a semiriemannian manifold, fix two points p and q in $\mathcal{M}$ and let $z\colon [0,1] \to \mathcal{M}$ be a geodesic joining p and q. It is well known that the second derivative $f''(z)\colon T_z\Omega^{1,2}(p,q;\mathcal{M}) \times T_z\Omega^{1,2}(p,q;\mathcal{M}) \to \mathbb{R}$ of the action integral at z is given by

$$f''(z)[\zeta,\zeta'] = \int_0^1 g(z)[D_s\zeta, D_s\zeta']ds - \int_0^1 g(z)[R(\zeta,\dot z)\dot z, \zeta']ds, \tag{1.3}$$

for any $\zeta,\zeta' \in T_z\Omega^{1,2}(p,q;\mathcal{M})$, where R denotes the curvature tensor for the metric g.

Formula (1.3) clearly shows as the index $\nu(g)$ influences on the spectral properties of $f''(z)$. Indeed if g is a Riemannian metric, then $f''(z)$ is a Fredholm operator and it is a compact perturbation of a positive definite bilinear form. On the other hand, if $\nu(g) > 0$, then $f''(z)$ is still a Fredholm operator, but now it is a compact perturbation of a symmetric bilinear form which is both negative definite and positive definite on an infinite dimensional subspace of $T_z\Omega^{1,2}(p, q; \mathcal{M})$. Then the Morse index of any geodesic is finite for Riemannian metrics, but it is equal to $+\infty$ if $\nu(g) > 0$. This makes difficult to apply the classical results of critical point theory (and in particular the Morse Theory) based on the deformation of the sublevels of the functional by the gradient flow, because critical points having Morse index equal to $+\infty$ do not change the homotopy type of the sublevels of a functional (we are attaching infinite dimensional handles and the infinite dimensional unit sphere on a Hilbert space is contractible !!).

A geodesic $z \in \Omega^{1,2}(p, q; \mathcal{M})$ is said nondegenerate if it is a nondegenerate critical point of the action integral f, i.e. the second derivative defines an invertible linear operator on the tangent space $T_z\Omega^{1,2}(p, q; \mathcal{M})$ with respect to some $H^{1,2}$ inner product on $T_z\Omega^{1,2}(p, q; \mathcal{M})$. Since $f''(z)$ defines a Fredholm operator of index 0, this is equivalent to require that the kernel of $f''(z)$ is trivial and this is equivalent to say that there are no solutions of the Jacobi equations $D_s^2\zeta + R(\zeta, \dot{z})\dot{z} = 0$ such that $\zeta(0) = 0$, $\zeta(1) = 0$.

Two points p and q of a semiriemannian manifold $(\mathcal{M}, g)$ are said *nonconjugate* if any geodesic joining p and q is nondegenerate. From a variational point view, the nonconjugation of the points p and q means that the action integral (1.2) is a Morse function, i.e. its critical points are nondegenerate. Using the Sard theorem it can be proved that all the couple of points in $\mathcal{M}$, except for a nowhere dense set, are nonconjugate (cf. [28]).

2 A review of Classical Critical Point Theory

We present the classical results on critical point theory and the applications to Riemannian Geometry, in particular to the geodesics joining two points on a complete Riemannian manifold, see [9, 25, 26, 28, 30].

Let (X, h) be a (possibly infinite dimensional) Riemannian manifold and $f : X \to \mathbf{R}$ a C^2 functional, a point $x \in X$ is said *critical point* of f if $f'(x) = 0$. A number $c \in \mathbf{R}$ is said *critical value* if there exists a critical point x of f such that $f(x) = c$, otherwise c is called *regular value.* Let x be a critical point of f and denote by T_xX the tangent space at x to X. The Hessian $f''(x) : T_xX \times T_xX \to \mathbf{R}$ at x is defined in the following way. For any $\xi \in T_xX$ we set

$$f''(x)\,[\xi,\xi] = \left(\frac{d^2 f(\gamma(s))}{ds^2}\right)_{s=0}$$

(where $\gamma :]-\varepsilon,\varepsilon[\to X$ is a smooth curve such that $\gamma(0) = x, \dot{\gamma}(0) = \xi$) and then we extend $f''(x)$ by polarization to any couple of tangent vectors.

The critical point x is said *nondegenerate* if the linear operator induced on T_xX by $f''(x)$ is an isomorphism. The functional f is said to be a *Morse function* if all its critical points are nondegenerate. The *Morse index* $m(x, f)$ is the maximal dimension of a subspace of T_xX where $f''(x)$ is negative definite. The *augmented Morse index* is given by $m^*(x, f) = m(x, f) + \dim Kerf''(x)$, where

$$Kerf''(x) = \{\xi \in T_xX \mid f''(x)\,[\xi,\xi'] = 0, \forall \xi' \in T_xX\}\,.$$

Clearly if $\dim X = +\infty$, the Morse index and the augmented Morse index can be infinite. We recall now the *Palais-Smale* (PS) compactness condition.

Definition 2.1 *Let $f : X \to \mathbf{R}$ be a C^1 functional defined on a Riemannian manifold (X, h) and let F be a closed subset of X, then the functional f satisfies the Palais-Smale condition on F if for any sequence $(x_m)_{m\in\mathbf{N}}$ of points of F, such that*

i)$\{f(x_m)\}_{m\in\mathbf{N}}$ is bounded;

ii)$\|\nabla f(x_m)\| \to 0$,

there exists a converging subsequence. Here $\|\cdot\|$ denotes the norm induced on the tangent bundle by the fixed Riemannian metric h on X.

For any $c \in \mathbf{R}$ we set

$$\begin{aligned} f^c &= \{x \in X \mid f(x) \le c\}\,, \\ f_c &= \{x \in X \mid f(x) \ge c\}\,. \end{aligned} \tag{2.1}$$

Moreover for any $a < b$ we set

$$f_a^b = \{x \in X \mid a \le f(x) \le b\}\,. \tag{2.2}$$

We present now the main results of Critical Point Theory. They are all based on the following deformation theorems which show the relations between the changes of the homotopy type of the sublevels of a functional and the presence of critical points of the functional itself.

Theorem 2.2 *Let $f : (X, h) \to \mathbb{R}$ be a C^1 functional defined on a complete Riemannian manifold (X, h), let $a < b$ be two regular values of f and assume that there are no critical points in f_a^b and f satisfies the Palais–Smale condition on the closed set f_a^b.*

Then there exists a continuous deformation of f^b onto f^a, i.e. there exists a continuous homotopy $H : [0, 1] \times f^b \to f^b$ such that

(i) $H(0, x) = x$, for any $x \in f^b$;

(ii) $H(t, y) = y$, for any $t \in [0, 1]$ and for any $y \in f^a$;

(iii) $H(1, f^b) = f^a$.

The proof of the previous theorem can be found in the book of J. Mawhin and M. Willem [27]. The idea is to construct the homotopy H using the flow lines of the gradient vector field ∇f of the functional f with respect to the Riemannian structure h of the manifold X. The absence of critical points of f on f_a^b and the Palais–Smale condition on the same set assure that the flow starting from f^b reaches the sublevel f^a in a finite time, remaining f^a fixed. Such idea works only for C^2 functionals, for which the gradient is locally Lipschitz continuous and the Cauchy problems for the gradient flow have an unique solution. The proof for functionals of class C^1 is obtained using the notion of *pseudogradient* field introduced by R. Palais.

The deformation lemma can be extended to in the case of presence of critical points of the functional. We present now a version describing the behavior of the functional nearby a critical value, see for instance [27] for the proof.

Theorem 2.3 *Let $f : (X, h) \to \mathbb{R}$ be a C^1 functional defined on the complete Riemannian manifold (X, h), let $c \in \mathbb{R}$ and let $K_c = \{x \in X : f(x) = c, f'(x) = 0\}$ be the set of the critical points of f at the level c (the Palais–Smale condition implies that K_c is a compact set).*

Then, for any neighborhood U of K_c, there exists a positive number ε_0 such that for any $\varepsilon \in]0, \varepsilon_0[$ there exists a continuous homotopy $H_\varepsilon : [0, 1] \times f^{c+\varepsilon} \setminus U \to f^{c+\varepsilon} \setminus U$ such that

(i) $H_\varepsilon(0, x) = x$, for any $x \in f^{c+\varepsilon} \setminus U$;

(ii) $H_\varepsilon(t, y) = y$, for any $t \in [0, 1]$ and $y \in f^{c-\varepsilon} \setminus U$;

(iii) $H_\varepsilon(1, f^{c+\varepsilon} \setminus U) = f^{c-\varepsilon}$.

The two previous lemma are the basic tool to deduce some results on the existence and the multiplicity of critical points of functionals bounded from below and satisfying the Palais–Smale condition.

Theorem 2.4 *Let $f : (X, h) \to \mathbb{R}$ be a C^1 functional defined on a complete Riemannian manifold (X, h), bounded from below and satisfying the Palais–Smale condition on X.*

Then the infimum is attained and there exists a point $x_0 \in X$ such that $f(x_0) = \inf_X f$.

The proof of this theorem is a simple consequence of the first deformation lemma. If the topology of the manifold X is rich, we have the following estimate from below of the number of critical points of a functional in terms of a topological invariant of the manifold, the *Ljusternik–Schnirelmann category* of X. We recall that for any topological space X, the Ljusternik–Schnirelmann category $\mathrm{cat}(X)$ is equal to the minimal number of closed and contractible subsets which cover X itself. If such a minimal number does not exist, it is $\mathrm{cat}(X) = +\infty$.

Theorem 2.5 *Let $f : (X, h) \to \mathbb{R}$ be a C^1 functional defined on a complete Riemannian manifold (X, h), bounded from below and satisfying the Palais–Smale condition.*

Then the functional f has at least $\mathrm{cat}(X)$ critical points. Moreover, if $\mathrm{cat}(X) = +\infty$, then there exists a sequence x_n of critical points of f such that $f(x_n) \to +\infty$.

The proof of this theorem was obtained by Ljusternik and Schnirelmann at the end of the twenties of the last century. A modern proof can be found in the book of Mawhin and Willem.

Finally we present the results of classical Morse Theory for a functional f bounded from below and satisfying the Palais–Smale condition. Morse Theory gives more precise estimates for the critical points of a functional defined on a Hilbert manifold, in particular it gives some estimates on the number of critical points having a fixed Morse index. However, in order to prove the results of Morse Theory, we have to pay two costs. Firstly we have to assume that the functional is of class C^2 and secondly all the critical points of f have to be nondegenerate, i.e. the functional f is *a Morse function.* We state these results using cohomology groups rather homology groups, because cohomology seems to be more useful in extensions to strongly indefinite functionals.

Let (A, B) be a topological pair, that is A is a topological space and B is a subspace of A and let $\mathcal{K}$ a field. For any $k \in \mathbf{N}$, $H^k(A, B; \mathcal{K})$ denotes the k-th relative cohomology group (with coefficients in $\mathcal{K}$) of the pair (A, B) (cf. [32]). Since $\mathcal{K}$ is a field, the cohomology group $H^k(A, B; \mathcal{K})$ is a vector field and its dimension $\beta^k(A, B; \mathcal{K})$ is called the k–th *Betti number* of (A, B) (with respect to $\mathcal{K}$). The Poincarè polynomial of the pair (A, B) is defined by setting

$$\mathcal{P}(A, B; \mathcal{K})(r) = \sum_{k=0}^{\infty} \beta^k(A, B; \mathcal{K}) r^k.$$

In general $\mathcal{P}$ is a formal series whose coefficients are positive cardinal numbers (possibly $+\infty$).

We state now the Morse Relations for a Morse functional, bounded from below and such that the Morse index of any critical point is finite. We deduce as a consequence the classical *Morse Inequalities* proved by Morse, which relate the numbers of critical points having index k with the k–th Betti number of the manifold. For the proof see the article of Bott [9] or [27].

Theorem 2.6 *Let $f : X \to \mathbb{R}$ be a C^2 functional defined on a complete Riemannian manifold (X, h). Assume that f is bounded from below and satisfies the Palais–Smale condition (PS) on X. Moreover assume that all the critical points of f are nondegenerate and the Morse index $m(x, f)$ of any critical point x of f is finite.*

Then for any field $\mathcal{K}$ there exists a formal series $Q(r)$, whose coefficients are positive cardinal numbers, such that

$$\sum_{x \in K(f)} r^{m(x,f)} = \mathcal{P}(X, \mathcal{K})(r) + (1 + r)Q(r). \tag{2.3}$$

Notice that under the assumption of the previous theorem, the number of critical points of the functional f is countable, because nondegenerate critical points are isolated, and the Palais–Smale condition holds on the whole manifold X.

We state now the classical Morse inequalities, whose proof is a trivial consequence of (2.3).

Theorem 2.7 *Under the assumptions of Theorem 2.6, let $\mathcal{K}$ be a field, let $k \in \mathbb{N}$ be a positive integer and denote by $\beta^k(X; \mathcal{K})$ the k–th Betti* *number of the manifold X (with respect to the field $\mathcal{K}$) and by $M(f,k)$ the number of critical points x of f such that $m(x,f)=k$.*

Then, for any $k \in \mathbb{N}$, the following Morse inequality holds:

$$M(f,k) \geq \beta^k(X;\mathcal{K}). \tag{2.4}$$

We apply now the abstract results above to the action integral of a Riemannian manifold, getting existence and multiplicity results and a Morse Theory for geodesics on a Riemannian manifold. These results were already the core of the results of Morse, Ljusternik and Schnirelmann, obtained essentially using finite dimensional reductions of the problem (see the classical book of J. Milnor [28] on Morse Theory). The infinite dimensional approach using Hilbert manifolds, gradient flows and the Palais–Smale condition was introduced by R. Palais in the celebrated paper [30].

Let $(\mathcal{M},g)$ be a complete Riemannian manifold, let p and q two points of $\mathcal{M}$ and consider the action integral $f(x)=\int_0^1 g(x(s))[\dot{x}(s),\dot{x}(s)]\mathrm{d}s$ on the manifold $\Omega^{1,2}(p,q;\mathcal{M})$. Then f is bounded from below and it satisfies the (PS) condition, see [30]. By Theorem 2.4, there exists a minimum of f, so there exists a minimal geodesic joining p and q. This is a variational proof of the geodesic connectedeness of a complete Riemannian manifold, which is usually proved as a consequence of the well known Hopf–Rinow Theorem in Riemannian Geometry. In order to obtain multiplicity results for the geodesics joining p and q, variational methods seem to be necessary. In particular Fadell and Husseini have proved [12] that the Ljusternik–Schnirelmann category $\mathrm{cat}(\Omega^{1,2}(p,q;\mathcal{M}))$ is equal to $+\infty$ whenever the manifold $\mathcal{M}$ is noncontractible into iteself. In this case, for any couple of points p and q of $\mathcal{M}$, there exists infinitely many geodesics joining p and q and there exists a sequence (x_n) of such geodesics such that the action integral $f(x_n)$ tends to $+\infty$. This result was already proved by Serre [31] in the case of a compact and simply connected manifold.

Finally, if p and q are nonconjugate (a condition which holds almost surely), the Morse Relations and the Morse inequalities hold for the geodesic joining p and q. So, the variational properties of the action integral can be completely described. In particular the Morse inequalities hold, then the number $G(p,q;k)$ of geodesics joining p and q and having index $k \in \mathbb{N}$ and the k–the Betti number $\beta^k(\Omega^{1,2}(p,q;\mathcal{M});\mathcal{K})$ are related by the formula

$$G(p,q;k) \geq \beta^k(\Omega^{1,2}(p,q;\mathcal{M});\mathcal{K}).$$

Moreover, since the infinite dimensional manifold $\Omega^{1,2}(p,q;\mathcal{M})$ is homotopically equivalent to the based loop space $\Omega(\mathcal{M})$, then their cohomology groups are isomorphic, so we have a full relation between the differential structure of the geodesics of the complete Riemannian metric g and the topological structure of the manifold $\mathcal{M}$ and the following inequality holds:

$$G(p,q;k) \geq \beta^k(\Omega(\mathcal{M});\mathcal{K}).$$

Whenever the $\mathcal{M}$ is contractible, then $\mathrm{cat}(\Omega^{1,2}(p,q;\mathcal{M})) = 1$, and the only nonnull Betti number of $\Omega(\mathcal{M})$ is $\beta^0(\Omega(\mathcal{M});\mathcal{K}) = 1$. Then both the Ljusternik–Schnirelmann and Morse Theory gives the existence of at least one geodesic joining two points p and q on $\mathcal{M}$, the minimal one. On the other hand there are situations in which the numbers of geodesics can be greater than one. Such cases are interesting in the study of the geometric causes of the so called *multiple image effect*, studied in Astrophysics to describe the *gravitational lens effect*. We cite the paper [20] for some results in these directions.

3 Variational properties of geodesics on Lorentzian manifolds

We consider now a semiriemannian manifold $(\mathcal{M},g)$ of index $\nu(g) > 0$, in particular a Lorentzian manifold. As we said in the Introduction, the geodesics for the metric g joining two points p and q on $\mathcal{M}$ are still the critical points of the action integral $f(z) = \int_0^1 g(z(s))[\dot{z}(s),\dot{z}(s)]ds$ on the manifold $\Omega^{1,2}(p,q;\mathcal{M})$, but now the variational properties of the functional f are completely different from the case of a Riemannian manifold. Now the functional f is *unbounded* both from below and from above, the Morse index $m(z,f)$ of any geodesic joining p and q is equal to $+\infty$ and finally, the functional f does not satisfy, in general, the (PS) condition. These three facts makes very difficult the problem to find a critical point for f, proving the geodesic connectedeness of a semiriemannian manifold. One finds further difficulties to prove multiplicity results or to develop a Morse Theory for the functional f.

These difficulties are not only of technical nature to develop variational arguments. Indeed, there are rather important counterexamples to the geodesic connectedeness of a Lorentzian manifold. For instance there exists Lorentzian manifolds which are geodesically complete, but not geodesically connected. We recall that a semiriemannian manifold is geodesically complete if and only if any maximal geodesic is defined on the whole real line $\mathbb{R}$. The Hopf–Rinow shows that a Riemannian manifold is geodesically complete if and only if it is complete, so any geodesically complete Riemannian manifold is geodesically connected.

Moreover, there exists a compact Lorentzian manifold which is not geodesically connected, see [26] and its references for such counterexamples. It is not still known a complete geometric theory for the geodesic connectedeness of a semiriemannian manifold, as for the Riemannian case. The only classical result is the Avez–Seifert theorem on the existence of a maximal causal geodesic joining two causally related points in a globally hyperbolic Lorentzian manifold, see [4]. Such a result was proved using a maximization argument not far from the proof of Theorem 2.4.

In the recent years the problem of the existence of geodesics (without any restriction on the causality of such geodesics) joining two points in a Lorentzian manifold has been studied by some authors using variational methods, see [26] for an almost exhaustive literature on the topic. Other results using arguments not variational in nature have been recently obtained for the class of *generalized Robertson–Walker spacetimes*, see [15]. Here we present the results obtained via variational methods and in particular we state the *Morse Inequalities* for the geodesics joining two nonconjugate points on a stationary or an orthogonal splitting Lorentzian manifold.

We introduce now the notion of *splitting Lorentzian manifold*, which is the most general class of Lorentzian manifold we shall study. Indeed, *orthogonal splitting* and *standard stationary* Lorentzian manifolds are two subclasses of it.

Definition 3.1 *A Lorentzian manifold $(\mathcal{M}, g)$ is said splitting if $\mathcal{M} = \mathcal{M}_0 \times \mathbf{R}$, where $\mathcal{M}_0$ is a smooth connected manifold, and the metric g has the following form. For any $z = (x,t) \in \mathcal{M}$ and for any $\zeta = (\xi, \tau) \in T_z\mathcal{M} = T_x\mathcal{M}_0 \times \mathbf{R}$,*

$$g(z)[\zeta,\zeta] = \langle \alpha(x,t)\xi, \xi\rangle + 2\langle \delta(x,t), \xi\rangle\tau - \beta(z)\,\tau^2, \tag{3.1}$$

where $\langle\cdot,\cdot\rangle$is a Riemannian metric on $\mathcal{M}_0$, $\alpha(x,t)$ is a positive linear operator on $T_x\mathcal{M}_0$, smoothly depending on z, $\delta(x,t)$ is a smooth vector field tangent to $\mathcal{M}_0$ and $\beta(z)$ is a smooth scalar field on $\mathcal{M}$.

The metric g is said orthogonal splitting if the vector field $\delta(x,t) = 0$, while the metric g is said standard stationary if the linear operator α, the vector field δ and the scalar field β do not depend on the variable t.

If the metric is stationary we can assume without any loss of generality that the the linear operator $\alpha(x)$ is equal to the identity map and the metric has the following form:

$$g(z)[\zeta,\zeta] = \langle\xi,\xi\rangle + 2\langle\delta(x),\xi\rangle\tau - \beta(x)\tau^2, \tag{3.2}$$

while an orthogonal splitting metric takes the form

$$g(z)[\zeta,\zeta] = \langle\alpha(x,t)\xi,\xi\rangle - \beta(x,t)\tau^2. \tag{3.3}$$

Remark 3.2 We recall a result of Geroch (see [17]) which states that any globally hyperbolic Lorentzian manifold is diffeomorphic to a splitting Lorentzian manifold. The problem on the geometric conditions which assure that a Lorentzian manifold is splitting has been widely studied (see for instance [4]).

We point out that the definition of splitting Lorentzian manifold can be given intrinsically. Indeed a Lorentzian manifold is splitting if and only if it admits a time function T on the manifold such that the timelike vector field ∇T is complete. Moreover a standard stationary Lorentzian manifold is a particular case of a *stationary Lorentzian manifold*, i.e. a Lorentzian manifold admitting a global timelike Killing field. We shall focus our attention on orthogonal splitting and standard stationary Lorentzian manifold. For the most general cases of splitting Lorentzian manifolds see [19] and for stationary Lorentzian manifolds see [23].

We introduce now a notion of regularity for orthogonal splitting and standard stationary Lorentzian manifolds. They require the completeness of the Riemannian factor of the manifold and some growth condition at the spacelike infinity of the coefficients of the metric.

Definition 3.3 *A standard stationary Lorentzian manifold* $(\mathcal{M},g)$, $\mathcal{M} = \mathcal{M}_0\times\mathbb{R}$, *is said to be regular if it satisfies the following assumptions:*

A$_1$) *The Riemannian manifold* $(\mathcal{M}_0,\langle\cdot,\cdot\rangle)$ *is complete;*

A$_2$) *There exists two positive constants* $0<\nu\le M$ *such that for any* $z\in\mathcal{M}$,

$$\nu\le\beta(z)\le M;$$

A$_3$) $\sup\{\langle\delta(x),\delta(x)\rangle_0, x\in\mathcal{M}_0\}<+\infty.$

Definition 3.4 *An orthogonal splitting Lorentzian manifold* $(\mathcal{M},g)$ *is said to be regular if it satisfies the following assumptions:*

B$_1$) *The Riemannian manifold* $(\mathcal{M}_0, \langle\cdot,\cdot\rangle)$ *complete;*

B$_2$) *There exists* $\lambda > 0$, *such that for any* $z = (x,t) \in \mathcal{M}$, *and for any* $\xi \in T_x\mathcal{M}_0$,

$$\langle \alpha(z)\xi, \xi\rangle \geq \lambda\langle\xi,\xi\rangle;$$

B$_3$) *there exists two positive constants* $0 < \nu \leq M$ *such that for any* $z \in \mathcal{M}$,

$$\nu \leq \beta(z) \leq M;$$

B$_4$) *there exists* $L > 0$ *such that for any* $z \in \mathcal{M}$,

$$|\langle\alpha_t(z)\xi,\xi\rangle| \leq L, \qquad |\beta_t(z)| \leq L,$$

where α_t *and* β_t *denote respectively the partial derivative, with respect to* t, *of* α *and* β;

B$_5$)

$$\begin{gathered}\mathrm{limsup}_{t\to+\infty}\langle\alpha_t(x,t)\xi,\xi\rangle \leq 0,\\ \mathrm{liminf}_{t\to-\infty}\langle\alpha_t(x,t)\xi,\xi\rangle \geq 0,\end{gathered}$$

uniformly in $x \in \mathcal{M}_0$ *and* $\xi \in T_x\mathcal{M}_0$, $\langle\xi,\xi\rangle = 1$.

The first main result about regular orthogonal splitting or regular standard stationary Lorentzian manifold is the following theorem.

Theorem 3.5 *Let* $(\mathcal{M}, g)$ *be a regular standard stationary Lorentzian manifold or a regular orthogonal splitting Lorentzian manifold.*

Then $(\mathcal{M}, g)$ *is geodesically connected, i.e. for any couple of points* $p = (x_0, t_0)$ *and* $q = (x_1, t_1) \in \mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$, *there exists a geodesic joining* p *and* q.

Theorem 3.5 has been proved in [5, 18] for standard stationary Lorentzian manifolds and in [6] for orthogonal splitting manifolds. The proofs of the results above are of variational nature. It is shown that if the Lorentzian manifold is regular standard stationary or regular orthogonal splitting, the action integral (1.2) has a critical point, which is an infinite dimensional saddle point. In [18] a variational principle for the geodesics on a stationary Lorentzian manifold is proved. Such a principle, obtained using a global saddle point reduction and the implicit function theorem, allows to characterize the spatial part of a geodesic as a critical point of a new functional J defined on the manifold $\Omega^{1,2}(x_0, x_1; \mathcal{M}_0)$. If the stationary Lorentzian manifold is regular, the functional J is bounded from below and satisfy the (PS) condition. So the critical point theory developed in Sect. 2 permits to obtain the geodesic connectedeness of regular stationary Lorentzian manifold. If the coefficients of the Lorentzian metric depend on the time variable, no global reduction permits to reduce the problem to a Riemannian one and genuine methods for strongly indefinite functionals have to be applied. In particular the geodesic connectedeness of a regular orthogonal splitting Lorentzian metric is obtained applying to the action integral (1.2) the so called *Rabinowitz saddle point theorem.* Indeed, the assumption for an orthogonal splitting Lorentzian metric to be regular imply that the action integral has the *saddle point geometry.* One finds further difficulties because in this case the action integral does not satisfy the (PS) condition and the results are obtained using a perturbation argument. We refer to [6, 26] for the details.

If the topology of the regular Lorentzian manifold is nontrivial, we have the following multiplicity result.

Theorem 3.6 *Let $(\mathcal{M}, g)$ be a regular standard stationary Lorentzian manifold or a regular orthogonal splitting Lorentzian manifold. Assume that the manifold $\mathcal{M}$ is noncontractible into itself.*

Then, for any couple of points $p = (x_0, t_0)$ and $q = (x_1, t_1) \in \mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$, there exist infinitely many geodesics joining them. Moreover there exists a sequence (z_m) of such geodesics such that $f(z_m) \to +\infty$

The previous results have been proved in [5, 18] for a regular stationary Lorentzian manifold and in [19] for an orthogonal splitting Lorentzian manifold. So, for these two classes of regular Lorentzian manifolds there hold the analogous existence and multiplicity results that for Riemannian manifolds. The proof for the stationary case is obtained applying the Ljusternik-Schnirelmann Theory to the functional J obtained by the global saddle point reduction described above. The case of regular, orthogonal splitting manifolds is more delicate. In order to get multiplicity results for critical points of strongly indefinite functionals, a new topological invariant, the *relative category* was introduced by some authors [16]. Fadell and Husseini [13] have shown that for a noncontractible manifold, this invariant takes arbitrarily large values on subsets of its based loop spaces. Then, their results can be applied to obtain multiplicity of geodesics, see [19].

Assume now that the points p and q are nondegenerate, so that the action integral is again a Morse function as for the Riemannian case. In order to develop a Morse Theory for the geodesics joining p and q, some further difficulty immediately arises. Indeed, since any geodesic joining p and q has Morse index equal to $+\infty$, the statements of Theorem 2.6

and of Theorem 2.7 do not make any sense.

Morse Theory for strongly indefinite functionals has been the object of several studies in the last years. In particular the definition of the index for a critical point of a functional such that the second differential at the critical point is a Fredholm operator of index 0 has been studied by many authors [1, 2, 3, 10, 11, 33].

We define here a *relative index* of a class of bilinear form on a Hibert space. Let H be a real Hilbert and let $a\colon H \times H \longrightarrow \mathbb{R}$ be a continuous, symmetric, nondegenerate bilinear form on H such that $a = a_0 + k$, where a_0 is another continuous, symmetric, nondegenerate bilinear form on H and k is a compact form. Let A and A_0 be the linear isomorphisms on H induced by the forms a and a_0 and denote by $V^+(A)$ and $V^-(A)$ the maximal A–invariant subspaces on which A is respectively positive definite and negative definite. Analogously, the A_0–invariant subspaces $V^+(A_0)$ and $V^-(A_0)$ on which A_0 is positive definite and negative definite are defined. Then the index of a relatively to a_0, denoted by $j(a, a_0)$ is defined as follows:

$$j(a, a_0) = \dim(V^-(A) \cap V^+(A_0)) - \dim(V^+(A) \cap (V^-(A_0))). \tag{3.4}$$

The relative index $j(a, a_0)$ is a relative integer number and coincides with the Morse index of the form a (i.e. the maximal dimension of a subspace where a is negative definite) if a_0 is positive definite.

Now, let $(\mathcal{M}, g)$ be an arbitrary semiriemannian manifold and let p and q two nonconjugate points of $\mathcal{M}$. Let z be a geodesic joining p and q, then it is a nondegenerate critical point of the action integral $f(z) = \int_0^1 g(z(s))[\dot{z}(s), \dot{z}(s)]ds$. The second derivative $f''(z)\colon T_z\Omega^{1,2}(p, q; \mathcal{M}) \times T_z\Omega^{1,2}(p, q; \mathcal{M}) \to \mathbb{R}$ of the action integral at z is given by (see (1.3))

$$f''(z)[\zeta, \zeta'] = \int_0^1 g(z)[D_s\zeta, D_s\zeta']ds - \int_0^1 g(z)[R(\zeta, \dot{z})\dot{z}, \zeta']ds, \tag{3.5}$$

Then $f''(z)$ defines a Fredholm operator of index 0 on the tangent space $T_z\Omega^{1,2}(p, q; \mathcal{M})$. Indeed we have that that $f''(z) = a_0(z) + k(z)$, where $a_0(z) = \int_0^1 g(z)[D_s\zeta, D_s\zeta']ds$ is nondegenerate and $k(z) = -\int_0^1 g(z)[R(\zeta, \dot{z})\dot{z}, \zeta']ds$ defines a compact linear operator on $T_z\Omega^{1,2}(p, q; \mathcal{M})$.

We can define the *relative index* for a geodesic joining two points on a semiriemannian manifold, see [2]

Definition 3.7 *Let $(\mathcal{M}, g)$ be a semiriemannian manifold and let $z\colon [0,1] \to \mathcal{M}$ be a geodesic joining $p = z(0)$ and $q = z(1)$. The relative index $j(z)$ of the geodesic z is defined setting*

$$j(z, f) = j(f''(z), a_0(z)), \tag{3.6}$$

where $f''(z)$ is the second differential evaluated at the geodesic z of the action integral f.

By the abstract definition of the relative index, it follows that the index for a semiriemannian geodesic z is a relative integer $j(z) \in \mathbb{Z}$, so it could be negative. If the metric g is Riemannian, then the bilinear form $a_0(z)$ is positive definite and so the relative index $j(z)$ reduces to the classical Morse index $m(x, f)$ of the geodesic. If the metric is Lorentzian, the spectral properties of $f''(z)$ are partially known and one can conclude that if z is a *causal geodesic*, then $j(z) \in \mathbb{N}$, see [4, 7]. It would be interesting to give examples or describe completely the (spacelike) geodesics with negative index.

We state now the Morse inequalities for Lorentzian geodesics on regular standard stationary or regular orthogonal splitting Lorentzian manifolds, see [2] for the proof.

Theorem 3.8 *Let $(\mathcal{M}, g)$ be a regular standard stationary or a regular orthogonal splitting Lorentzian manifold and let p and q two nonconjugate points of $\mathcal{M}$. Moreover, for any $k \in \mathbb{N}$, let $G(p, q; k)$ be the number of geodesics z for the metric g, joining p and q and such that the relative index $j(z, f)$ is equal to k.*

Then, for any $k \in \mathbb{N}$ and for any field $\mathcal{K}$ we have

$$G(p, q; k) \geq \beta^k(\Omega^{1,2}(p, q; \mathcal{M})), \tag{3.7}$$

where $\beta^k(\Omega^{1,2}(p, q; \mathcal{M}))$ is the k–th Betti number of the manifold $\Omega^{1,2}(p, q; \mathcal{M})$ with respect to the field $\mathcal{K}$.

Since the manifold $\Omega^{1,2}(p, q; \mathcal{M})$ is homotopically equivalent to the based loop space $\Omega(\mathcal{M})$ of the manifold $\mathcal{M}$, the Morse inequalities (3.8) can be stated as $G(p, q; k) \geq \beta^k(\Omega(\mathcal{M}))$ and this relates the differential structure of the geodesics for the Lorentzian metric g joining two nonconjugate points and the topological structure of the manifold $\mathcal{M}$.

Under the assumptions of the previous theorem, whenever the manifold $\mathcal{M}$ is noncontractible into itself, there exits a sequence of geodesics (z_m) of geodesics joining the points p and q. The topological properties of the based loop space $\Omega(\mathcal{M})$ (cf. [12]) allows to estimate the relative index $j(z_m, f)$ of such geodesics showing that $j(z_m, f) \to +\infty$ as $m \to +\infty$, see [2].

The Morse inequalities allows to estimate the number of geodesics having *positive* index in terms of the singular cohomology groups of the based loop space. If one wants to give some estimates on the number of geodesics with *negative* index, classical homological or cohomological theories (as singular homology or singular cohomology) do not work. From the variational point of view, the classical theories are not affected by attaching an infinite dimensional cell, because the infinite dimensional unit sphere is contractible. The study of cohomology theories for strongly indefinite functionals is an active field of interaction between Algebraic Topology and the Critical Point Theory (see [1, 33] for some recent result).

The proof of Theorem 3.8 is a consequence on some abstract results on a Morse Theory for a class of strongly indefinite functionals, developed in [2]. The action integral for a standard stationary or a orthogonal splitting Lorentzian manifold belongs to such a class of functionals. The case of a regular standard stationary follows by a direct argument, because the action integral (1.2) of a regular standard stationary Lorentzian manifold

$$f(z) = f(x,t) = \int_0^1 \left[\langle \dot{x}, \dot{x}\rangle + \langle \delta(x), \dot{x}\rangle \dot{t} - \beta(x)\dot{t}^2\right] ds$$

satisfies a variant of the (PS) condition, the so called $(\mathrm{PS})^*$ condition, which is very useful for the study of strongly indefinite functionals. The proof in the case of regular orthogonal splitting Lorentzian manifolds is quite involved. Indeed, in this case the action integral (1.2),

$$f(z) = f(x,t) = \int_0^1 \left[\langle \alpha(x,t)\dot{x}, \dot{x}\rangle - \beta(x,t)\dot{t}^2\right] ds$$

does not satisfy the (PS) or the $(\mathrm{PS})^*$ condition, see for instance [26]. In this case an approximation scheme with a family of functionals converging in the bounded sets to the functional f and satisfying the $(\mathrm{PS})^*$ condition, allows to pass to the limit in the approximation and to prove the Morse inequalities for f as limit if the Morse inequalities of the approximating functionals. We refer to [2] for the details.

While the classical Morse Inequalities hold both for regular standard stationary and orthogonal splitting Lorentzian manifolds, the *Morse Relations* as in Theorem 2.6 hold for regular standard stationary Lorentzian manifolds, but actually it is an open problem to prove them for regular orthogonal splitting Lorentzian manifolds. This is due to the fact that the lost of the $(\mathrm{PS})^*$ for the action integral f does not allow to control the growth of the infinite dimensional cohomology (introduced by Szulkin in [33]) of the sublevels f^c of the functional f, as $c \to +\infty$, see [2].

Previous results on a Morse Theory for Lorentzian geodesics have been proved by some authors. In [34] a Morse Theory for timelike geodesics joining two chronologically related fixed points in a globally hyperbolic Lorentzian manifold. In this paper the set of such timelike geodesics is related to the topology of the space of the timelike curves joining the fixed points. We would like to point out the evaluation of the topological invariants of such a space, as for instance the Betti numbers, is not known.

In some papers a Morse Theory for the lightlike geodesics joining an event with a timelike curve representing the world line of a light source is developed [14, 21, 34]. These results are obtained using a relativistic version of the Fermat principle of classical optics and it has permitted to produce a mathematical model of the *gravitational lens effect* in Astrophysics. Such results have been extended to timelike geodesics in [22].

Some kind of Morse Relations for the geodesics joining two non-conjugate points on stationary Lorentzian manifolds and on Generalized Robertson–Walker spacetimes have proved in some papers [8, 15, 24]. The results of these papers are based on a reduction argument which allows to deduce the Morse Relations by applying the classical Morse Theory to a suitable functional, bounded from below, satisfying (PS) and whose critical points have finite Morse index. However the index associated is always positive, so such Morse Relations are not obtained with the natural Morse Index $j(z, f)$ of a geodesic z. It is an open problem to understand if these results are equivalent to the results presented in this note. If the geodesic is causal, the equivalence between the theories is true.

4 Conclusion

We have presented some recent results on the variational theory for geodesics on Lorentzian manifolds and we have presented also some open problem. For metrics g of index $\nu(g)$ greater that 2, we do not know any result in these directions and much work must be done to understand the variational properties of the action integral $f(z) = \int_0^1 g(z(s))[\dot{z}(s), \dot{z}(s)]ds$.

References

[1] A. Abbondandolo, *Morse theory for Hamiltonian systems.* CRC research Notes in Mathematics **425**, Chapman and Hall UK, London (2001).

[2] A. Abbondandolo, V. Benci, D. Fortunato, A. Masiello *Morse Relations for geodesics on Lorentzian manifolds*, preprint.

[3] H. Amann and E. Zehnder, *Nontrivial solutions for a class of nonresonant problems and applications to nonlinear differential equations*, Ann. Scuola Normale Superiore Cl. Sci. (4), **8**, 539–603, (1980).

[4] J. K. Beem, P. H. Ehrlich and K. L. Easley, *Global Lorentzian Geometry*, Marcel Dekker Inc., New York (1996).

[5] V. Benci and D. Fortunato, *On the existence of infinitely many geodesics in space–time manifolds*, Adv. Math, **105**, 1–25, (1994).

[6] V. Benci, D. Fortunato and A. Masiello, *On the geodesic connectedeness of Lorentzian manifolds*, Math. Z, **217**, 73–93, (1994).

[7] V. Benci, F. Giannoni and A. Masiello, *Some properties of the spectral flow in semiriemannian geometry*, J. Geom. Phys, **27**, 267–280, (1998).

[8] V. Benci and A. Masiello, *A Morse index for geodesics in static Lorentz manifolds*, Math. Ann, **293**, 433-442, (1992).

[9] R. Bott, *Morse Theory old and new*, Bull. Am. Math. Soc, **7**, 331-358, (1982).

[10] K. C. Chang, J. Q. Liu and M. J. Liu, *Nontrivial periodic solutions for strong resonance Hamiltonian systems*, Ann. Inst. H. Poincaré, Analyse Nonlineaire, **14**, 103–117, (1997).

[11] C. Conley and E. Zehnder, *Morse type index theory for flows and periodic solutions for Hamiltonian systems*, Comm. Pure Appl. Math, **37**, 207–253, (1984).

[12] E. Fadell and S. Husseini, *Category of loop spaces of open subsets of Euclidean spaces*, Nonlinear Anal. T.M.A, **17**, 1153–1161, (1991).

[13] E. FADELL AND S. HUSSEINI, *Relative category, products and co-products*, Rend. Sem. Mat. Fis. Univ. Milano, **LXIV**, 99–117, (1994)

[14] D. FORTUNATO, F. GIANNONI AND A. MASIELLO, *A Fermat principle for stationary space–times with applications to light rays*, J. Geom. Phys **15**, 159–188, (1995).

[15] J. L. FLORES AND M. SANCHEZ, *Geodesic connectedeness and conjugate points in GRW space–times*, J. Geom. Phys, **36**, 285–314, (2000).

[16] G. FOURNIER AND M. WILLEM, *Relative category and the calculus of variations*, IN *Variational Problems*, H. BERESTICKY, J. M. CORON AND I. EKELAND ED, BIRKHAUSER, BASEL, 95-104, (1990).

[17] R. GEROCH, *Domains of dependence*, J. Math. Phys, **11**, 437-439, (1970).

[18] F. GIANNONI AND A. MASIELLO, *On the existence of geodesics in stationary Lorentz manifolds with convex boundary*, J. Funct. Anal, **101**, 340-369, (1991).

[19] F. GIANNONI AND A. MASIELLO, *Geodesics on product Lorentzian manifolds*, Ann. Inst. Henri Poincaré Analyse Nonlineaire, **12**, 27-60, (1995).

[20] F. GIANNONI, A. MASIELLO AND P. PICCIONE, *Convexity and the finiteness of geodesics. Applications to the multiple image effect*, Class. Quant. Grav, **69**, 731–748, (1999).

[21] F. GIANNONI, A. MASIELLO AND P. PICCIONE, *The Fermat principle in General Relativity and applications*, J. Math. Phys, **43**, 563–596, (2002).

[22] F. GIANNONI, A. MASIELLO AND P. PICCIONE, *A Morse Theory for massive particles and photons in General Relativity*, J. Geom. Phys, **35**, 1–34, (2000).

[23] F. GIANNONI AND P. PICCIONE *An intrinsic approach to the geodesical connectedenss of stationary Lorentzian manifolds* , Comm. Anal. Geom, **7**, 157–197, (1999).

[24] F.GIANNONI, A.MASIELLO, P. PICCIONE AND D. TAUSK, *A Generalized index theorem for Morse-Sturm systems and applications to semi-riemannian geometry*, Asian J. Math, **3**, 441–472, (2001).

[25] W. KLINGENBERG, *Riemannian Geometry. 2nd edition*, W. De Gruyter, Berlin (1995).

[26] A. MASIELLO, *Variational Methods in Lorentzian Geometry* , Pitman Research Notes in Mathematics **309**, Longman, London (1994).

[27] J. MAWHIN, M. WILLEM, *Critical Point Theory and Hamiltonian Systems*, Springer Verlag, Berlin (1989).

[28] J. MILNOR, *Morse Theory.* Ann. of Math. Studies **51**, Princeton Univ. Press, Princeton (1963).

[29] B. O'NEILL, *Semiriemannian Geometry with applications to Relativity*, Academic Press, New York (1983).

[30] R. PALAIS *Morse Theory on Hilbert manifolds*, Topology, **2**, 299-340, (1963).

[31] J. P. SERRE, *Homologie singuliere des espaces fibres*, Ann. of Math, **54**, 425-505, (1951).

[32] E. H. SPANIER, *Algebraic Topology*, Mc Graw Hill, New York (1966).

[33] A. SZULKIN, *Cohomology and Morse Theory for strongly indefinite functionals*, Math. Z, **209**, 375-418, (1992).

[34] K. UHLENBECK, *A Morse Theory for geodesics on Lorentz manifolds*, Topology, **14**, 69-90, (1975).

Proceedings of the meeting
Lorentzian Geometry-Benalmádena 2001
Benalmádena, Málaga, Spain
Pub. de la RSME, Vol. 5 (2003), 205-214

A uniqueness result for Willmore surfaces in the Minkowski space

José A. Pastor[1]

[1] *Departamento de Matemáticas, Universidad de Murcia, 30100 Espinardo, Murcia, Spain*

Abstract

We prove that a Willmore surface in the Minkowski space with spherical boundary and which intersects the plane of the circle in a constant angle is a hyperbolic cap or a flat disc.

1 Introduction

On the last years, there has been some great efforts trying to determine the compact surfaces satisfying a certain geometric condition in the Euclidean space and having their boundary in a round circle. The surfaces which satisfy such geometric condition – for example, constant mean curvature surfaces or Willmore surfaces – admit an alternative characterization as solutions of a variational problem. Therefore, the problem mentioned above may be reformulated in the following terms: *does a solution of a variational problem necessarily inherit the symmetries of the boundary?*

The same situation can be considered in the Minkowski space for spacelike compact surfaces with boundary. The author, jointly with Alías and López, gave a uniqueness result for constant mean curvature surfaces in the Minkowski space with circular boundary (see [1]). In this paper, we consider Willmore spacelike compact surfaces with circular boundary and we prove the following uniqueness result:

Theorem 1.1 *Let $\psi : \Sigma \to \mathbb{L}^3$ be a compact spacelike Willmore surface in the Minkoski space with circular boundary. Assume that the surface intersects the plane of the circle in a constant angle. Then, the image $\psi(\Sigma)$ is a hyperbolic cap or a flat disc.*

The corresponding problem for the Euclidean space has been solved by Palmer in [5]. Our approach here is similar but there exist some differences: for instance, our result is true independently of the surface topology. Moreover, we present here a proof based in vector fields and the divergence theorem in contrast with the forms and the flux formula given in Palmer's work.

2 Preliminaries

Let us denote by $\mathbb{L}^3$ the three-dimensional Minkowski space, that is, the real vector space $\mathbb{R}^3$ endowed with the Lorentzian metric

$$\langle , \rangle = (\mathrm{d}x_1)^2 + (\mathrm{d}x_2)^2 - (\mathrm{d}x_3)^2,$$

where (x_1, x_2, x_3) are the canonical coordinates in $\mathbb{R}^3$. A smooth immersion $\psi : \Sigma \to \mathbb{L}^3$ of a 2-dimensional connected manifold Σ is said to be a *spacelike surface* if the induced metric via ψ is a Riemannian metric on Σ, which, as usual, is also denoted by $\langle , \rangle$. If we denote by $\{e_1, e_2, e_3\}$ the canonical basis in $\mathbb{L}^3$, we must observe that e_3 is a unit timelike vector field globally defined on $\mathbb{L}^3$ which determines a time-orientation on $\mathbb{L}^3$. Thus, we can choose a unique unit normal vector field N on Σ which is a future-directed timelike vector in $\mathbb{L}^3$, and hence we may assume that Σ is oriented by N.

Next, we will denote by ∇^0 and ∇^Σ the Levi-Civita connections of $\mathbb{L}^3$ and Σ, respectively. Let A stand for the Weingarten endomorphism associated to N. Then the Gauss and Weingarten formulas for the surface Σ are written respectively as

$$\nabla^0_X Y = \nabla^\Sigma_X Y - \langle A(X), Y \rangle N,$$

and

$$A(X) = -\nabla^0_X N,$$

for all tangent vector fields X, Y to Σ. The mean curvature function on Σ is defined by $H = -(1/2)\mathrm{trace}A$ – hence the mean curvature vector field is given by $\overrightarrow{H} = HN$. Let K be the Gaussian curvature of Σ. The Gauss equation for Σ in $\mathbb{L}^3$ is given by

$$K = -\det A.$$

In this context, we will say that a surface $\psi : \Sigma \to \mathbb{L}^3$ is a *Willmore surface* if it is a critical point of the functional

$$W(\psi) = \int_\Sigma \left(H^2 + K\right) \mathrm{d}A$$

with respect to compactly supported variations of the surface. Here $\mathrm{d}A$ stands for the area element of Σ with respect to the induced metric and the chosen orientation.

Throughout this paper we will deal with *compact* spacelike surfaces immersed in $\mathbb{L}^3$. Let us remark that there exists no closed spacelike surface in $\mathbb{L}^3$. In order to see this, let $a \in \mathbb{L}^3$ be a fixed arbitrary vector, and consider the height function $\langle a, x\rangle$ defined on the spacelike surface Σ. The gradient – in the induced metric – of $\langle a, x\rangle$ is

$$\nabla\langle a, x\rangle = a^{\mathrm{T}} = a + \langle a, N\rangle N,$$

where a^{T} is tangent to Σ, so that

$$|\nabla\langle a, x\rangle|^2 = \langle a, a\rangle + \langle a, N\rangle^2 \geq \langle a, a\rangle.$$

In particular, when a is spacelike the height function has no critical points in Σ, so that Σ cannot be closed. Therefore, every compact spacelike surface Σ necessarily has non-empty boundary ∂M. As usual, if Γ is a simple closed curve in $\mathbb{L}^3$, a spacelike surface $\psi : \Sigma \to \mathbb{L}^3$ is said to be a surface *with boundary* Γ if the restriction of the immersion ψ to the boundary $\partial\Sigma$ is a diffeomorphism onto Γ.

In what follows, $\psi : \Sigma \to \mathbb{L}^3$ will be a compact spacelike surface with boundary $\psi(\partial\Sigma) = \Gamma$, and we will consider Σ oriented by a unit future-directed timelike normal vector field N. The orientation of Σ induces a natural orientation on the curve $\partial\Sigma$ as follows: a tangent vector $v \in T_p(\partial\Sigma)$ is positively oriented if and only if $\{u, v\}$ is a positively oriented basis for $T_p\Sigma$, whenever $u \in T_p\Sigma$ is outward pointing. We will denote by ν the outward pointing unit conormal vector field along $\partial\Sigma$, whereas τ will denote the unit tangent vector field to the boundary such that $\{\nu, \tau\}$ is a positively oriented orthonormal frame. Thus, in each point of the boundary, we can consider a positively oriented orthonormal basis in $\mathbb{L}^3$ given by $\{\nu, \tau, N\}$.

Throughout this work we will assume that the boundary $\Gamma = \psi(\partial\Sigma)$ is contained in a fixed plane Π of $\mathbb{L}^3$. Since Γ is a closed curve, it follows that the plane Π is spacelike (we can assume without loss of generality that Π is the plane $x_3 = 0$). Moreover, in this particular case we have the following result (see, for example, [1]) concerning the topology of the surface.

Lemma 2.1 *Any compact spacelike surface $\psi : \Sigma \to \mathbb{L}^3$ with boundary a planar simple closed curve Γ is a spacelike graph over the planar domain bounded by Γ. Therefore, the surface is a topological disc.*

3 Proof of the Theorem

An important construction concerning Willmore surfaces is the *conformal Gauss map.* Let $\mathbb{H}^4_1$ denote the anti deSitter space given by

$$\mathbb{H}^4_1 = \{Y = (Y_0, Y_1, Y_2, Y_3, Y_4) \in \mathbb{E}^5_2 \quad | \quad Y \cdot Y = -1\},$$

where $\cdot$ stands for the natural product in $\mathbb{E}^5_2$ with signature $(+,+,+,-,-)$. If $\psi : \Sigma \to \mathbb{L}^3$ is a spacelike surface, then the conformal Gauss map $Y : \Sigma \to \mathbb{H}^4_1$ assigns to each $p \in \Sigma$ a certain oriented 2-sphere (see [2] for the details). The map Y can be expressed in terms of canonical coordinates on $\mathbb{E}^5_2$ as

$$Y(p) = -H(p)\left(\frac{\langle\psi(p),\psi(p)\rangle - 1}{2}, \psi(p), \frac{\langle\psi(p),\psi(p)\rangle + 1}{2}\right)$$

$$+ \left(\langle\psi(p), N(p)\rangle, N(p), \langle\psi(p), N(p)\rangle\right).$$

It can be proved that, except for umbilical points, Y defines a conformal spacelike immersion from Σ into $\mathbb{H}^4_1$. Moreover, it was shown in [2] that ψ is a Willmore surface if and only if Y defines a zero mean curvature immersion on Σ minus the umbilic set. In this case we have that the i-th (metric) component of the map Y on the canonical basis $\{e_i\}_{0\le i\le 4}$, which we will denote by $Y_i = Y \cdot e_i$, satisfies the equation

$$\triangle Y_i - 2(H^2 + K)Y_i = 0 \qquad (3.1)$$

where $0 \leq i \leq 4$ and $\triangle$ stands for the Laplacian on Σ with respect to the induced metric. Now, let us consider the following vector fields over Σ,

$$X_{ij} = Y_i \nabla Y_j - Y_j \nabla Y_i$$

where $0 \leq i < j \leq 4$. Since ψ is a Willmore immersion, it is clear from (3.1) that X_{ij} are divergence free, that is, $\mathrm{div} X_{ij} = 0$ where div stands for the divergence operator on Σ with respect to the induced metric. Now, applying the divergence theorem we have

$$0 = \int_\Sigma \mathrm{div} X_{ij} \mathrm{d}A = \int_{\partial\Sigma} \langle X_{ij}, \nu \rangle \mathrm{d}s \tag{3.2}$$

where $\mathrm{d}s$ is the induced line element on $\partial\Sigma$.

Our aim is to develop the formula (3.2) for certain particular cases. For example, let us begin with $i = 0, j = 4$. In that case we have

$$\begin{aligned} Y_0 &= -H\frac{\psi^2 - 1}{2} + \langle \psi, N \rangle, \\ \nabla Y_0 &= -H\psi^{\mathrm{T}} - \nabla H \frac{\psi^2 - 1}{2} - A(\psi^{\mathrm{T}}), \\ Y_4 &= H\frac{\psi^2 + 1}{2} - \langle \psi, N \rangle, \\ \nabla Y_4 &= H\psi^{\mathrm{T}} + \nabla H \frac{\psi^2 + 1}{2} + A(\psi^{\mathrm{T}}). \end{aligned}$$

From now on, let us assume that Γ is a unit circle in the x_3-plane, that is, $\langle \psi, e_3 \rangle = 0$ and $\langle \psi, \psi \rangle = 1$. With such hypothesis we have

$$X_{04} = \langle \psi, N \rangle \nabla H + H^2 \psi^{\mathrm{T}} + HA(\psi^{\mathrm{T}}). \tag{3.3}$$

Let us observe that the vector position ψ is a unit vector such that $\langle \psi, \tau \rangle = 0$. Moreover, for every boundary point $p \in \partial\Sigma$, we have that $\{\tau(p), \psi(p), e_3\}$ is a basis for the Minkowski space satisfying

$$\begin{cases} \tau &= \tau, \\ \psi &= \langle \nu, \psi \rangle \nu - \langle N, \psi \rangle N, \\ e_3 &= \langle \nu, e_3 \rangle \nu - \langle N, e_3 \rangle N. \end{cases} \tag{3.4}$$

Along the boundary, we can define a hyperbolic angle function β given by the expression $\cosh\beta = -\langle N, e_3 \rangle$. Then, equation (3.4) may be rewrited as

$$\begin{cases} \psi & = \cosh\beta\nu + \sinh\beta N, \\ e_3 & = \sinh\beta\nu + \cosh\beta N. \end{cases} \tag{3.5}$$

In order to simplify the equation (3.3) we may write

$$\psi^{\mathrm{T}} = \cosh\beta\nu$$

whereas

$$\langle\psi, N\rangle = -\sinh\beta.$$

Then

$$\begin{aligned} \langle X_{04}, \nu\rangle =& \langle -\sinh\beta\nabla H, \nu\rangle + H^2\cosh\beta + H\langle A(\cosh\beta\nu), \nu\rangle = \\ =& -\sinh\beta\langle\nabla H, \nu\rangle + H^2\cosh\beta + H\cosh\beta\langle A(\nu), \nu\rangle. \end{aligned}$$

Taking into account that $-2H = \mathrm{trace}A = \langle A(\nu), \nu\rangle + \langle A(\tau), \tau\rangle$ we can write

$$\langle X_{04}, \nu\rangle = -\sinh\beta\langle\nabla H, \nu\rangle - H^2\cosh\beta - H\cosh\beta\langle A(\tau), \tau\rangle.$$

If we define the normal curvature of the curve $\partial\Sigma$ as $\kappa_n = -\langle A(\tau), \tau\rangle$ and we apply the divergence theorem we get finally

$$\begin{aligned} 0 &= \int_{\partial\Sigma}\langle X_{04}, \nu\rangle \mathrm{d}s = \\ &= \int_{\partial\Sigma}\left(-\sinh\beta\langle\nabla H, \nu\rangle - H\cosh\beta(H - \kappa_n)\right)\mathrm{d}s. \end{aligned} \tag{3.6}$$

Let us compute an analogous formula for the indexes $i = 0, j = 3$. In this case, for Y_3 we have the expression

$$Y_3 = -H\langle\psi, e_3\rangle + \langle N, e_3\rangle$$

whereas for its gradient we get

$$\nabla Y_3 = -He_3^{\mathrm{T}} - \langle\psi, e_3\rangle\nabla H - A(e_3^{\mathrm{T}}).$$

Remember that, since Γ is a planar curve contained in the x_3-plane, we have that $\langle\psi, e_3\rangle = 0$. Moreover, from (3.5) we have $e_3^{\mathrm{T}} = \sinh\beta\nu$ and $\psi^{\mathrm{T}} = \cosh\beta\nu$, so we get

$$\begin{aligned} X_{03} =& H\sinh\beta e_3^{\mathrm{T}} + \sinh\beta A(e_3^{\mathrm{T}}) - \cosh\beta H\psi^{\mathrm{T}} - \\ & - \cosh\beta A(\psi^{\mathrm{T}}) = -H\nu - A(\nu). \end{aligned} \tag{3.7}$$

Applying the divergence theorem we have

$$\begin{aligned} 0 &= \int_{\partial\Sigma} \langle X_{03}, \nu\rangle \mathrm{d}s = \int_{\partial\Sigma} \left(\langle H\nu + A(\nu), \nu\rangle\right) \mathrm{d}s = \\ &= \int_{\partial\Sigma} \left(H + \langle A(\nu), \nu\rangle\right) \mathrm{d}s = \int_{\partial\Sigma} (\kappa_n - H)\, \mathrm{d}s. \end{aligned} \tag{3.8}$$

Finally, let us consider now the case $i = 3, j = 4$. Then, we have

$$Y_3 \nabla Y_4 = (-\cosh\beta)(H\psi^{\mathrm{T}} + \nabla H + A(\psi^{\mathrm{T}}))$$

whereas

$$Y_4 \nabla Y_3 = -(H + \sinh\beta)(He_3^{\mathrm{T}} + A(e_3^{\mathrm{T}})).$$

Finally

$$X_{34} = -H\nu - A(\nu) - \cosh\beta \nabla H + \sinh\beta H(H\nu + A(\nu)).$$

The divergence theorem for this vector field give us

$$0 = \int_{\partial\Sigma} \langle X_{34}, \nu\rangle \mathrm{d}s = \int_{\partial\Sigma} \left(H\sinh\beta(\kappa_n - H) - \cosh\beta\langle \nabla H, \nu\rangle\right) \mathrm{d}s. \tag{3.9}$$

When the angle along the boundary curve is constant we are able to obtain some interesting integral formulas involving the equations (3.6), (3.8) and (3.9). Therefore, let us assume that the function β is constant. The first integral formula can be obtained multiplying by $\cosh\beta$ the equation (3.6) and subtracting $\sinh\beta$ times equation (3.9). This give us

$$0 = \int_{\partial\Sigma} \left(H(\kappa_n - H)\right) \mathrm{d}s. \tag{3.10}$$

On the other hand, if we multiply by $\sinh\beta$ the equation (3.6) and subtract $\cosh\beta$ times equation (3.9) we get

$$0 = \int_{\partial\Sigma} \langle \nabla H, \nu\rangle \mathrm{d}s. \tag{3.11}$$

Now, let us denote by "prime" the differentiation with respect to arc length on the boundary $\partial\Sigma$. Since Γ is a circle we have that $\psi' = \tau$ and $\psi'' = -\psi$. Then we get

$$\kappa_n = -\langle A(\tau), \tau\rangle = \langle \nabla^0_\tau N, \tau\rangle = \langle N', \psi'\rangle = -\langle N, \psi''\rangle = \langle N, \psi\rangle = -\sinh\beta.$$

So, we have that κ_n is constant along the boundary. Using a classical theorem by Joachimsthal adapted to the Minkowski space we have that, since the angle between the surface Σ and the plane $x_3 = 0$ is constant along the boundary Γ, the boundary is a line of curvature. Thus, let $\kappa_1 = -\kappa_n$ and $\kappa_2 = -2H + \kappa_n$ denote the principal curvatures of Σ along $\partial\Sigma$. Then (3.8) and (3.10) yield

$$0 = \int_{\partial\Sigma} (\kappa_1 - \kappa_2)\,\mathrm{d}s \tag{3.12}$$

and

$$0 = \int_{\partial\Sigma} \left(\kappa_1^2 - \kappa_2^2\right)\mathrm{d}s. \tag{3.13}$$

Applying Hölder's inequality, we get for $i = 1, 2$

$$\left|\int_{\partial\Sigma} \kappa_i \mathrm{d}s\right| \leq \int_{\partial\Sigma} |\kappa_i| \mathrm{d}s \leq \left(\int_{\partial\Sigma} \kappa_i^2 \mathrm{d}s\right)^{1/2} (2\pi)^{1/2}. \tag{3.14}$$

Let us observe now that, since κ_1 is constant, we have equality in (3.14) for $i = 1$. Using equations (3.12) and (3.13) it is easy to conclude that κ_2 is also a constant. Thus we get that H is constant along the boundary and every boundary point is umbilic.

From now on, the proof follows an argument given by Palmer in [5]. Nevertheless, we include it here for the sake of completeness. Let II_Y denote the second fundamental form of the immersion Y. Let us consider the 4-form

$$Q := II_Y^{(4,0)}.$$

This form was introduced by Bryant in [3] when the conformal Gauss map was considered for the Riemannian space forms. In this Lorentzian context, Q verifies the same properties as in the Riemannian one: it defines a holomorphic quartic differential on any Willmore surface which is given locally in terms of a complex coordinate on Σ by $Q = q\mathrm{d}z^4$ where (see [4])

$$q = \begin{cases} (\phi^2/4)(H^2 + \triangle \log \phi), & \text{if } \phi \neq 0, \\ \\ \phi_z H_z, & \text{if } \phi = 0. \end{cases}$$

Here, $\phi \mathrm{d}z^2$ is the Hopf differential which is the $(2,0)$ part of the second fundamental form of ψ. In this complex coordinate, the Codazzi equation on Σ takes the form

$$\phi_{\bar{z}} = e^{\rho} H_z \tag{3.15}$$

where $e^{\rho}\mathrm{d}z^2$ is the local expression of the metric. It is a well known fact that the zeros of ϕ correspond to umbilics on Σ. Thus, we have that ϕ vanishes on $\partial\Sigma$. Since the surface Σ is a topological disc, we may parametrize Σ globally by the complex coordinate $z = re^{i\theta}$. Therefore, since ϕ along the boundary is zero we get

$$0 = \frac{\partial \phi}{\partial \theta} = i(z\phi_z - \overline{z}\phi_{\overline{z}}). \tag{3.16}$$

On the other hand, since H is constant along the boundary we have

$$0 = \frac{\partial H}{\partial \theta} = i(zH_z - \overline{z}H_{\overline{z}}). \tag{3.17}$$

Now, for the boundary points we get using the equations (3.15),(3.16) and (3.17) that

$$\begin{aligned} q =&\phi_z H_z = \overline{z}z\phi_z H_z = (\overline{z})^2\phi_{\overline{z}}H_z = \\ =&(\overline{z})^2 e^{\rho}H_z^2 = (\overline{z})^3 e^{\rho} z H_z^2 = (\overline{z})^4 e^{\rho} H_z H_{\overline{z}} \end{aligned} \tag{3.18}$$

where we have used this expression for q since along the boundary $\phi = 0$.

Since q is holomorphic on the disc Σ, it then follows that the function $z^4 q$ is holomorphic on the disc and is real valued on the boundary by (3.18). By elementary theory on complex analysis we get that $z^4 q = a$ for some real constant a. In particular, for $z = 0$ we get $a = 0$ and then $q = 0$ on Σ. Applying the results on [3], it then follows that either $\psi(\Sigma)$ is, after a conformal transformation, a maximal immersion or $\psi(\Sigma)$ is part of a hyperbolic space. In the first case, we have a maximal surface with a boundary made up entirely of umbilics. It follows then that the surface is a flat disc since its Hopf differential vanishes identically. In the the remaining case, we have that the surface is a hyperbolic cap.

Acknowledgments

The author was partially supported by Dirección General de Investigación (MCYT), Grant N° BFM2001-2871 and Consejería de Educación y Universidades (CARM) Programa Séneca, Grant N° PI-3/00854/FS/01.

References

[1] L. J. Alías, R. López and J. A. Pastor *Compact Spacelike Surfaces with Constant Mean Curvature in the Lorentz-Minkowski 3-space*, Tohoku Math. J., **50**, 491-501, (1998).

[2] L. J. Alías and B. Palmer,*Conformal Geometry of Surfaces in Lorentzian Space Forms*, Geom. Dedicata, **60**, 301-315, (1996).

[3] R. L. Bryant,*A Duality Theorem for Willmore Surfaces*, J. Diff. Geom, **20**, 23-53, (1984).

[4] B. Palmer,*The Conformal Gauss map and the stability of Willmore Surfaces*, Ann. Global Analysis Geom., **9**, 305-317, (1991).

[5] B. Palmer, *Uniqueness Theorems for Willmore Surfaces with Fixed and Free Boundaries*, Indiana Univ. Math. J., **49**, (2000).

Proceedings of the meeting
Lorentzian Geometry-Benalmádena 2001
Benalmádena, Málaga, Spain
Pub. de la RSME, Vol. 5 (2003), 215-238

A survey on geodesic connectedness

Miguel Sánchez[1]

[1] *Dept. Geometría y Topología, Fac. Ciencias, Univ. Granada. Avda. Fuentenueva s/n, E-18071 Granada (Spain).*

Abstract

The question whether a semi-Riemannian manifold is geodesically connected (i.e.: if each two points can be joined by a geodesic) is a basic geometric property. Moreover, in the Lorentzian case problems as "when a pair of causally related points can be joined by means of a causal geodesic" becomes natural from a physical viewpoint.

We will summary the different techniques and concepts relevant for these problems. This includes geometrical, variational and topological techniques, as well as comparations with the Riemannian and affine cases. Lorentzian manifolds which serve as models of relativistic spacetimes are specially considered.

1 Introduction

In this talk, we will wonder the following question: *Let (M, g) be a semi-Riemannian manifold. Which hypotheses are natural to ensure that it is geodesically connected?*

There are several reasons to study it. From a purely geometrical viewpoint, geodesic connectednes is a basic property, closely related to other elementary properties of a manifold. This is the reason why, in different contexts and decades, authors with very diverse viewpoints have obtained results on geodesic connectedness; say, from Hopf-Rinow theorem to the results in [43], [15], [32], [12], [35] or [24], to be commented below. So, it is not strange that, recently, this problem has been studied very widely.

On the other hand, geodesic connectedness is the more representative one of a series of geometrical problems, some of them with clear applications to General Relativity. For example, in a spacetime (i.e., time-oriented Lorentzian manifold):

- When can two chronologically or causally related points (events) be joined by means of a causal geodesic? The interpretation of this question is straightforward: if, say, p lies in the chronological past q, can we go from p to q freely falling, or must we accelerate? A standard result is the following one, obtained independently by Avez and Seifert [3], [42]: *in a globally hyperbolic spacetime, each two causally related events can be joined by a causal geodesic.* Of course, it is interesting either to extend this result to other spacetimes or to wonder when the connecting geodesic is unique.

- When can an event p and a timelike curve γ be joined by means of a lightlike geodesic? Or, say: if γ is the trajectory of a star, can this star be observed from p? This problem is related to topological properties of the set of timelike curves joining p and the points in γ, and was studied in globally hyperbolic spacetimes by Uhlenbeck [44].

- In previous problem, if there exists a connecting lightlike geodesic, is it unique? Otherwise, the *lens gravitational effect* appears, being the oddity of the images of the observed stars a well-known experimental fact, with some theoretical justifications (see for example, [31]).

The purpose of this article is to give a brief survey about the results and techniques for the problem of geodesic connectedness. We start by the Riemannian case, where sharper results can be obtained, Section 2. Next, we consider manifolds endowed only with an affine connection; the results are then applicable to the Levi-Civita connection of any semi-Riemannian manifold, Section 3. Then, two very particular cases of Lorentzian manifolds (but also very interesting cases from a mathematical viewpoint) are studied, Lorentzian surfaces and spaceforms. For Lorentzian surfaces, we rewrite in a modern language all previous results known by us, and discuss them, Section 4. The results for spaceforms were obtained in the Lorentzian case by Calabi and Markus [15], with some extensions to the semi-Riemannian case in [46]; these results are briefly recalled in Section 5. Finally, in the last two sections we summarize the results obtained recently by using variational and topological techniques. This survey also summarizes and updates pedagogically the one by the author in [41], including new references and discussions.

2 Riemannian case

Let $M \equiv (M, \langle\cdot,\cdot\rangle)$ be a n-dimensional connected Riemannian manifold. As in any semi–Riemannian or even affine manifold, we can wonder if it is *geodesically connected*, that is:

> For any $p, q \in M$, there exists some geodesic connecting them.

But because of the existence of a distance canonically associated to any Riemannian metric, now we can wonder if the manifold is *(weakly) convex*, i.e.:

> For any $p, q \in M$ there is some *distance-minimizing* connecting geodesic.

(If the distance–minimizing geodesic is unique the Riemannian manifold will said to be *strongly convex*).

When M is *complete*, it is well–known:

1. M is convex (Hopf–Rinow).
2. If M is not contractible[1], then each $p, q \in M$, can be joined by *infinitely many geodesics*, with diverging lenghts (Serre-type result obtained by using Ljusternik–Schnirelman theory; see for example [35]).

These results are quite definitive; so, in the Riemannian case one has to wonder just what happen in the incomplete case. On the other hand, this case is natural because:

(i) In the semi-Riemannian indefinite case, there is no associated distance; moreover, there is no any relation between (geodesic) completeness and connectedness.

(ii) Consider a trajectory $x(s)$ for an *autonomous potential*, that is, a solution of $x'' = -\nabla V$ for some function V on M. Any such trajectory

[1] that is, there exists $x_0 \in M$ and a continuous map $H : [0,1] \times M \to M$ such that: $H(1,x) = x_0 = H(t,x_0), \forall t \in [0,1], \forall x \in M$.

will have a constant energy $E = (1/2)\langle x'(s), x'(s)\rangle + V(x(s))$. When $E > V$, this trajectories are pregeodesics for the Jacobi metric $\langle\cdot,\cdot\rangle_E = (E-V)\langle\cdot,\cdot\rangle$; of course, this metric maybe incomplete even if $\langle\cdot,\cdot\rangle$ is complete, or maybe studied just in the open subsets where $E > V$. So, the possibility of connecting each $p, q \in M$ by trajectories for V with a fixed energy $E > V$ is equivalent to the geodesic connectedness for a (possibly incomplete) Riemannian metric. This problem is also related to other variational problems, as the existence of a closed trajectory for a potential with either a fixed energy or a fixed period (see for example [34]).

First, let us consider the following particular (simplest) case. Assume that M is complete, and $\mathcal{D} \subset M$ is a domain (open connected subset) of M with differentiable boundary $\partial\mathcal{D}$; $\overline{\mathcal{D}} = \mathcal{D} \cup \partial\mathcal{D}$. One expects that "good properties" of $\partial\mathcal{D}$ should imply convexity of $\mathcal{D}$. These good properties are related to the following different notions of convexity for a boundary. Fix $p \in \partial\mathcal{D}$.

- $\partial\mathcal{D}$ is *Infinitesimally convex at p* ($\mathbf{IC}_p$) if $\sigma_p \geq 0$, that is, the second fundamental form, σ_p, with respect to the interior normal, is positive semidefinite.

- $\partial\mathcal{D}$ is *variationally convex at p* ($\mathbf{VC}_p$) if for one (and then for all) C^2 function $\phi : U \cap \overline{\mathcal{D}} \longrightarrow \mathbb{R}$, where $U \subset M$ is a neighborhood of p, such that (i) $\phi^{-1}(0) = U \cap \partial\mathcal{D}$, (ii) $\phi > 0$, on $U \cap \mathcal{D}$ and (iii) $d\phi(q) \neq 0, \forall q \in U \cap \partial\mathcal{D}$ one has:

$$H_\phi(p)[v,v] \leq 0 \quad (\text{resp. } < 0) \quad \forall v \in T_p\partial\mathcal{D}.$$

- $\partial\mathcal{D}$ is *locally convex at* $p \in \partial\mathcal{D}$ ($\mathbf{LC}_p$) if there exists a neighborhood $U \subset M$ of p such that

$$\exp_p(T_p\partial\mathcal{D}) \cap (U \cap \mathcal{D}) = \emptyset.$$

- $\partial\mathcal{D}$ is *geometrically convex* (**GC**) if any in $\overline{\mathcal{D}}$ with endpoints in $\mathcal{D}$ is contained in $\mathcal{D}$.

- $\mathcal{D}$ is *(geodesically) pseudoconvex* (**PC**) if for each compact set $K \subseteq \mathcal{D}$ there is a compact set $H \subseteq \mathcal{D}$ such that each geodesic segment with extremes in K lies in H.

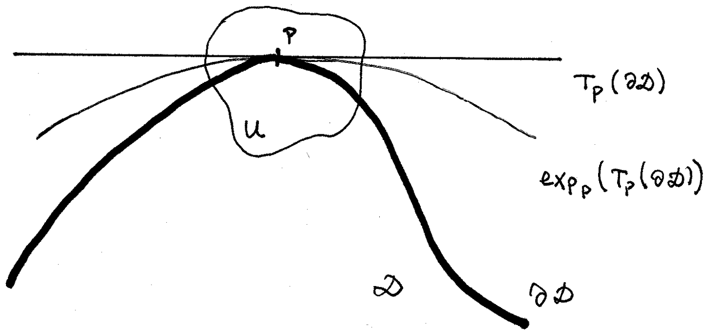


Figure 2: Locally convex domain $\mathcal{D}$ at p.

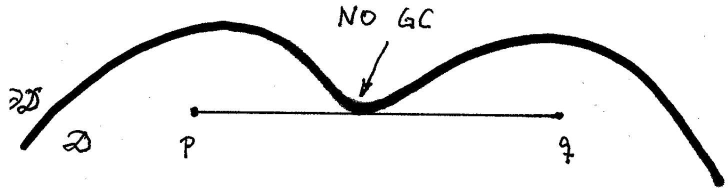


Figure 3: Non-geometrically convex domain.

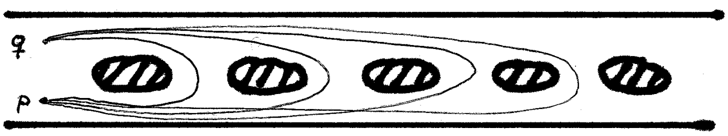


Figure 4: A complete surface with infinite holes is not PC (put $K = \{p, q\}$).

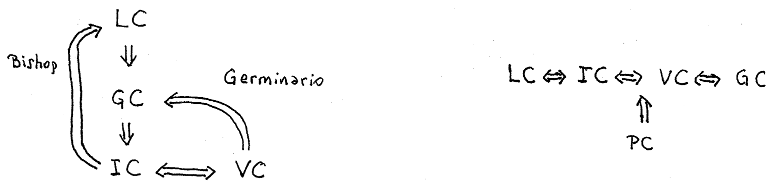


Figure 5: Summary for differentiability C^4.

Recall that the three first notions are applied to each point of the boundary; $\partial\mathcal{D}$ will be called IC (resp. VC, LC) if it is IC_p (resp. VC_p, LC_p) $\forall p \in \partial\mathcal{D}$. For the definition of VC, recall that a function ϕ globally defined on $\partial\mathcal{D}$ satisfying (i), (ii) and (iii) can be found. The concept GC is globally defined on all the boundary, and PC applies to the domain or manifold, rather than to its boundary.

It is straightforward to check that IC_p and VC_p are equivalent. Moreover, $\mathrm{IC}_p \Leftarrow \mathrm{LC}_p$ but the converse does not hold; for a counterexample, just take: $\mathcal{D} = \{(x,y) \in \mathbb{R}^2 : y > x^3\}, p = (0,0)$. For the whole boundary, it is not difficult to show that PC is a more restrictive concept than IC, VC, GC, LC. Essentially, these last four concepts are equivalent, but some of the equivalences are not as trivial as may sound. Bishop [14] showed $IC \Rightarrow LC$ (using explicitly differentiability C^4), and Germinario [27] showed, when $\overline{\mathcal{D}}$ is complete, $VC \Rightarrow GC$ (using differentiability C^2; as a technical simplification she also uses Nash's theorem which needs C^3). Taking into account the straightforward implications, one has Fig. 4 as a summary (see [41], [6] for further discussions). The expected result for convexity is then the following:

Theorem 2.1 *When $\overline{\mathcal{D}} = \mathcal{D} \cup \partial\mathcal{D}$ is complete, then $\mathcal{D}$ is convex if and only if $\partial\mathcal{D}$ is convex.*

Of course, the hypothesis on completeness is essential. Recall that if M is complete then so is $\overline{\mathcal{D}}$. Even though the converse is not true, there is no loss of generality if we assume it (if M is not complete, then the metric can be deformed out of $\bar{D}$ to obtain the completeness of this new metric on M).

Sketch of geometrical proof for Theorem 2.1. Fixed $p, q \in \mathcal{D}$, consider piecewise smooth curves in $\overline{\mathcal{D}}$ joining p, q. The infimum L of the lengths of these curves is equal to the distance in $\mathcal{D}$ between p, q, and one can prove: (i) if only piecewise geodesics are taken into account, the infimum is still L, (ii) a curve γ of minimum length in $\overline{\mathcal{D}}$ must exist; out of the boundary, this curve must be a geodesic, and (iii) as $\sigma \geq 0$, γ cannot touch $\partial\mathcal{D}$.

□

Sketch of variational proof for Theorem 2.1. Essentially, geodesics joining p and q are the critical points of the functional:

$$f(x) = \frac{1}{2}\int_0^1 \langle \dot{x}(s), \dot{x}(s)\rangle ds \tag{2.1}$$

defined on absolute continuous curves $x : [0,1] \to \mathcal{D}$, $x(0) = p, x(1) = q$. This functional is positive definite, but, because of the lack of completeness of $\mathcal{D}$, we cannot ensure the existence of a minimum a priori. Nevertheless, the functional can be penalized by using the function ϕ in the definition of VC:

$$f_\varepsilon(x) = \frac{1}{2}\int_0^1 \left(\langle \dot{x}(s), \dot{x}(s)\rangle + \frac{\varepsilon}{\phi^2}\right) ds.$$

When a curve approaches $\partial\mathcal{D}$, f_ε diverges; this is the key why f_ε attains a minimum x_ε for each ε. A priori estimates ensure that the x_ε's lie uniformly far from the boundary, and a minimum of f can be found by taking $\varepsilon \to 0$.

□

Now, let us consider the general case when either $\partial\mathcal{D}$ may be non-differentiable or $\overline{\mathcal{D}}$ is not complete. This case was first studied in [32] and, with full generality, in [6]. From Theorem 2.1, it is straightforward (see Fig. 5):

Corollary 2.2 *If there exists a sequence $(\mathcal{D}_m), m \in \mathbf{N}$ of complete open submanifolds of $\mathcal{D}$ with convex (differentiable) boundary such that*

$$\mathcal{D}_m \subset \mathcal{D}_{m+1} \quad \textit{and} \quad \mathcal{D} = \bigcup_{m\in\mathbf{N}} \mathcal{D}_m, \tag{2.2}$$

then $\mathcal{D}$ is geodesically connected.

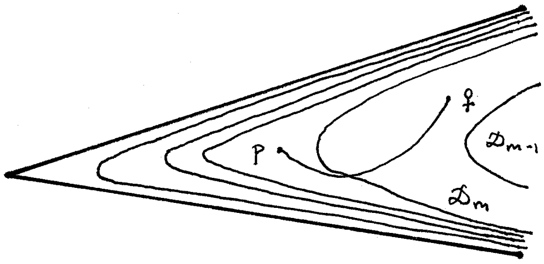


Figure 6: Application of Corollary 2.2.

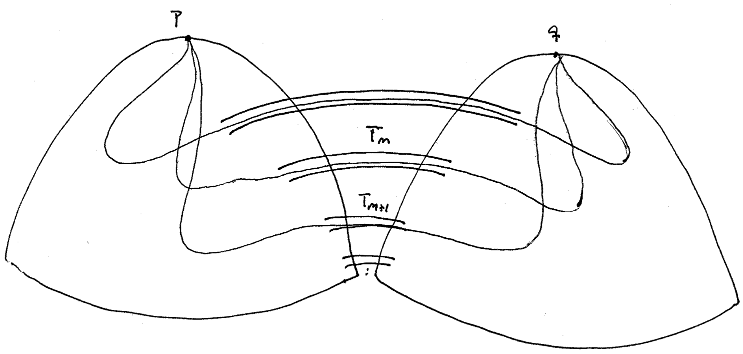


Figure 7: When Corollary 2.2 is applicable, $\mathcal{D}$ may be non-convex, if $\mathcal{D}$ is not an open subset of a complete Riemannian manifold. (The length of the tubes can be controlled in such a way that any geodesic from p to q through the tube T_m is strictly *longer* than the geodesic through the tube T_{m+1}.)

Nevertheless, $\mathcal{D}$ is not necessarily convex; in fact, geometrical arguments as those in the proof of Theorem 2.1 yields: *if* $\overline{\mathcal{D}}$ *is complete then* $\mathcal{D}$ *is convex* (Fig. 6). From these elementary considerations, the results in [32] can be re-proven and extended.

Variational methods permit to extend these results. In fact, when M is complete, if the boundaries $\partial\mathcal{D}_m$'s in Corollary 2.2 are not convex but their lack of convexity goes to zero and can be suitably controlled, then the convexity of $\mathcal{D}$ still holds [6, Theorem 1.6]. When M is not complete, the geodesic connectedness of $\mathcal{D}$ can be studied intrinsically by using the Cauchy boundary $\partial_c\mathcal{D}$ (this concept is also useful for the complete case, [6]). Let $\overline{\mathcal{D}}_c = \mathcal{D} \cup \partial_c\mathcal{D}$ be the canonical Cauchy completation of $\mathcal{D}$

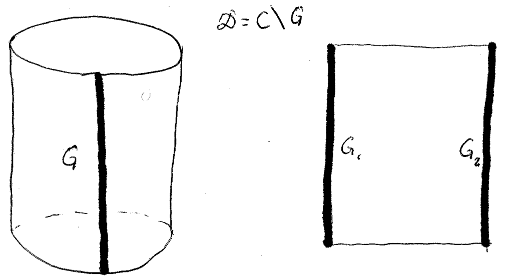


Figure 8: Each point of the boundary $\partial\mathcal{D} = G$ corresponds to two points of the Cauchy boundary $\partial_c\mathcal{D} = G_1 \cup G_2$.

(obtained by means of Cauchy sequences). As a difference with $\partial\mathcal{D}$, Cauchy boundary $\partial_c\mathcal{D}$ has an intrinsic meaning for $\mathcal{D}$. For example, let $M = C$ be an (infinite) cylinder and $\mathcal{D}$ the domain obtained by removing the generatrix G. Clearly $\partial\mathcal{D} = G$ but $\partial_c\mathcal{D}$ is the set of two lines, which can be seen as the boundary of a strip in $\mathbb{R}^2$ (see Fig 7). In general, if M is complete, then a quotient set of $\partial_c\mathcal{D}$ is identificable to $\partial\mathcal{D}$; if M is not complete then there are points in $\partial_c\mathcal{D}$ with no relation with $\partial\mathcal{D}$. Because of the intrinsic meaning of $\partial_c\mathcal{D}$ we can assume $M = \mathcal{D}$. The following result is then an extension of previous ones:

Theorem 2.3 *Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold, and $\overline{M}_c = M \cup \partial_c M$ its canonical Cauchy completation. Assume that there exists a positive differentiable function ϕ on M such that:*

(i) $\lim_{x\to\partial_c M} \phi(x) = 0$*;*

(ii) each $y \in \partial_c M$ admits a neighbourhood $U \subset \overline{M}_c$ and constants $a, b > 0$ such that

$$a \leq \|\nabla\phi(x)\| \leq b \quad \forall x \in M \cap U;$$

(iii) each $y \in \partial_c M$ admits a neighbourhood $U \subset \overline{M}_c$ and a constant $m \in \mathbb{R}$ such that inequality

$$H_\phi(x)[v, v] \leq m\langle v, v\rangle\phi(x)$$

holds for all $x \in M \cap U$ and for all $v \in T_xM$.

Then M is convex.

On the other hand, by using Ljusternik–Schnirelman theory, multiplicity results can be obtained in all previous results when the (geodesically connected) manifolld is not contractible.

3 Affine case

Let M be a (connected) manifold endowed with an affine connection ∇; as we are interested in geodesic connectedness, there is no loss of generality assuming that ∇ is symmetric. In comparison with the Riemannian case, we have the following facts:

1. All previous notions on convexity for $\partial\mathcal{D}$ can be translated directly, but IC. Nevertheless, this concept was completely equivalent in the Riemannian case to VC, which still makes sense.

2. It is not difficult to show the equivalences LC $\Longrightarrow$ GC $\Longrightarrow VC$ (we do not know counterexamples to the converses).

3. Good conditions on the boundary $\partial\mathcal{D}$ of $\mathcal{D}$ (even under good conditions on geodesic completeness) *do not imply connectedness.* In fact, a complete affine manifold is not necessarily geodesically connected. Even more:

   - For each $n > 0$ there exist a complete affine manifold with the following property: in order to connect each two points with broken geodesics, it is necessary to use a broken geodesic with at least n breaks (Hicks, [33]).
   - Geodesically disconnected complete affine tori exist (Bates, [9]).

It is not difficult to construct Bates' torus. Just consider on $\mathbb{R}^2$ the moving frame ($X_1 = \cos x\partial_x + \sin x\partial_y, X_2 = -\sin x\partial_x + \cos x\partial_y$) and define the connection ∇ which parallelizes it. Any geodesic $\gamma(s) = (x(s), y(s))$ is a integral curve of a linear combination of X_1 and X_2; thus, γ is complete and $x(s)$ lies in an interval of length $\leq 2\pi$. So, the induced connection of the torus $T^2 = \mathbb{R}^2/4\pi\mathbb{Z}^2$ is geodesically complete and geodesically disconnected.

Some relevant concepts for connectedness where introduced by Beem, Parker and Low (see for example [12], [11]). Among them, the concepts

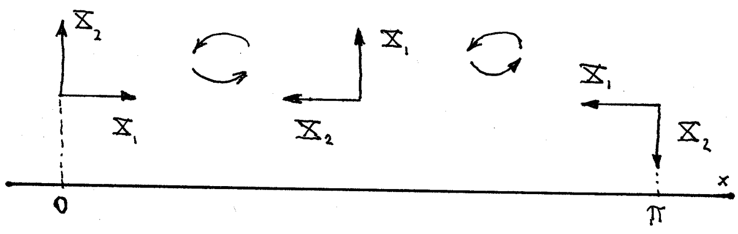


Figure 9: Affine connection such that $\nabla_{X_i}X_j \equiv 0$. No geodesic which crosses the line $x = 0$ can reach $x = \pi$.

of pseudoconvex manifold, as defined in Section 2, and geodesically disprisoning manifold, i.e.: for any (inextendible) geodesic $\gamma : (a, b) \to M$ and any compact subset $K \subseteq M$,

$$\exists\{t_n\} \to a^+, \{s_n\} \to b^- : \gamma(t_n), \gamma(s_n) \notin K, \forall n \in \mathbf{N}.$$

On the other hand, it is also relevant the space of unreparametrized geodesics $G(M)$. In this space, *tangential convergence* is considered: a sequence converges tangentially, when it is possible to choose (inextendible) reparametrizations $\gamma_n :]a_n, b_n[\to M$ for the sequence and $\gamma :]a, b[\to M$ for the candidate to limit such that $\gamma_n'(t_0) \to \gamma'(t_0)$ (for some $t_0 \in]a, b[$ contained in all $]a_n, b_n[$ but a finite number). Remarkably:

- In this case: $\limsup\{a_n\} \leq a < b \leq \liminf\{b_n\}$ and $\{\gamma_n'\}$ converges uniformly on compact subsets of $]a, b[$ to γ'. Nevertheless, there is no relation between the completeness of the elements of the sequence and the completeness of the limit (even in a Lorentzian torus [37]).
- It is easy to see that a sequence may have more than one limit. Nevertheless, if $G(M)$ is Hausdorff then the *sky of p* is closed, where, following [11],

$$Sky(p) := \{q \in M \text{ connectable to } p \text{ by means of a geodesic}\}.$$

Disprisoning and pseudoconvexity implies that $G(M)$ is Hausdorff. Thus, one have the following implications, valid for any affine manifold.

$$\left.\begin{array}{r} \text{Dispr. + psdoc.} \Rightarrow G(M) \text{ Hausdorff} \Rightarrow \text{Sky}(p) \text{ closed }, \forall p \in M \\ \text{Inexistence of conjugate points} \Rightarrow \text{Sky}(p) \text{ open }, \forall p \in M \end{array}\right\} \Rightarrow$$

$$M \text{ geodesically connected.}$$

Some extensions of these results are: (1) previous implications can be extended to a class of (disprisoning and pseudoconvex) *sprays*, [21], and (2) $G(M)$ Hausdorff implies geodesic connectedness if the affine manifold is *unitrace* (and, thus, it satisfy that, for each $p \in M$ there exists a neighborhood U such that any geodesic which enters U either leaves and never returns or retraces the same path every time it returns) [11].

4 Lorentzian surfaces

Now, let us consider the simplest semi-Riemannian case, that is, a Lorentzian manifold with dimension 2 (surface). Recall that, among indefinite semi-Riemannian manifolds, only in the Lorentzian case there is something related to a distance, the *time-separation* or *Lorentzian distance*, based in the local maximazing properties of causal curves (see, for example, [10], [36]). The properties of this distance are essential to prove the Avez-Seifert result on geodesic connectedness for causally related points (Section 1). But, in principle, it cannot be used for the case of non-causally related points. Moreover, as in the affine case, neither completeness nor compactness imply connectedness.

For an orientable and time-orientable Lorentzian surface there is a pair of transverse *null foliations*, generated by lightlike geodesics. Two Lorentzian metrics on a surface are pointwise conformal if and only if their pairs of null foliations coincide. A *Lorentz surface* is defined as a surface endowed with a class of conformally equivalent (oriented and time-oriented) Lorentzian metrics. Their systematic study is carried out in [45], and their properties depends on the properties of their pairs of null foliations. By taking into account these foliations, the following result on geodesic connectedness was proven in [43]:

Theorem 4.1 *A Lorentzian surface (S, g) is geodesically connected if it is globally conformal to Lorentz-Minkowski plane $\mathbb{L}^2$.*

Proof. This result can be proven nowadays as a corollary of Avez-Seifert one. In fact, recall first that global hyperbolicity is a conformal invariant and, thus, g as well as $-g$ are globally hyperbolic metrics. Therefore, any $p, q \in S$ can be joined either by a causal curve or by a spacelike curve (with non-vanishing velocity). In the first case, p and q are causally related for g, in the latter case, they are causally related for $-g$; thus, one has to apply Avez-Seifert result either to g or to $-g$.

□

Remark 4.2 Recall that, in previous proof, it is essential that g as well as $-g$ are both globally hyperbolic. But a simply connected globally hyperbolic Lorentz surface maybe non-geodesically connected, as the universal covering of 2-dimensional de Sitter spacetime $\widetilde{\mathbb{S}}_1^2$ shows (see next section). In fact, this spacetime is a counterexample to the main result in [30].

Theorem 4.1 can be applied to some non-trivial examples. Consider first the following result in [40].

Theorem 4.3 *Let (T^2, g) be a Lorentzian torus with a Killing vector field $K(\not\equiv 0)$. The following properties are equivalent:*

(i) (T^2, g) is globally conformally flat.

(ii) The sign of $g(K, K)$ is constant (either positive or negative or 0 on all T^2).

(iii) (T^2, g) is (geodesically) complete.

(iv) All the timelike (resp. lightlike; spacelike) geodesics are complete.

If (i)—(iv) holds the universal covering of the torus is globally conformal to $\mathbb{L}^2$. Thus, as a consequence of the last two thheorems, one reobtains the following result in [40]:

Corollary 4.4 *Any complete Lorentzian torus with a Killing vector field $K(\not\equiv 0)$ is geodesically connected.*

Nevertheless, in the incomplete case, i.e., if $\mathrm{sign}(g(K, K))$ is not constant, then the torus may be geodesically disconnected. Remarkably, this happens if we consider Bates' example in Section 3, and take the Lorentzian metric g such that X_1, X_2 are lightlike and $g(X_1, X_2) = -1$. The resulting torus is geodesically disconnected; in fact, there are two points in T^2 such that no curve with a definite causal character (i.e., no timelike, no lightlike and no spacelike -with non-vanishing velocity- curve) can connect them. Nevertheless, as a relevant difference with Bates' example, this torus is *not* geodesically complete ($g(K, K)$ is not constant for the Killing vector $K = \partial_y$). In fact, we do not know any example of complete and geodesically disconnected Lorentzian torus.

5 Indefinite spaceforms

We mean by a indefinite *spaceform* a complete semi–Riemannian n–manifold M of index $\nu \in \{1, \ldots, n-1\}$ and constant curvature C. It is well known about them (ssee for example [36]):

(1) M is covered by the model (1-connected) space of the same dimension, index and curvature $M(n,\nu,C)$, that is: $M = M(n,\nu,C)/\Gamma$, where Γ is the fundamental group of M.

(2) The case $C = 0$ is trivial for our problem (the model space $M(n,\nu,0) \equiv \mathbb{R}^n_\nu$ is geodesically connected); thus, it will not be taken into account in what follows.

(3) Up to a homothety, we can assume $C = 1$ (the homothetic factor may be positive as well as negative).

(4) If $n \geq 3$, the model space is then the *pseudosphere* $\mathbb{S}^n_\nu$ (spacelike vectors of "norm" 1 in $\mathbb{R}^{n+1}_\nu$). The Lorentzian pseudosphere $\mathbb{S}^n_1$ is also called *de Sitter* spacetime. This spacetime is globally hyperbolic, thus, if $M = \mathbb{S}^n_1/\Gamma$, causally related points are connectable by causal geodesics. For $n = 2$ the pseudosphere $\mathbb{S}^2_1$ is not 1-connected; recall that its universal covering $\widetilde{\mathbb{S}}^2_1$ is globally hyperbolic.

(5) From a direct computation of the geodesics, *no indefinite pseudosphere* $\mathbb{S}^n_\nu$, $0 < \nu < n$ *is geodesically connected.* In fact, for $n = 1$, two points $p, q \in \mathbb{S}^n_1$ are connectable by a geodesic if and only if $\langle p, q\rangle_1 > -1$, where $\langle \cdot, \cdot\rangle_1$ is the usual Lorentzian product of $\mathbb{L}^{n+1}(\equiv \mathbb{R}^{n+1}_1)$.

The main result on geodesic connectedness is given in the following theorem [15]. Recall that M is called *starshaped* from a point $p \in M$ if the exponential at p, $\exp_p : T_pM \to M$, is onto.

Theorem 5.1 *For $n \geq 2$, $(\nu = 1)$:*

(1) $\mathbb{S}^n_1$ is not starshaped from any point.

(2) Any spaceform $M = \mathbb{S}^n_1/\Gamma, M \neq \mathbb{S}^n_1$ is starshaped from some $p \in M$.

(3) A spaceform $M = \mathbb{S}^n_1/\Gamma$ is geodesically connected if and only if it is not time-orientable.

The points (2) and (3) show a possibility which is striking from the Riemannian viewpoint: a complete semi-Riemannian manifold maybe starshaped from a point, but not from another point.

Theorem 5.1 solves completely the geodesic connectedness of Lorentzian spaceforms with positive curvature and $n \geq 3$. Extensions of this result for arbitrary index ν (including the case $\nu = n-1$, which is equivalent to the Lorentzian case of constant negative curvature) and $n \geq 3$ are given in [46]. The additional assumptions to obtain a result are:

1. The fundamental group Γ is finite. This assumption is necessary in order to work with the barycenter of the orbits. It is automatically satisfied if $2\nu \leq n$.

2. The non time-orientability must be replaced by the inexistence of a *proper time-axis*. By a time-axis T we mean a one-dimensional, Γ-invariant, negative-definite linear subspace of $\mathbb{R}^{n+1}_{\nu}$. T is proper if Γ acts trivially (i.e., as the identity) on T.

6 Variational Methods for the Lorentzian case

As in the Riemannian case, geodesics joining two fixed points p, q in a semi–Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ are critical points of the action functional (2.1). Nevertheless, this functional is strongly indefinite and does not have *a priori* good properties from a variational point of view. The application of critical point theory started with some papers by Benci, Fortunato and Giannoni (see the book [35]); since then, many authors have used variational methods. Essentially, two classes of spacetimes has been studied, stationary and splitting type ones. We review briefly these results, pointing out some new references since [41].

A. STATIONARY SPACETIMES. These spacetimes are Lorentzian manifolds admitting a (globally defined) timelike Killing vector field K. A *standard stationary spacetime* is a product manifold $M = \mathbb{R} \times M_0$ endowed with a metric

$$\langle \cdot, \cdot \rangle = -\beta dt^2 + 2\omega \otimes dt + g_0,$$

where dt^2 is the usual metric on $\mathbb{R}$, $\beta : M_0 \to \mathbb{R}$ is a positive function, ω is a 1-form on M_0 (if $\omega \equiv 0$, the spacetime is *static*), and g_0 is a Riemannian metric on M_0. Locally, every stationary spacetime looks like a standard one ($K \equiv \partial_t$).

In order to see the key for the variational approach, consider the standard case. Let $z : [0, 1] \to M$, $z(s) \equiv (t(s), x(s))$ be a curve joining two fixed points $(t_1, x_1), (t_2, x_2) \in M$. The action functional $f(z)$ in (2.1) is invariant by the flow of $K = \partial_t$. Thus, if $z(s) \equiv (t(s), x(s))$ is a critical point then:

$$\langle \partial_t, z' \rangle \left(= -\beta(x(s))\dot{t}(s) + 2\omega(\dot{x}(s)) + g_0(\dot{x}(s), \dot{x}(s))\right) \equiv C_z \quad \text{(constant).} \tag{6.1}$$

Thus, from (6.1): (i) the value of $\dot{t}(s)$ can be written in terms of $x(s)$ and C_z, and (ii) the condition $\int_0^1 \dot{t}(s)ds = t_2 - t_1$ allows to obtain C_z in terms of $x(s)$. This suggest to replace the inicial functional $f(z)$ by a new functional J defined only on curves $x : [0,1] \to M_0$ which join x_1 and x_2, say $J(x) = f(z)$ where the part $t \equiv t(s)$ of $z = (t,x)$ is computed from (6.1) (by taking into account previous comment (ii) on the value of C_z). One can check that critical points of J are in bijective correspondence with critical points of f.

Summing up, the initial functional f is replaced by a new functional J on the "Riemannian part" M_0. The expression of J is not as simple as the expression of f (J is non-local, recall (ii) above), but it is bounded from below and, under reasonable assumptions, it satisfies good variational properties as the well-known "condition C of Palais–Smale". As a consequence, one obtains (d_0, $\| \cdot \|_0$ will denote, resp. the g_0-distance and norrm on M_0):

Theorem 6.1 *A stationary spacetime is geodesically connected, if: (i) g_0 is complete, (ii) $0 <\mathrm{Inf}(\beta) \leq \mathrm{Sup}(\beta) < \infty$, and (iii) the g_0-norm of $\omega(x)$ has a sublinear growth in M_0, that is, $\| \omega(x) \|_0 \leq A \cdot d_0(x,p_0)^\alpha + B$, for some $A, B \in \mathbb{R}, \alpha \in [0,1[, p_0 \in M_0$.*

It is worth pointing out:

1. When M is not contractible, infinitely many (spacelike) connecting geodesics can be obtained. Moreover, the existence of infinitely many timelike geodesics connecting a point $z \in M$ and a line $\mathrm{L}[x] = \{(t,x) | t \in \mathbb{R}\}$, $x \in M_0$, can be proven (see [35]).

2. One can also wonder when the number of connecting geodesics is finite; in the case of lighlike geodesics, this is related to the multiple image and gravitational lens effects [28].

3. More intrinsic hypotheses (valid for non–standard stationary spacetimes) can be found [29].

4. Assume that $\mathcal{D}_0 \subset M_0$ is a domain with differentiable boundary, and consider the open submanifold $\mathbb{R} \times \mathcal{D}_0 \subset M$. In this case VC $\Longleftrightarrow$ GC and, under the assumptions in Theorem 6.1, $\mathbb{R} \times \mathcal{D}_0$ is geodesically connected (the problems relative to the boundary are exhaustively studied in [4], see also [7]). These results admit some extensions to non-differentiable boundary, at least in the static

case. In fact, outer Reissner-Nordström and outer Schwarzschild spacetimes are shown to be geodesically connected [13].

5. Related techniques can be applicated to other spacetimes, as Gödel type (where two independent Killing vector fields -none necessarily timelike- span a Lorentzian plane) [20] or gravitational waves [16].

6. A natural extension to the problem of geodesic connectedness is the existence of geodesics connecting two submanifolds. It is not difficult to check that, among these geodesics, those which are orthogonal to the submanifolds are also critical points of the functional (2.1); for this problem see [19]. It is specially interesting the case of lightlike geodesics joining the two submanifolds (say, one of the submanifolds may be the worldline of a star), [18].

7. Finally, it is worth pointing out that connectedness by geodesics can be generalized to the problem of connectedness by trajectories of more general Lagrangian systems as, for example, trajectories under an electromagnetic field. Static manifolds with potential vector fields independent of time were studied in [5], for more general results, see [8] and references therein.

B. Splitting type spacetimes. In what follows, a Lorentzian splitting manifold $(M, \langle\cdot,\cdot\rangle)$ will be a product manifold $M = \mathbb{R}\times M_0$ endowed with a Lorentzian metric type:

$$\langle\cdot,\cdot\rangle = -\beta(t,x)dt^2 + 2\omega(t,x)\otimes dt + g_{(t,x)}, \tag{6.2}$$

where β is a function on all M, ω is a 1-form on M_0 which depends on each point $(t,x)\in M$, and $g_{(t,x)}$, is an Euclidean scalar product in the tangent space to each slice $\{t\}\times M_0 \ni (t,x)$. We can put:

$$g_{(t,x)}(\cdot,\cdot) = g_0(\cdot,\alpha(t,x)[\cdot]),$$

where g_0 is a complete Riemannian metric on M_0. When $\omega\equiv 0$ the splitting is *orthogonal.*

In this case, a reduction to a "Riemannian" problem is not possible, and the key for the variational approach is to use Rabinowitz's Saddle Point Theorem, but taking into account: (i) to solve the possible absence of Palais–Smale condition, the action functional is approximated by a family of penalized functionals, and (ii) In Rabinowitz's theorem, the independent directions where the functional goes to $-\infty$ are finite; so, a *Galerkin finite-dimensional approximation* is carried out. A standard general result is the following (see [35] and references therein):

Theorem 6.2 *A splitting spacetime* $(M, \langle\cdot,\cdot\rangle$ *is geodesically connected, if:*

(1) g_0 *is complete and there exists* $\lambda > 0$ *such that* $g_t > \lambda g_0$ *for all* t.

(2) $0 <$*Inf*(β) *and* $\beta(x,0), \| \omega(x,0) \|_0$ *are bounded.*

(3) $g_t/\beta(t,x)$ *(resp.* $\omega/\beta(t,x)$*) is bounded by a function on* M *type:* $b_0(x) + b_1(x)|t|^\mu$, *with* $\mu \in [0,1[$ *(resp.* $\mu \in [0,2[$*)*

(4) Consider the natural derivatives $\partial_t\alpha, \partial_t\beta, \partial_t\delta$ *of* α, β, δ *with respect to* t*. Then the* g_t*–norms of* $\partial_t\alpha/\alpha, \partial_t\beta/\beta, \partial_t\delta$ *are bounded at each hypersurface with constant* t*, and its supremum when* $t \to \pm\infty$ *goes to 0.*

Remarks:

1. The case with boundary becomes more complicated. Nevertheless, strips type $\mathcal{D} =]a,b[\times M_0$ with variationally convex boundary are shown to inherit geodesic connectedness. When M is not contractible, the existence of infinitely many connecting geodesics can be ensured by using the *relative category*, a topological invariant somewhat subtler than Ljusternik-Schnirelman category.

2. More accurate results for the orthogonal splitting case can be given, [2]. Connectedness of two submanifolds by normal geodesics can be also studied, [17]; for trajectories under an electromagnetic potential, see [1].

7 A topological method

Recently, the geodesic connectedness of some spacetimes have been proven by using topological arguments [24], [26]. This method is explained in another talk of these proceedings, [25]. Thus, we will give here just some comments in relation to the results in previous sections.

¿From the viewpoint of differential equations, fixed two points p, q, of a Lorentzian manifold, the existence of a connecting geodesic is just the existence of a solution for a system of equations. Except in very particular cases, the complexity of this system of differential equations make impossible to ensure the existence of a solution, even in spacetimes where the geodesic equations can be integrated (notice that a solution with the "initial" condition p and the "final" condition q is needed).

Roughly, this problem is equivalent to the existence of zeroes for some function $\mathcal{F}$. And it is well-known that, under certain conditions, the existence of such zeroes is ensured by using Brower's topological degree.

At any case, in order to apply this degree, it is necessary to ensure some boundary conditions on function $\mathcal{F}$. Thus, some qualitative knowledge on the behaviour of the geodesics is needed, and the existence of some partial integration of their equations becomes specially useful. This is the reason why the topological method works well for "classical relativistic spacetimes" (say, spacetimes with a "proper name", which present some symmetries). In fact, the topological method has been used to ensure the connectedness of spacetimes as the Schwarzschild black hole or the exterior part of Kerr spacetime, which were not covered by previous techniques. Summing up, the obtained results are the following:

1. Consider a multiwarped spacetime $M = I \times F_1 \times \cdots \times F_n$,

$$g = -dt^2 + \sum_i f_i(t)^2 g_i.$$

   where $I \subseteq \mathbb{R}$ is an interval and the fibers (F_i, g_i) are convex Riemannian manifolds.

   - In the case $n = 1$ (Generalized Robertson-Walker spacetimes) a very general sufficient condition for geodesic connectedness can be given. In fact, this condition becomes necessary and sufficient if the fibers are strongly convex [22].
   - When $n > 1$, a sufficient condition very close to a necessary one can be given (including the case with boundary), [24]. In particular, Schwarzschild black hole and generalizations of Intermediate Reissner Nordström spacetimes are shown to be geodesically connected.
   - The connectedness by causal geodesics can be studied, extending Avez-Seifert result [38].

   Summing up, the problem is solved completely for this type of spacetimes.

2. Consider the exterior region of low rotating Kerr spacetime, that is the region with radial coordinate r greater than the radius r_+ of the first even horizont. This region is not stationary; in fact, one can show that the stationary part is not geodesically connected [23].

Nevertheless, an accurate use of the topological technique shows that it is geodesically connected [26].

In conclusion, we can say that variational methods give a general and rough estimate about when stationary or splitting spacetimes are geodesically connected. But, frequently, classical spacetimes need more accurate estimates. This can be seen clearly in Gödel type spacetimes. The geodesic connectedness of many such spacetimes can be proven by using variational methods. But classical Gödel spacetime itself is not covered in this way. This spacetime needs a specific proof taking into account the first integrals of its geodesic equations and some topological arguments [20]. For Gödel spacetime these topological arguments become rather trivial, but for other classical spacetimes as the ones above, they become subtler.

Finally, it is worth pointing out the case of causal geodesics. In general, Avez-Seifert type-results (obtained by using either the time separation or topological ideas) are sharper than the results obtained by using variational methods [39]. Neverhteless, when these methods are applicable, the multiplicity of connecting geodesics can also be ensured.

Acknowledgments

This work has been partially supported by a MCyT-FEDER grant number BFM2001-2871-C04-01.

References

[1] F. Antonacci, F. Giannoni and Paola Magrone *On the problem of the existence for connecting trajectories under the action of gravitational and electromagnetic fields*, Diff. Geom. Appl., **13**, 1-17, (2000).

[2] F. Antonacci and R. Sampalmieri, *On a class of geodesically connected Lorentzian manifolds*, J. Diff. Equat., **138**, 171-187, (1997).

[3] A. Avez, *Formule the Gauss-Bonnet-Chern en métrique de signature quelconque*, C.R. Acad. Sci., **255**, 2049-2051, (1962).

[4] R. Bartolo, *Curvas críticas en variedades riemannianas y lorentzianas con borde*, Ph.D thesis, Univ. Granada (2000).

[5] R. Bartolo, *Trajectories connecting two events of a Lorentzian manifold in the presence of a vector field*, J. Diff. Equat., **153**, 82-95, (1999).

[6] R. Bartolo, A. Germinario and M. Sánchez, *Convexity of domains of Riemannian manifolds*, Ann. Glob. Anal. Geom., **21**, 63-84, (2002).

[7] R. Bartolo, A. Germinario and M. Sánchez, *A note on the boundary of a static spacetime*, Diff. Geom. Appl., **16**, 121-131, (2002).

[8] R. Bartolo, A. Germinario and M. Sánchez, *Trajectories connecting two submanifolds on non-complete Lorentzian manifolds*, preprint.

[9] L. Bates, *You can't get there from here,* Diff. Geom. Appl., **8**, 273-274, (1998).

[10] J.K. Beem, P.E. Ehrlich and K. Easley, *Global Lorentzian Geometry*, Marcel Dekker Inc., N.Y. (1996).

[11] J.K. Beem, R.J. Low, P.E. Parker, *Spaces of geodesics: products, coverings, connectedness*, Geom. Dedicata, **59**, 51–64, (1996).

[12] J.K. Beem and P.E. Parker, *Pseudoconvexity and geodesic connectedness*, Ann. Mat. Pura Appl., **155**, 137–142, (1989).

[13] V.Benci, D.Fortunato and F.Giannoni, *On the existence of geodesics in static Lorentz manifolds with singular boundary,* Ann. Scuola Norm. Sup. Pisa Cl. Sci., **XIX**, 255-289, (1992).

[14] R.L. Bishop, *Infinitesimal convexity implies local convexity*, Indiana Math. J. **24**, 169-172, (1974).

[15] E. Calabi and L. Markus, *Relativistic space forms*, Ann. Math., **75**, 63-76, (1962).

[16] A.M. Candela, J.L. Flores and M. Sánchez, *On general plane fronted waves. Geodesics* Gen. Relat. Gravit. **35**, 631-649, (2003).

[17] A.M. Candela, A. Masiello and A. Salvatore, *Existence and multiplicity of normal geodesics in Lorentzian manifolds*, J. Geom. Anal. **10**, 591-619, (2000).

[18] A.M. Candela and A. Salvatore, *Light rays joining two submanifolds in space-times*, J. Geom. Phys. **22** 281-297, (1997).

[19] A.M. Candela and A. Salvatore, *Normal geodesics in stationary Lorentzian manifolds with unbounded coefficients*, preprint.

[20] A.M. Candela and M. Sánchez, *Geodesic connectedness in Gödel-type spacetimes*, Diff. Geom. Appl. **12**, 105-120, (2000).

[21] L. Del Riego and P.E. Parker, *Pseudoconvex and disprisoning homogeneous sprays*, Geom. Dedicata, **55**, 211-220, (1995).

[22] J.L. Flores and M. Sánchez, *Geodesic connectedness and conjugate points in GRW spacetimes*, J. Geom. Phys., **36**, 285-314, (2000).

[23] J.L. Flores and M. Sánchez, *Geodesics in stationary spacetimes. Application to Kerr spacetime*, Int. J. Theor. Phys. Group Th. Nonlin. Opt., **8**, 319-336, (2002).

[24] J.L. Flores and M. Sánchez, *Geodesic connectedness of multiwarped spacetimes*, J. Diff. Equat., **186**, 1-30, (2002).

[25] J.L. Flores and M. Sánchez, *A topological technique for the geodesic connectedness in some Lorentzian manifolds*, these proceedings.

[26] J.L. Flores and M. Sánchez, *A topological method for geodesic connectedness of spacetimes: outer Kerr spacetime*, J. Math. Phys., **43**, 4861-4885, (2002).

[27] A. Germinario, *Homoclinics on Riemannian manifolds with convex boundary*, Dynam. Syst. Appl. **4**, 549-566, (1995).

[28] F. Giannoni, A. Masiello and P. Piccione, *On the finiteness of light rays between a source and an observer on conformally stationary space-times*, Gen. Relat. Gravit. **33**, 491–514, (2001).

[29] F. Giannoni and P. Piccione, *An intrinsic approach to the geodesical connectedness of stationary Lorentzian manifolds*, Comm. Anal. Geom. **7**, 157–197, (1999).

[30] M. Guediri, *On the geodesic connectedness of simply connected Lorentz surfaces*, Ann. Fac. Sci. Toulouse Math., **6**, 499-510, (1997).

[31] F. Giannoni and M. Lombardi, *Gravitational lenses: odd or even images?*, Class. Quant. Grav., **16**, 1689-1694, (1999).

[32] W.B. Gordon, *The existence of geodesics joining two given points*, J. Diff. Geom., **9**, 443-450, (1974).

[33] N. Hicks, *An example concerning affine connexions*, Proc. Amer. Math. Soc., **11**, 952-956, (1960).

[34] V.V. Kozlov, *Calculus of variations in the large and classical mechanics*, Russian Math. Surveys **440**, 37-71, (1985).

[35] A. Masiello, *Variational methods in Lorentzian Geometry*, Longman Sc. Tech., Harlow, Essex (1994).

[36] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York, (1983).

[37] A. Romero and M. Sánchez, *On the completeness of geodesics obtained as a limit*, J. Math. Phys., **34**, 3768–3774, (1993).

[38] M. Sánchez, *Geodesic connectedness in Generalized Reissner–Nordström type Lorentz manifolds*, Gen. Relat. Gravit., **29**, 1023-1037, (1997).

[39] M. Sánchez, *Some remarks on Causality and Variational Methods in Lorentzian manifolds*, Conf. Sem. Mat. Univ. Bari, **265**, 1-12, (1997).

[40] M. Sánchez, *Structure of Lorentzian tori with a Killing vector field*, Trans. AMS., **349**, 1063-1080, (1997).

[41] M. Sánchez, *Geodesic connectedness of semi-Riemannian manifolds*, Nonlinear Anal., **47**, 3085-3102, (2001).

[42] H.J. Seifert, *Global connectivity by timelike geodesics*, Z. Naturefor. **22a**, 1356, (1970).

[43] J.W. Smith, *Lorentz structures on the plane*, Trans. AMS., **95**, 226-237, (1960).

[44] K. Uhlenbeck, *A Morse theory for geodesics on a Lorentz manifold*, Topology, **14**, 69-90, (1975).

[45] T. WEINSTEIN, *An introduction to Lorentz surfaces*, Walter de Gruyter, New York (1996).

[46] J.A. WOLF, *Spaces of constant curvature*, MacGraw-Hill, New York (1967).

Proceedings of the meeting
Lorentzian Geometry-Benalmádena 2001
Benalmádena, Málaga, Spain
Pub. de la RSME, Vol. 5 (2003), 239-246

On the intersection of geometrical structures

Ignacio Sánchez Rodríguez[1]

[1] *Departmento de Geometría y Topología*
Universidad de Granada

Abstract

We give a theorem about intersection of reductions of a principal fiber bundle. As an application, we show that the intersections of conformal and volume structures, considered as G–structures of first order, are precisely the (semi)Riemannian structures. Also, we can apply it to the intersection of both, a projective structure and the first prolongation of a volume structure, considered as G–structures of second order. A possible application for a better understanding of the General Relativity theory is pointed out.

1 Introduction

The most of differential geometrical structures commonly used can be understood as G–structures of first or second order. A G–structure of first order (or, simply, a G–structure) on a manifold M is a reduced bundle of the linear frame bundle LM with structure group a subgroup G of $GL(n, \mathbb{R})$. Examples of G–structures that we are interested in are (semi)Riemannian, conformal and volume structures. A G–structure of second order is a reduced bundle of the second order frame bundle $F^2(M)$ with structure group a subgroup G of $G^2(n)$. Examples of it are symmetric linear connections, projective structures and the first prolongations of G–structures of first order which admit symmetric connections.

In this communication we state a result about intersection of reductions of principal fiber bundles, which has an immediate lecture in terms of G–structures of first or second order. Two applications of this result are given. The first shows that (semi)Riemannian structures belonging to a given conformal structure on a manifold are in bijective correspondence with the volume structures on the manifold (for us, volume structure refers to a little generalization of volume element, which does not need the orientability of the manifold to be defined). A second application shows that a given volume structure selects a symmetric linear connection belonging to a given projective structure.

The General Relativity theory maintains that the space–time geometry is given by a Lorentzian metric structure. It is well understood ([3]) that the physical phenomenon of light propagation determines a Lorentzian conformal structure. Then, the first application suggests us to investigate in the physical motivation that would conduce to the introduction of a volume structure as an ingredient of the space-time geometry.

2 A theorem on intersection of reduced bundles

We will understand a manifold M as a C^∞, second countable, manifold of dimension n. Let G be a Lie group. Let H be a closed subgroup of G. Let $\mu^{\mathrm{G,H}} : \mathrm{G} \times (\mathrm{G/H}) \to \mathrm{G/H}$, $\mu^{\mathrm{G,H}}(a, b\mathrm{H}) \equiv \mu_a^{\mathrm{G,H}}(b\mathrm{H}) := ab\mathrm{H}$, be the natural left action of G on the homogeneous manifold G/H.

It is well known ([7, Ch.I, Prop.5.6]) the bijective correspondence between the H–reductions of a principal bundle, $P(M, \mathrm{G})$, and the sections of its associated bundle which corresponds to the left action $\mu^{\mathrm{G,H}}$. We already know that the sections of an associated bundle are in bijective correspondence with the equivariant functions of the principal bundle into the typical fibre of the associated bundle, G/H in our case. Then we can prove the following result. This result can be obtained as a consecuence of the work of Bernard ([2, Sec. I.6]) but we prefer this approach technically more clear and perfectly adapted to the applications which we are interested in.

Theorem 2.1 *Let* H, K *be two closed subgroups of a Lie group* G *such that* $\mathrm{G} = \mathrm{HK}$ *(i.e.* $\forall a \in \mathrm{G}$, $\exists b \in \mathrm{H}$, $c \in \mathrm{K}\colon a = bc$*). Let* $Q(M, \mathrm{H})$ *and* $R(M, \mathrm{K})$ *be two reductions of a principal bundle* $P(M, \mathrm{G})$*. Then,* $Q \cap R$ *is a reduced bundle of* P*, with* $\mathrm{H} \cap \mathrm{K}$ *as structure group.*

We give a previous lemma.

Lemma 2.2 *Let* H, K *be two closed subgroups of a Lie group* G *such that* G = HK. *Then, the application* $\rho\colon \mathrm{G}/\mathrm{K} \to \mathrm{H}/(\mathrm{H}\cap\mathrm{K})$, $\rho(a\mathrm{K}) := b(\mathrm{H}\cap\mathrm{K})$, *with* $b^{-1}a \in \mathrm{K}$, *is a diffeomorphism.*

Proof. We prove that ρ is well defined: (i) Since G = HK, given $a \in \mathrm{G}$, there exists $b \in \mathrm{H}$, with $b^{-1}a \in \mathrm{K}$, and, if other $\hat{b} \in \mathrm{H}$ also verifies $\hat{b}^{-1}a \in \mathrm{K}$ then, $\mathrm{H} \ni b^{-1}\hat{b} = (b^{-1}a)(\hat{b}^{-1}a)^{-1} \in \mathrm{K}$, which implies that $b(\mathrm{H}\cap\mathrm{K}) = \hat{b}(\mathrm{H}\cap\mathrm{K})$. (ii) If $a\mathrm{K} = \hat{a}\mathrm{K}$ and $b^{-1}a,\ \hat{b}^{-1}\hat{a} \in \mathrm{K}$, with $b,\ \hat{b} \in \mathrm{H}$, then $\mathrm{H} \ni b^{-1}\hat{b} = (b^{-1}a)(a^{-1}\hat{a})(\hat{b}^{-1}\hat{a})^{-1} \in \mathrm{K}$, which implies that $b(\mathrm{H}\cap\mathrm{K}) = \hat{b}(\mathrm{H}\cap\mathrm{K})$.

The application ρ is bijective: (i) It is clearly onto. (ii) If $a, \hat{a} \in \mathrm{G}$ and $b(\mathrm{H}\cap\mathrm{K}) = \hat{b}(\mathrm{H}\cap\mathrm{K})$, with $b,\ \hat{b} \in \mathrm{H}$ and $b^{-1}a,\ \hat{b}^{-1}\hat{a} \in \mathrm{K}$, then $a^{-1}\hat{a} = (b^{-1}a)^{-1}(b^{-1}\hat{b})(\hat{b}^{-1}\hat{a}) \in \mathrm{K}$, which implies that $a\mathrm{K} = \hat{a}\mathrm{K}$.

The application ρ^{-1} maps $b(\mathrm{H}\cap\mathrm{K})$ into $b\mathrm{K}$. This is an immersion ([5, Ch.II, Prop.4.4(a)]). But a bijective immersion is a diffeomorphism ([9, Ch.I, Exer.6]). Thus ρ is a diffeomorphism.

$\square$

Proof of the theorem. Let $f\colon P \to \mathrm{G}/\mathrm{K}$ be the function $\mu^{\mathrm{G,K}}$–equivariant corresponding to $R(M,\mathrm{K})$. This means that, $\forall a \in \mathrm{G}$, $f \circ \mathcal{R}^{P}_{a} = \mu^{\mathrm{G,K}}_{a^{-1}} \circ f$, with $\mathcal{R}^{P}$ being the principal right action of G on P, and $f^{-1}(\{\mathrm{K}\}) = R$. We will prove that $Q \cap R$ is a reduction of Q, which corresponds to the equivariant function $\rho \circ f|_{Q}\colon Q \to \mathrm{H}/(\mathrm{H}\cap\mathrm{K})$:

(i) Let $d \in \mathrm{H}$ and $a \in \mathrm{G}$ be given, and let $b \in \mathrm{H}$ be such that $b^{-1}a \in \mathrm{K}$. We obtain that

$$\begin{aligned}(\rho \circ \mu^{\mathrm{G,K}}_{d^{-1}})(a\mathrm{K}) &= \rho(d^{-1}a\mathrm{K}) = \rho(d^{-1}b\mathrm{K}) = d^{-1}b(\mathrm{H}\cap\mathrm{K}) \\ &= d^{-1}\rho(a\mathrm{K}) = (\mu^{\mathrm{H,H\cap K}}_{d^{-1}} \circ \rho)(a\mathrm{K})\,.\end{aligned}$$

Now, given $q \in Q$, $d \in \mathrm{H}$, we obtain that

$$\begin{aligned}(\rho \circ f|_{Q} \circ \mathcal{R}^{Q}_{d})(q) &= (\rho \circ f \circ \mathcal{R}^{P}_{d})(q) = (\rho \circ \mu^{\mathrm{G,K}}_{d^{-1}} \circ f)(q) = \\ &= (\mu^{\mathrm{H,H\cap K}}_{d^{-1}} \circ \rho \circ f|_{Q})(q)\,.\end{aligned}$$

Thus the function $\rho \circ f|_{Q}$ is $\mu^{\mathrm{H,H\cap K}}$–equivariant.

(ii) Given $q \in Q$, if $(\rho\circ f|_{Q})(q) = \mathrm{H}\cap\mathrm{K}$, then we have $f(q) = f|_{Q}(q) = \rho^{-1}(\mathrm{H}\cap\mathrm{K}) = \mathrm{K}$, which implies that $q \in R$. Thus $(\rho \circ f|_{Q})^{-1}(\{\mathrm{H}\cap\mathrm{K}\}) = Q \cap R$.

The theorem follows from the fact that a reduction, $Q \cap R$, of a reduction Q of P is a reduction of P.

$\square$

As an $(\mathrm{H} \cap \mathrm{K})$–reduction of P trivially extends to an H–reduction and to a K–reduction of P, it is inmediate to prove the following result.

Corollary 2.3 *Let* H, K *be two closed subgroups of a Lie group* G *such that* $\mathrm{G} = \mathrm{HK}$. *Let* $P(M, \mathrm{G})$ *be a principal bundle. The* $(\mathrm{H}\cap\mathrm{K})$*–reductions of* P *are precisely the intersections of* H*–reductions with* K*–reductions of* P.

3 Conformal and volume structures

Let G be a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$. A G–*structure of first order* on a manifold M is a G–reduction of the linear frame bundle LM.

Let η be the standard scalar product on $\mathbb{R}^n$ of a fixed signature. We define the adjoint with respect to η, $a^\dagger$, of $a \in \mathrm{GL}(n, \mathbb{R})$ as the unique matrix such that $\eta(v, a^\dagger w) = \eta(av, w)$, $\forall v, w \in \mathbb{R}^n$. A *conformal structure* on M is a G–structure with $\mathrm{G} = \mathrm{CO}(n) := \{a \in \mathrm{GL}(n, \mathbb{R})\colon a^\dagger a = kI,\ k > 0\}$, where I is the identity matrix in $\mathrm{GL}(n, \mathbb{R})$.

We define a *volume structure* on M as a G–structure with $\mathrm{G} = \mathrm{SL}^{\pm}(n) := \{a \in \mathrm{GL}(n, \mathbb{R})\colon |\mathrm{d}et(a)| = 1\}$. Note that the existence of volume structures does not depend on the orientability of M as in the case of $\mathrm{SL}(n)$–structures.

Theorem 3.1 *The (semi)Riemannian structures on* M *are the intersections of conformal and volume structures on* M.

Proof. It is clear that

$$\mathrm{CO}(n) \cap \mathrm{SL}^{\pm}(n) = \mathrm{O}(n) := \{a \in \mathrm{GL}(n, \mathbb{R})\colon a^\dagger a = I\}.$$

Then, by the results of the previous section, we only need to prove that $\mathrm{GL}(n, \mathbb{R}) = \mathrm{CO}(n)\mathrm{SL}^{\pm}(n)$. This follows from the fact that

$$a = (|\mathrm{d}et(a)|^{1/n} I)\,(|\mathrm{d}et(a)|^{-1/n} a),\ \forall a \in \mathrm{GL}(n, \mathbb{R})\ .$$

$\square$

It is usual to define a conformal structure on M as the set $[g]$ of metric tensors which are proportional by a positive factor to a given metric tensor g on M, i.e. $g' \in [g]$ if and only if $g' = \omega g$, with $\omega\colon M \to \mathbb{R}^+$. In this context, the $\mathrm{CO}(n)$–structure P corresponding to $[g]$ is composed of all the linear frames $l \in LM$, that considered as basis of the tangent space T_mM in some point $m \in M$, are orthonormal for some $g' \in [g]$.

We can also understand a volume structure $Q \subset LM$ as a selection, for every point $m \in M$, of a maximal set of basis of T_mM with the same unoriented volume, in the sense of linear algebra.

It is intuitively clear that if we intersect a conformal structure P or, equivalently, $[g]$ and a volume structure Q, we are selecting, at each point $m \in M$, all the linear frames of P with the same unoriented volume defined by Q. But this procedure is equivalent to select the tensor metric in $[g]$ for which these linear frames are orthonormal. This metric is unique because two tensor metrics, which are proportional and distinct in a point $m \in M$, have orthonormal basis in T_mM with different volume.

4 G–structures of second order

The *second order frame bundle* $F^2(M)$ over a manifold M (see [6, Ch.4, Sec.5] and see [8] for more details) is the principal fibre bundle, whose fibre over each point $m \in M$ is the set of *second order frames* at m, i.e. the set of 2–jets

$$\{j^2_0(x^{-1})\colon x \text{ is a chart of } M \text{ with } x(m) = 0\}.$$

Its structure group is the Lie group of 2–jets

$$\mathrm{G}^2(n) := \{j^2_0(\phi)\colon \phi \text{ is a local diffeomorphism of } \mathbb{R}^n \text{ with } \phi(0) = 0\}.$$

Let G be a subgroup of $\mathrm{G}^2(n)$. A G*–structure of second order* on M is a G–reduction of $F^2(M)$.

Let $\mathrm{S}^2(n)$ be the set of symmetric bilinear maps $t\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, considered as an additive Lie group. For each $t \in \mathrm{S}^2(n)$, we will write $t^i_{jk} \equiv u^i(t(e_j, e_k))$, with $\{e_1, \dots, e_n\}$ and $\{u^1, \dots, u^n\}$ being the usual basis of $\mathbb{R}^n$ and $\mathbb{R}^{n*}$, respectively. We set $\mathrm{GL}(n,\mathbb{R}) \otimes \mathrm{S}^2(n)$ for the semidirect product of Lie groups, whose product law is given by $(a, t)\cdot(a', t') := (aa', a'^{-1}t(a', a') + t')$. There is a canonical isomorphism ([8, Lem.1], see also [1, Sec.4]) between $\mathrm{G}^2(n)$ and $\mathrm{GL}(n,\mathbb{R}) \otimes \mathrm{S}^2(n)$ given by the application that maps $j^2_0(\phi)$ into $(\left.D\phi\right|_0 \,,\, \left.D\phi\right|_0^{-1} \left.D^2\phi\right|_0)$. We will identify both groups.

Examples of second order G–structures are the following:

- A symmetric linear connection of M can be identified ([6, Ch.4, Prop.7.1]) as a G–structure of second order on M, with $\mathrm{G} = \mathrm{GL}(n, \mathbb{R}) \otimes \{0\}$. It is composed of the 2–jets at 0 of the inverse of all normal charts for the symmetric linear connection ([8]).

- A projective structure on M ([6, Ch.4, Prop.7.1]) can be identified with a G–structure of second order on M, with $\mathrm{G} = \mathrm{GL}(n, \mathbb{R}) \otimes \mathfrak{p}$, where

$$\mathfrak{p} := \{t \in \mathrm{S}^2(n) \colon t^i_{jk} = \delta^i_j p_k + \delta^i_k p_j, \text{ for some } (p_1, \ldots, p_n) \in \mathbb{R}^{n*}\}.$$

  It is composed of the union of the G–structures of second order corresponding to the projectively equivalent symmetric connections which belong to the projective structure.

- We say that a G–structure of first order P is 1–*integrable* if it admits a symmetric linear connection. Semiriemannian, conformal and volume structures are examples of 1–integrable G–structures. The first prolongation P_1 of an 1–integrable G–structure P is ([8, Ch.4, Sec.3.3]) unique and it can be identified with an H–structure of second order with $\mathrm{H} = \mathrm{G} \otimes \mathfrak{g}_1$, where $\mathfrak{g}_1$ denote the first prolongation of the Lie algebra $\mathfrak{g}$ of G. We can see P_1 as the set of 2–jets at 0 of the inverse of all normal charts for all symmetric linear connections that can be defined in P.

With the identifications introduced above, it can be shown the following result.

Theorem 4.1 *The intersection of a projective structure on M and the first prolongation of a volume structure on M gives a symmetric linear connection of M.*

Proof. Since the first prolongation of the Lie algebra $\mathfrak{sl}(n)$ of $\mathrm{SL}^{\pm}(n)$ is $\mathfrak{sl}(n)_1 = \{t \in \mathrm{S}^2(n) \colon t^h_{hk} = 0\}$, then

$$0 = t^h_{hk} = \delta^h_h p_k + \delta^h_k p_h = (n+1)p_k, \ \forall k \in \{1, \ldots, n\},$$

thus $t = 0$. This implies that

$$(\mathrm{GL}(n, \mathbb{R}) \otimes \mathfrak{p}) \cap (\mathrm{SL}^{\pm}(n) \otimes \mathfrak{sl}(n)_1) = \mathrm{SL}^{\pm}(n) \otimes \{0\}.$$

Moreover, it is readily verified that

$$\mathrm{GL}(n, \mathbb{R}) \otimes \mathrm{S}^2(n) = (\mathrm{GL}(n, \mathbb{R}) \otimes \mathfrak{p}) \cdot (\mathrm{SL}^{\pm}(n) \otimes \mathfrak{sl}(n)_1)$$

since $(a,t) = (a,r) \cdot (I,s), \quad \forall (a,t) \in \mathrm{GL}(n,\mathbb{R}) \otimes \mathrm{S}^2(n)$, where

$$r^i_{jk} := \frac{1}{n+1}(\delta^i_j t^h_{hk} + \delta^i_k t^h_{hj}) \text{ and } s^i_{jk} := t^i_{jk} - \frac{1}{n+1}(\delta^i_j t^h_{hk} + \delta^i_k t^h_{hj})\,.$$

Then, by Theorem 1, the intersection of a projective structure on M and the first prolongation of a volume structure on M is a $\mathrm{SL}^{\pm}(n) \otimes \{0\}$–structure of second order. This extends trivially to a $\mathrm{GL}(n,\mathbb{R}) \otimes \{0\}$–structure of second order naturally included in the projective structure.

□

In other words, this result says that a volume structure on M select a connection of a projective class of linear symmetric connections.

5 Remarks on Lorentzian geometry and General Relativity

Several geometrical structures can be derived from a (semi)Riemannian metric: a conformal structure, a symmetric linear connection and a projective structure. The fact that a metric is the intersection of a conformal structure and a volume structure allows the metric to be considered derived from the conformal and volume structures. From my point of view, the understanding of a metric structure as being composed by these two pieces can be used to gain an insight into the meaning of the General Relativity theory.

The phenomenon of light propagation is described geometrically by a field of light cones which determines a Lorentzian conformal structure. It would be very interesting to identify some substantial physical phenomenon as being represented by a volume structure. In this way, two physical principles will lead up to the Lorentzian metric proposed by the Relativity theory.

Some authors ([3], [4]) have tried to give an axiomatic approach to General Relativity by deriving the metric structure from the conformal and projective structures. The projective structure would explain the mouvement of free particles. But this approach is mathematically more complicated because some extra conditions are needed to determine a Lorentzian metric, except for a constant factor.

Acknowledgments

This work has been partially supported by Junta de Andalucía P.A.I.: FQM0203.

References

[1] E. Aguirre Dabán and I. Sánchez Rodríguez, *Explicit formulas for the 3–jet lift of a matrix group. Aplications to conformal geometry*, in: A. Pereira do Vale and M.R. Pinto, eds., *Proceedings of the 1st International Meeting on Geometry and Topology* (Universidade do Minho, Braga, Portugal, 1998) 191–205.

[2] D. Bernard, *Sur la géométrie différentielle des G-structures*, Ann. Inst. Fourier, Grenoble, **10**, 151-270, (1960).

[3] J. Ehlers, F.A.E. Pirani and A. Schild, *The geometry of free fall and light propagation*, in: L. O'Raifeartaigh, ed., *General Relativity* (Oxford, 1972) 63–84.

[4] J. Ehlers, *Survey of General Relativity Theory*, in: W. Israel, ed., *Relativity, Astrophysics and Cosmology* (Dordrecht–Holland, 1973) 1–124.

[5] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces* (Academic Press, New York, 1978).

[6] S. Kobayashi, *Transformation Groups in Differential Geometry* (Springer, Heidelberg, 1972).

[7] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol.I* (John Wiley, New York, 1963).

[8] I. Sánchez Rodríguez, *Conexiones en el fibrado de referencias de segundo orden. Conexiones conformes* (Thesis, Department of Geometry and Topology, Complutense University, Madrid, 1994).

[9] F.W. Warner, *Foundations of Differentiable Manifolds and Lie Groups* (Scott, Foresman and Co., Glenview, Illinois, 1971).

Proceedings of the meeting
Lorentzian Geometry-Benalmádena 2001
Benalmádena, Málaga, Spain
Pub. de la RSME, Vol. 5 (2003), 247-260

The Hyperbolic-AntiDeSitter-DeSitter triality

Mariano Santander[1]

[1]*Departamento de Física Teórica, Facultad de Ciencias*
Universidad de Valladolid, E47011-Valladolid, Spain
email: santander@fta.uva.es

Abstract

A triality $\mathcal{T}$ relating the four two-dimensional spaces of real type with constant non-zero curvature and non-degenerate metric is described. The sphere $\mathbf{S^2}$ is invariant under $\mathcal{T}$, yet the remaining three spaces, the hyperbolic plane $\mathbf{H^2}$, the AntiDeSitter sphere $\mathbf{AdS}^{1+1}$ and the DeSitter sphere $\mathbf{dS}^{1+1}$ are related *cyclically* by triality: any geometric property in either of these spaces might be ultimately reformulated in terms of any other. Our approach is based on Lie algebras, but possible alternative approaches through Hopf projections and likely relations to the octonionic triality when the base field is extended from $\mathbb{R}$ to $\mathbb{C}, \mathbb{H}, \mathbb{O}$ are also suggested.

1 Introduction

The aim of the present work is to introduce and discuss a *triality* between three well known two-dimensional spaces: the Hiperbolic plane $\mathbf{H}^2$ and the two AntiDeSitter $\mathbf{AdS}^{1+1}$ and DeSitter $\mathbf{dS}^{1+1}$ spheres in $1+1$ dimensions (We conform to the 'physical' notation; the 'mathematical' notation for these AntiDeSitter and DeSitter spheres —which for $1+1$ only differ by the interchange between time-like and space like lines— is $\mathbf{H}^2_1$ and $\mathbf{S}^2_1$). This triality follows from rather elementary properties of the Lie group $SO(2,1)$ and Lie algebra $\mathfrak{so}(2,1)$ behind these geometries, but it seems to have passed unnoticed. It is related to a triality recently discussed by Arnol'd [1] in relation with the geometry of spherical curves on $\mathbf{S}^2$. We focus here on the algebraic description, at the Lie algebra level, and include only a descriptive account of results for the geometry of the three spaces.

2 Homogeneous spaces of Cayley-Klein type

We first briefly recall the algebraic structure of the nine 2D Cayley-Klein (CK) spaces (for more details see [2]). These *nine* spaces are 2-dimensional homogeneous spaces of rank one and real type, and are distinguished by two geometric properties: the sign of their *constant curvature* —either positive, zero or negative—, and the character of their metric —either definite positive (riemannian), degenerate or indefinite (hence lorentzian in this case)—. According to these two threefold alternatives, the nine spaces could be arranged in a diagram as in Table 1.

For the reasons explained below it is adviseable to describe all these possibilities through two *real* coefficients, κ_1, κ_2 and to denote $SO_{\kappa_1,\kappa_2}(3)$ the corresponding groups of motion. The commutation relations of the CK algebra $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$ in the basis $\{P_1, P_2, J\}$ (where the notation intends to convey the idea of P_1, P_2 generating translations along two orthogonal lines l_1, l_2 through O and J generating rotations around O) read:

$$[P_1, P_2] = \kappa_1 J \qquad [J, P_1] = P_2 \qquad [J, P_2] = -\kappa_2 P_1 \tag{2.1}$$

The constants κ_1, κ_2 can be reduced to $+1, 0, -1$ by rescaling the generators. The CK algebras in the quasi-orthogonal family $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$ can be endowed with a $\theta_2 \otimes \theta_2$ group of commuting automorphisms generated by:

$$\begin{aligned} \Pi_{(1)} &: (P_1, P_2, J) \to (-P_1, -P_2, J), \\ \Pi_{(2)} &: (P_1, P_2, J) \to (P_1, -P_2, -J). \end{aligned} \tag{2.2}$$

The two remaining involutions are the composition $\Pi_{(02)} = \Pi_{(1)} \cdot \Pi_{(2)}$ and the identity. Each involution Π determines a subalgebra of $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$, denoted $\mathfrak{h}$, whose elements are invariant under Π; the subgroups generated by these subalgebras will be denoted by H with the same subindices as the involution.

The following 3D real matrix representation of the CK algebra $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$:

$$P_1 = \begin{pmatrix} 0 & -\kappa_1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; P_2 = \begin{pmatrix} 0 & 0 & -\kappa_1\kappa_2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\kappa_2 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.3}$$

gives rise through exponentiation to a natural realization of the CK group $SO_{\kappa_1,\kappa_2}(3)$ as a group of linear transformations in an ambient linear space $\mathbb{R}^3 = (x^0, x^1, x^2)$. A generic element $R \in SO_{\kappa_1,\kappa_2}(3)$ satisfies

$$R^T\,\Lambda_{\kappa_1,\kappa_2}\,R = \Lambda_{\kappa_1,\kappa_2}, \qquad \det R = 1, \tag{2.4}$$

and therefore $SO_{\kappa_1,\kappa_2}(3)$ acts in $\mathbb{R}^3$ as linear isometries of a bilinear form with a diagonal $\Lambda_{\kappa_1,\kappa_2}$ matrix whose entries are $\{+1, \kappa_1, \kappa_1\kappa_2\}$.

The exponential of the matrices (2.3) leads to a representation of the one-parametric subgroups $H_{(2)}$, $H_{(02)}$ and $H_{(1)}$ generated by P_1, P_2 and J as:

$$\begin{aligned}
\exp(\alpha P_1) &= \begin{pmatrix} C_{\kappa_1}(\alpha) & -\kappa_1 S_{\kappa_1}(\alpha) & 0 \\ S_{\kappa_1}(\alpha) & C_{\kappa_1}(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\exp(\beta P_2) &= \begin{pmatrix} C_{\kappa_1\kappa_2}(\beta) & 0 & -\kappa_1\kappa_2 S_{\kappa_1\kappa_2}(\beta) \\ 0 & 1 & 0 \\ S_{\kappa_1\kappa_2}(\beta) & 0 & C_{\kappa_1\kappa_2}(\beta) \end{pmatrix} \\
\exp(\gamma J) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{\kappa_2}(\gamma) & -\kappa_2 S_{\kappa_2}(\gamma) \\ 0 & S_{\kappa_2}(\gamma) & C_{\kappa_2}(\gamma) \end{pmatrix}
\end{aligned} \tag{2.5}$$

where the 'cosine' $C_\kappa(x)$ and 'sine' $S_\kappa(x)$ functions with 'label' κ are [2]:

$$C_\kappa(x) := \begin{cases} \cos\sqrt{\kappa}\,x \\ 1 \\ \cosh\sqrt{-\kappa}\,x \end{cases} \qquad S_\kappa(x) := \begin{cases} \frac{1}{\sqrt{\kappa}}\sin\sqrt{\kappa}\,x & \kappa > 0 \\ x & \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}}\sinh\sqrt{-\kappa}\,x & \kappa < 0 \end{cases} \tag{2.6}$$

These 'labelled' trigonometric functions coincide with the usual elliptic and hyperbolic ones for $\kappa = 1$ and $\kappa = -1$ respectively; the case $\kappa = 0$ provides the parabolic or galilean functions: $C_0(x) = 1$, $S_0(x) = x$ [2].

The CK *plane* (as the *space of points*) corresponds to the 2D symmetric homogeneous space

$$S^2_{[\kappa_1],\kappa_2} = SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_2}(2) \qquad SO_{\kappa_2}(2) = \langle J \rangle, \tag{2.7}$$

hence the generator J leaves a point O (the origin) invariant, thus generating rotations around O, while P_1, P_2 generate translations that move O along two basic orthogonal directions, as implied by the notation. As any symmetric homogeneous space it has a canonical connection and an invariant metric (see below).

Alternatively, one may consider the *space of (actual) lines*, defined as the 2D symmetric homogeneous space

$$S^2_{\kappa_1,[\kappa_2]} = SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_1}(2) \qquad SO_{\kappa_1}(2) = \langle P_1 \rangle. \tag{2.8}$$

One may also introduce a third related space of *ideal* lines, by taking the quotient by the subgroup generated by the subalgebra invariant under the third involution $\Pi_{(02)}$. Both spaces of lines come also equipped with a canonical connection (as any symmetric homogeneous space) and an invariant metric; indeed the lines (as points in the line spaces) can be identified to *geodesics* of the invariant metric in the space of points, and conversely so everything may be ultimately formulated in terms of the single *space of points* $S^2_{[\kappa_1],\kappa_2}$.

Table 1: The nine two-dimensional CK spaces $S^2_{[\kappa_1],\kappa_2} = SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_2}(2)$.

Spherical: $\mathbf{S}^2$ $S^2_{[+],+} = SO(3)/SO(2)$	Euclidean: $\mathbf{E}^2$ $S^2_{[0],+} = ISO(2)/SO(2)$	Hyperbolic: $\mathbf{H}^2$ $S^2_{[-],+} = SO(2,1)/SO(2)$
Oscillating NH: $\mathbf{NH}^{1+1}_+$ (Co-Euclidean) $S^2_{[+],0} = ISO(2)/ISO(1)$	Galilean: $\mathbf{G}^{1+1}$ $S^2_{[0],0} = IISO(1)/ISO(1)$	Expanding NH: $\mathbf{NH}^{1+1}_-$ (Co-Minkowskian) $S^2_{[-],0} = ISO(1,1)/ISO(1)$
Anti-de Sitter: $\mathbf{AdS}^{1+1}$ (Co-Hyperbolic) $S^2_{[+],-} = SO(2,1)/SO(1,1)$	Minkowskian: $\mathbf{M}^{1+1}$ $S^2_{[0],-} = ISO(1,1)/SO(1,1)$	De Sitter: $\mathbf{dS}^{1+1}$ (Doubly Hyperbolic) $S^2_{[-],-} = SO(2,1)/SO(1,1)$

A *linear model* of the CK space is obtained through the natural linear action of $SO_{\kappa_1,\kappa_2}(3)$ on $\mathbb{R}^3$ which is not transitive, since it conserves the quadratic form $(x^0)^2 + \kappa_1(x^1)^2 + \kappa_1\kappa_2(x^2)^2$. The subgroup $H_{(1)}$, whose matrix representation is $\exp(\gamma J)$ (2.5), is the isotropy subgroup of the point $O \equiv (1,0,0)$, that is, the origin in the space $S^2_{[\kappa_1],\kappa_2}$. The action becomes transitive on the orbit in $\mathbb{R}^3$ of the point O, which is contained in the 'sphere' Σ:

$$\Sigma \equiv (x^0)^2 + \kappa_1(x^1)^2 + \kappa_1\kappa_2(x^2)^2 = 1; \tag{2.9}$$

this orbit can be identified with the CK space $S^2_{[\kappa_1],\kappa_2} \equiv SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_2}(2)$. The invariant metric in this space comes as the quotient by κ_1 of the restriction of the flat ambient metric $dl^2 = (dx^0)^2 + \kappa_1(dx^1)^2 + \kappa_1\kappa_2(dx^2)^2$ to Σ, and the canonical connection is, for $\kappa_2 \neq 0$ the associated Levi-Civita connection (for explicit expressions and further details, see [3]). This scheme unifies all the familiar embeddings of the sphere, hyperbolic plane, anti-de Sitter and de Sitter spaces in a linear 3D ambient space, with a flat metric of either euclidean or lorentzian type (compare the usual models for the sphere in $\mathbb{R}^3$ and for hyperbolic space, deSitter and AntiDeSitter in $\mathbb{R}^{2+1}$).

The curvature and metric signature of these CK spaces are determined directly by κ_1, κ_2: the curvature of the canonical *metric* in the space $S^2_{[\kappa_1],\kappa_2}$ is *constant* and equals κ_1 (written inside square brackets in the space notation) and at the origin O and in the basis P_1, P_2 of the tangent space at O, the metric matrix is $\mathrm{diag}(1, \kappa_2)$; therefore κ_2 determines the metric *signature*.

For the purposes of the present work we shall consider only the four *generic* spaces (at the corners in Table 1) where the two CK constants are different from zero; these correspond to *simple* groups, either isomorphic to $SO(3)$ when $\kappa_1, \kappa_2 > 0$ or to a $SO(2,1)$ when any constant κ_1, κ_2 or both are negative.

3 Duality from a group theoretical point of view

A fundamental property of the scheme of CK geometries is the existence of an 'automorphism' of the whole family, called *ordinary duality* $\mathcal{D}$. This is well defined for any dimension, and in the 2D case is given by:

$$\mathcal{D} : (P_1, P_2, J) \to (\mathcal{P}_1, \mathcal{P}_2, \mathcal{J}) = (-J, -P_2, -P_1) \tag{3.1}$$

$$\mathcal{D} : (\kappa_1, \kappa_2) \to (\mathsf{k}_1, \mathsf{k}_2) = (\kappa_2, \kappa_1). \tag{3.2}$$

The map $\mathcal{D}$ leaves the general commutation rules (2.1) invariant while it interchanges the space of points with the space of actual lines, $S^2_{[\kappa_1],\kappa_2} \leftrightarrow S^2_{\kappa_1,[\kappa_2]}$, as well as the corresponding constants $\kappa_1 \leftrightarrow \kappa_2$.

From a purely formal viewpoint, this Lie algebra duality can be considered as a *family automorphism* in the whole CK family of Lie algebras: the image of the Lie algebra $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$ is another Lie algebra in the family, corresponding to a different choice of parameters $\mathsf{k}_1, \mathsf{k}_2$; for a fixed algebra $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$ in the family, the duality $\mathcal{D}$ establishes an *isomorphism* $\mathfrak{so}_{\kappa_1,\kappa_2}(3) \approx \mathfrak{so}_{\kappa_2,\kappa_1}(3)$.

When carried to the CK spaces through the exponential map and the realization (2.5), the duality $\mathcal{D}$ relates in general *two* different CK geometries. Taking into account the interpretation of κ_1 and κ_2 in terms of the differential geometry of the space $S^2_{[\kappa_1],\kappa_2}$ we obtain a result suggesting a kind of duality between *curvature* and *signature* which seems worth studying:

Theorem 3.1 *The dual of a CK space with curvature κ_1 and metric of signature type $(+, \kappa_2)$ is the CK space with curvature κ_2 and metric of signature type $(+, \kappa_1)$.*

Let us now ask how this duality relates the four generic CK spaces. Starting from either $\mathbf{S}^2, \mathbf{H}^2, \mathbf{dS}^{1+1}, \mathbf{AdS}^{1+1}$, whose associated CK constants reduced to their fiducial values ± 1 are $(1,1), (-1,1), (1,-1)$, $(-1,-1)$, it suffices to recall the action (3.2) of the duality $\mathcal{D}$ and the involutivity of $\mathcal{D}$ to conclude:

$$\mathcal{D} : \underset{(1,1)}{\mathbf{S}^2} \longleftrightarrow \underset{(1,1)}{\mathbf{S}^2}\ , \quad \mathcal{D} : \underset{(-1,1)}{\mathbf{H}^2} \longleftrightarrow \underset{(1,-1)}{\mathbf{AdS}^{1+1}}\ , \quad \mathcal{D} : \underset{(-1,-1)}{\mathbf{dS}^{1+1}} \longleftrightarrow \underset{(-1,-1)}{\mathbf{dS}^{1+1}}\ .$$

Hence the sphere $\mathbf{S}^2$ and the DeSitter space $\mathbf{dS}^{1+1}$ are autodual; hyperbolic plane $\mathbf{H}^2$ and the AntiDeSitter space $\mathbf{AdS}^{1+1}$ are mutually dual. In general, duality relates two geometries placed in symmetrical positions relative to the main diagonal in table 1.

This difference between $\mathbf{dS}^{1+1}$ and $\mathbf{AdS}^{1+1}$ may seem surprising as the two $1+1$ de Sitter spaces only differ by a change of sign in the metric but this difference turns out important when duality is considered. To carry the description of duality (and triality to be introduced later) from the Lie algebra to the space level requires a detailed exposition, not to be done here, whose main idea is that in any CK geometry, points and lines may be either of a single type (to be considered simultaneously as *actual* or *ideal*), of a generic actual type and a limiting (*final*) type or of *two different* generic types (actual *and* ideal) separated by a common limiting type (final). Which alternative applies depend on whether the associated label —κ_1 for points, κ_2 for lines— is $>, =0, <0$. For instance, if $\kappa_2 < 0$ (AntiDeSitter, Minkowski and DeSitter), the existence of three types of lines, either actual (time-like), final (light-like) and ideal (space-like) is well known. A similar situation applies for points according to κ_1; in $\mathbf{S}^2$ all points are of a single type (actual *or* ideal), in $\mathbf{E}^2$ there are actual points and also points at infinity (final), and in $\mathbf{H}^2$ further to actual and final points there are points at ‘ultrainfinity’ or ideal points, where ultraparallel lines ‘meet’, and which are different from the actual ones.

Points/lines have associated measures of separation (distances/angles), which have also an associated label. For instance, the fiducial point O is invariant under the subgroup generated by J, and the measure of separation (angle) between two actual or ideal lines through O may be *defined* as the value of the canonical parameter θ of the unique element in the one-parameter subgroup $\mathrm{e}^{\theta J}$ which carries a line onto the other (this definition is alternative but consistent to the usual one through the metric); thus θ is related to $J; \kappa_2$. The measures of separation (distances) between points can be similarly *defined* as the canonical parameters of suitable one-parameter subgroups carrying the first point to the second. They are either associated to $P_1; \kappa_1$ (actual distances) or to $P_2; \kappa_1\kappa_2$ (ideal distances). Non-intersecting actual lines have a (generically unique) common ideal perpendicular, and non-intersecting ideal lines have a common actual perpendicular; for these the natural measure of separation is a distance, associated to either $P_2; \kappa_1\kappa_2$ or $P_1; \kappa_1$.

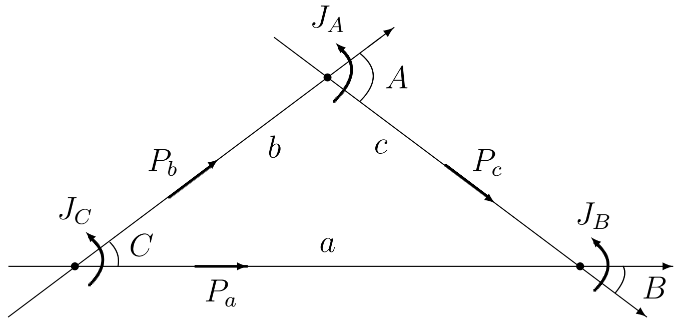
The Table contains the relationships between the geometric elements in the given space S and its dual $\mathcal{D}(S)$ in a schematic form. For clarity the Table makes reference only to the fiducial choices for O, l_1, l_2, but duality applies to all points, actual lines and ideal lines in the given CK geometry and relate them to objects in the dual space.

Dual CK space $\mathcal{D}$(S)	***versus***	**Original CK space S**
Points (invariant under $\mathcal{J} = -P_1$)	•	Actual lines (invariant under P_1)
Distance between points (along $\mathcal{P}_1 = -J$)	•	Angle between actual lines (along J)
Actual lines (invariant under $\mathcal{P}_1 = -J$)	•	Points (invariant under J)
Angle between actual lines (along $\mathcal{J} = -P_1$)	•	Distance between points (along P_1)
Ideal lines (invariant under $\mathcal{P}_2 = -P_2$)	•	Ideal lines (invariant under P_2)
Angle between ideal lines (along $\mathcal{J} = -P_1$)	•	Actual distance between ideal lines (along P_1)

Theorem 3.2 *The set of points of the dual $\mathcal{D}(S)$ of a CK space S is the set of actual lines in the original space S, with the former angle between lines as measure of separation between points. The set of actual lines in the dual $\mathcal{D}(S)$ of a CK space S is the set of points in the original space S, with the former distance between points as measure of separation between lines.*

4 Self-dual Trigonometry of Cayley-Klein spaces

In order to provide a working example within this scheme [2], [4], let us display how the trigonometry of the nine CK spaces can be *encapsulated in a single self-dual equation*, which formally applies for for the nine CK spaces.



Let us consider a generic triangle with vertices A, B, C and whose sides are three actual lines (this is no restriction when $\kappa_2 > 0$ but for $\kappa_2 < 0$ this would restrict to 'time-like' sides). Let a, b, c denote the side lengths, and let A, B, C denote the angles at the three vertices, with the labelling chosen so that a is the largest side and the opposite angle at A is the *external angle* (see figure). The three generators of translations along the three sides, denoted P_a, P_b, P_c, are all conjugate to the fiducial generator P_1 of translations along the line l_1. The three generators of rotations J_A, J_B, J_C around the three vertices are conjugates of the fiducial rotation generator J around O. Now, we have:

Theorem 4.1 *[4]. The sides a, b, c and angles A, B, C of any triangle in the CK space $S^2_{[\kappa_1],\kappa_2} = SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_2}(2)$ are related by the following* exponential identities *in the group $SO_{\kappa_1,\kappa_2}(3)$ which further are equivalent:*

$$\mathrm{e}^{-aP_a}\mathrm{e}^{BJ_B}\mathrm{e}^{cP_c}\mathrm{e}^{-AJ_A}\mathrm{e}^{bP_b}\mathrm{e}^{CJ_C} = 1$$

$$\mathrm{e}^{-aP_a}\mathrm{e}^{cP_c}\mathrm{e}^{bP_b} = e^{-(-A+B+C)J_C}, \qquad \mathrm{e}^{BJ_B}\mathrm{e}^{-AJ_A}\mathrm{e}^{CJ_C} = e^{-(-a+b+c)P_a}$$

$$1 = \mathrm{e}^{aP_a}\mathrm{e}^{-BJ_C}\mathrm{e}^{-cP_a}\mathrm{e}^{AJ_C}\mathrm{e}^{-bP_a}\mathrm{e}^{-CJ_C}$$

There are no explicit κ_1, κ_2 constants in these equations, which therefore hold exactly in the *same form* for all CK spaces at once. And they are explicitly invariant under the interchange $P_1 \leftrightarrow J$ and $a, b, c \leftrightarrow A, B, C$, hence they are *self-dual.* All these four equations have a similar structure: on the left hand side there are 'triangular' translations along the three sides and/or rotations around the three vertices. These are either all six translations and rotations (1st row), only three translations *or* rotations (2nd row), or none at all (3rd row). On each equation, all transformations 'missing' in the left hand side appear at the right hand side under a 'fiducial' form, as translations along *a single side* a and/or rotations around *a single vertex* C.

If in the last equation one replaces the exponentials by means of (2.5) one gets directly nine real equations (coming from the nine elements of a 3×3 matrix identity). From these one obtains directly the trigonometric equations of the CK space (the limiting cases where either κ_1, κ_2 vanish are easily dealt with, see [2]):

- Three **cosine theorems** for sides:

$$\begin{aligned} C_{\kappa_1}(a) &= C_{\kappa_1}(b)C_{\kappa_1}(c) - \kappa_1 S_{\kappa_1}(b)S_{\kappa_1}(c)C_{\kappa_2}(A) \\ C_{\kappa_1}(b) &= C_{\kappa_1}(a)C_{\kappa_1}(c) + \kappa_1 S_{\kappa_1}(a)S_{\kappa_1}(c)C_{\kappa_2}(B) \\ C_{\kappa_1}(c) &= C_{\kappa_1}(a)C_{\kappa_1}(b) + \kappa_1 S_{\kappa_1}(a)S_{\kappa_1}(b)C_{\kappa_2}(C) \end{aligned}$$

- Three **cosine dual theorems** for angles:

$$\begin{aligned} C_{\kappa_2}(A) &= C_{\kappa_2}(B)C_{\kappa_2}(C) - \kappa_2 S_{\kappa_2}(B)S_{\kappa_2}(C)C_{\kappa_1}(a) \\ C_{\kappa_2}(B) &= C_{\kappa_2}(A)C_{\kappa_2}(C) + \kappa_2 S_{\kappa_2}(A)S_{\kappa_2}(C)C_{\kappa_1}(b) \\ C_{\kappa_2}(C) &= C_{\kappa_2}(A)C_{\kappa_2}(B) + \kappa_2 S_{\kappa_2}(A)S_{\kappa_2}(B)C_{\kappa_1}(c) \end{aligned}$$

- A **sine theorem**: $$\frac{S_{\kappa_1}(a)}{S_{\kappa_2}(A)} = \frac{S_{\kappa_1}(b)}{S_{\kappa_2}(B)} = \frac{S_{\kappa_1}(c)}{S_{\kappa_2}(C)}$$

Spherical $S^2_{[+],+}$	**Euclidean** $S^2_{[0],+}$	**Hyperbolic** $S^2_{[-],+}$
$\cos a = \cos b \cos c - \sin b \sin c \cos A$	$a^2 = b^2 + c^2 + 2bc \cos A$	$\cosh a = \cosh b \cosh c + \sinh b \sinh c \cos A$
$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$	$\frac{\sinh a}{\sin A} = \frac{\sinh b}{\sin B} = \frac{\sinh c}{\sin C}$
$\cos A = \cos B \cos C - \sin B \sin C \cos a$	$A = B + C$	$\cos A = \cos B \cos C - \sin B \sin C \cosh a$
Anti Newton-Hooke $S^2_{[+],0}$	**Galilean** $S^2_{[0],0}$	**Newton-Hooke** $S^2_{[-],0}$
$a = b + c$	$a = b + c$	$a = b + c$
$\frac{\sin a}{A} = \frac{\sin b}{B} = \frac{\sin c}{C}$	$\frac{a}{A} = \frac{b}{B} = \frac{c}{C}$	$\frac{\sinh a}{A} = \frac{\sinh b}{B} = \frac{\sinh c}{C}$
$A^2 = B^2 + C^2 + 2BC \cos a$	$A = B + C$	$A^2 = B^2 + C^2 + 2BC \cosh a$
AntiDeSitter $S^2_{[+],-}$	**Minkowskian** $S^2_{[0],-}$	**DeSitter** $S^2_{[-],-}$
$\cos a = \cos b \cos c - \sin b \sin c \cosh A$	$a^2 = b^2 + c^2 + 2bc \cosh A$	$\cosh a = \cosh b \cosh c + \sinh b \sinh c \cosh A$
$\frac{\sin a}{\sinh A} = \frac{\sin b}{\sinh B} = \frac{\sin c}{\sinh C}$	$\frac{a}{\sinh A} = \frac{b}{\sinh B} = \frac{c}{\sinh C}$	$\frac{\sinh a}{\sinh A} = \frac{\sinh b}{\sinh B} = \frac{\sinh c}{\sinh C}$
$\cosh A = \cosh B \cosh C + \sinh B \sinh C \cos a$	$A = B + C$	$\cosh A = \cosh B \cosh C + \sinh B \sinh C \cosh a$

Table 2. Cosine, dual cosine and sine theorem for the nine CK spaces

We remark the appearance of the label κ_1 in all trigonometric functions of the actual sides, where all functions of angles carry the label κ_2, in accordance to the previous discussion. As an example, we collect in Table 2 the cosine and dual cosine theorems relative to the side a or angle A for the nine spaces with the CK constants reduced to ± 1; the choice for the external angle at A makes some sign differences when compared with the usual approach in the three riemannian spaces $\kappa_2 > 0$. The complete table can be found in [2] and its extension to the complex spaces in [4].

5 Triality from a group theoretical point of view

Now one may ask for another possibilities of *family automorphisms* in the whole CK family of Lie algebras, groups and spaces. The duality esssentially amounted to an interchange $P_1 \leftrightarrow J$, with P_2 unchanged, and the three minus signs in (3.1) serve to fulfil the automorphism property; the involutivity of duality is clearly related to the interchange $P_1 \leftrightarrow J$, as iterating it one gets again the identity automorphism. This suggest the search of further automorphisms which may *cyclically permute* the three generators. Do they exist? Let us try as an ansatz a transformation wich, up to factors, carries (P_1, P_2, J) into (J, P_1, P_2) and then try to fix the factors so that this mapping is an automorphism of the family of Lie algebras (2.1). This leads to the transformation:

$$\mathcal{T} : (P_1, P_2, J) \to (\mathcal{P}_1, \mathcal{P}_2, \mathcal{J}) = (J, -\kappa_2 P_1, -P_2) \tag{5.1}$$

which for the CK Lie algebras or spaces with $\kappa_2 \neq 0$ has the required properties. Indeed, by direct checking $\mathcal{T}$ can be shown to be a family automorphism of the CK algebras and the image of the Lie algebra $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$ is another Lie algebra in the family, corresponding to a new choice of parameters $\mathrm{k}_1, \mathrm{k}_2$ given by:

$$\mathcal{T} : (\kappa_1, \kappa_2) \to (\mathrm{k}_1, \mathrm{k}_2) = (\kappa_2, \kappa_1\kappa_2). \tag{5.2}$$

For a fixed algebra $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$ in the family (with $\kappa_2 \neq 0$), $\mathcal{T}$ establishes an *isomorphism* between $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$ and $\mathfrak{so}_{\kappa_2,\kappa_1\kappa_2}(3)$. As the mapping permutes cyclically three generators, it is not involutive, but instead of order 3; iterating $\mathcal{T}$ *three* times one gets the identity automorphism (up to rescaling of generators); hence a suitable name for this transformation would be *triality*. The standard usage of this term in mathematics is related to octonions [5], see the comments in the conclusions section.

Theorem 5.1 *The trial of a CK space of curvature κ_1 and metric of signature type $(+, \kappa_2)$ is the CK space with curvature κ_2 and metric of type $(+, \kappa_1\kappa_2)$.*

Let us now ask how this triality relates the four generic CK spaces, where we can again assume the constants κ_1, κ_2 have been reduced to their fiducial values ± 1. Starting from $\mathbf{S}^2$ we get:

$$\mathcal{T} : \underset{(1,1)}{\mathbf{S}^2} \xrightarrow{\mathcal{T}} \underset{(1,1)}{\mathbf{S}^2} \tag{5.3}$$

hence the sphere is invariant under triality. If howewer we start from either space $\mathbf{H}^2, \mathbf{dS}^{1+1}, \mathbf{AdS}^{1+1}$, whose associated CK constants are $(-1, 1)$, $(1, -1)$, $(-1, -1)$, it suffices to recall the action (5.2) of the triality $\mathcal{T}$ on the constant κ_1, κ_2 to conclude:

$$\mathcal{T} : \underset{(-1,1)}{\mathbf{H}^2} \xrightarrow{\mathcal{T}} \underset{(1,-1)}{\mathbf{AdS}^{1+1}} \xrightarrow{\mathcal{T}} \underset{(-1,-1)}{\mathbf{dS}^{1+1}} \xrightarrow{\mathcal{T}} \underset{(-1,1)}{\mathbf{H}^2} \tag{5.4}$$

Hence the three spaces $\mathbf{H}^2, \mathbf{AdS}^{1+1}, \mathbf{dS}^{1+1}$ are cyclically permuted by the triality $\mathcal{T}$. This means that any property in either of these spaces has a 'trial' property in the others and any of them codifies completely, albeit in a 'holographic' form, any geometric property from each other. In this sense, all geometry in deSitter spaces in $1 + 1$ dimensions can be completely obtained through a suitable trial hyperbolic reformulation.

To explore what this triality means, one should uncover with some detail the geometrical meaning hidden under the automorphism (5.1). This is summed up in the following table, which mimicks the pattern found for $\mathcal{D}$:

Trial CK space $\mathcal{T}$(S)	***versus*** **Original CK space S**
Points (invariant under $\mathcal{J} = -P_2$)	• Ideal lines (invariant under P_2)
Distance between points (along $\mathcal{P}_1 = J$)	• Angle between ideal lines (along J)
Actual lines (invariant under $\mathcal{P}_1 = J$)	• Points (invariant under J)
Angle between actual lines (along $\mathcal{J} = -P_2$)	• Ideal distance between points (along P_2)
Ideal lines (invariant under $\mathcal{P}_2 = -\kappa_2 P_1$)	• Actual lines (invariant under P_1)
Angle between ideal lines (along $\mathcal{J} = -P_2$)	• Ideal distance between actual lines (along P_2)

For the sake of clarity the table makes reference to the fiducial choice for a point O, an actual line l_1 and an ideal line l_2, but it applies to all points, actual lines and ideal lines in the given CK geometry and in its trial.

Theorem 5.2 *The set of points of the trial $\mathcal{T}(S)$ of a CK space S is the set of ideal lines in the original space S, with the former angle between ideal lines as measure of separation. The set of actual lines of the trial $\mathcal{T}(S)$ is the set of points in the original space, with the former ideal distance between points as measure of separation. The set of ideal lines in the trial space $\mathcal{T}(S)$ is the set of actual lines in the original space, with the ideal distance between them as a measure of separation.*

Rather surprisingly, this interpretation can be applied in the same literal form to either the hyperbolic space, the AntiDeSitter sphere or to the deSitter sphere, and in each case it leads to the next geometry in the cycle (5.4).

A more explicit description would require the introduction of suitable models which can be used 'simultaneously' for all the three spaces. The 'stereocentral' projection from the CK vector models afford such a tool but for lack of space this cannot be done here (stereographic projection is *not* really adequate to discuss this issue; for the hyperbolic plane $\mathbf{H}^2$ this projection afford the Poincaré conformal disk model, where the ideal points *cannot* be represented; the exterior of the disk is simply another copy of $\mathbf{H}^2$ and together they represent the conformal completion of the hyperbolic plane [3]).

6 Closing Comments

As an application of this triality, one could translate all the geometric properties of hyperbolic geometry centered around the *parallelism angle* into properties related to the existence of horizons holding in the $1+1$ kinematics of relativistic DeSitter spaces. This reinforces the analogy pointed out in the present paper between final points in hyperbolic geometry and light-like lines in lorentzian geometries. Generally speaking, the CK scheme, with its threefold alternatives related to each CK constant, should shed light on any classification problem for specific properties in different spaces in a given CK family.

Duality can be directly seen in the trigonometric equations displayed. This is not the case for triality simply because we have not considered any ideal line as a possible side; should these had been considered, the triality invariance of these trigonometric equations would be also clear.

A few years ago V. I. Arnol'd [1] has considered a curious triality between any curve Γ on $\mathbf{S}^2$, its 'dual' $\Gamma^\wedge$ and its 'derivative' Γ'; these three curves can be considered as three projections of a single Legendrian curve in $\mathbf{S}^3$ (the set of pure imaginary unit quaternions) by means of the three Hopf projections $\mathbf{S}^3 \to \mathbf{S}^2$ associated to three unit quaternions. The triality described here coincides in $\mathbf{S}^2$ with the one proposed by Arnol'd and this suggest an alternative approach to the triality for the triple $\mathbf{H}^2, \mathbf{AdS}^{1+1}, \mathbf{dS}^{1+1}$ through three ('pseudo'Hopf) projections associated to the three unit pseudoquaternions from the set of pure imaginary unit pseudoquaternions $\mathbf{S}^3_1$ into, respectively $\mathbf{H}^2, \mathbf{H}^2_1, \mathbf{S}^2_1$.

Lie algebra family automorphisms in the CK families other than ordinary duality also exist for higher dimensions and relate geometries with the same group, but the two spaces of DeSitter in $n+1$ dimensions, $n > 1$ have different groups. Thus, as far as 'hyperbolic-AntiDeSitter-DeSitter' are concerned, the triality described here is a 'two-dimensional' phenomenon, whose high dimensional analogues relate other spaces.

A last question concerns the relation of $\mathcal{T}$ with the usual octonionic triality; it seems there is indeed a relation. The CK family of spaces of rank one, real dimension 2 and *real* type has *complex, quaternionic and octonionic relatives* [6], [7]; the compact spaces, analogous in these cases to $\mathbf{S^2} \equiv SO(3)/SO(2)$ (or $O(3)/(O(1)\otimes O(2)) \equiv \mathbb{R}P^2$) are respectively $U(3)/(U(1)\otimes U(2)) \equiv \mathbb{C}P^2$, $Sp(3)/(Sp(1)\otimes Sp(2)) \equiv \mathbb{H}P^2$ and $F_4/Spin(9) \equiv \mathbb{O}P^2$; the last space is the Cayley plane (the three $\mathbb{R}, \mathbb{C}, \mathbb{H}$ families also exist for higher dimensions, but octonionic do not). The triality introduced here for the real case can likely be extended to all these CK families. A reasonable expectation is that in the last case, the octonionic version of $\mathcal{T}$ would permute the 52 generators of F_4 in a way compatible with the standard octonionic triality. We recall that group theoretically, this exceptional triality property comes from the unique position of $\mathfrak{so}(8)$ among the orthogonal (even) Lie algebras; the vector representation and the two disequivalent spinor representations have the same dimension only in this case; triality also can be guessed from the threefold symmetry of the corresponding Dynkin diagram for the Cartan series D_4.

Acknowledgments

This work has been partially supported by Ministerio de Ciencia y Tecnología, Spain (Project PB98-0370). Part of the work reported here has

been done in collaboration with F. J. Herranz and R. Ortega.

References

[1] V. I. ARNOL'D, *Uspekhi Mat. Nauk* **50**:1 3–68; English transl. in *Russian Math. Surveys* **50**:1 1–68, (1995).

[2] F.J. HERRANZ, R. ORTEGA AND M. SANTANDER, *J. Phys. A: Math. Gen.* **33** 4525, (2000).

[3] F. J. HERRANZ AND M. SANTANDER, *Conformal compactification and cycle-preserving symmetries of spacetimes*, `math-ph/0110019` (2001).

[4] R. ORTEGA, Ph. D. Thesis, Universidad de Valladolid (2002), `math-ph/0112030`

[5] J. BAEZ, *The Octonions*, `math.RA/0105155`, (2001).

[6] M. SANTANDER, *Symmetric homogeneous spaces in Classical and Quantum Mechanics: a viewpoint for classification*, in 'Symmetries in Quantum Mechanics and Quantum Optics', ed A Ballesteros *et al* (Burgos: University of Burgos) 335–352, (1999).

[7] V. I. ARNOL'D, *Polymathematics: Is mathematics a single science or a set of arts?*, in 'Mathematics, Frontiers and perspectives', V. Arnold, M. Atiyah, P. Lax, B. Mazur, Eds, IMU, AMS, (2000).

Proceedings of the meeting
Lorentzian Geometry-Benalmádena 2001
Benalmádena, Málaga, Spain
Pub. de la RSME, Vol. 5 (2003), 261-283

Superenergy tensors and their applications

José M. M. Senovilla[1]

[1]*Departamento de Física Teórica, Universidad del País Vasco, Apartado 644, 48080 Bilbao, Spain*

Abstract

In Lorentzian manifolds of any dimension the concept of *causal tensors* is introduced. Causal tensors have positivity properties analogous to the so-called "dominant energy condition". Further, it is shown how to build, from any given tensor A, a new tensor quadratic in A and "positive", in the sense that it is causal. These tensors are called *superenergy tensors* due to historical reasons because they generalize the classical energy-momentum and Bel-Robinson constructions. Superenergy tensors are basically unique and with multiple and diverse physical and mathematical applications, such as: a) definition of new divergence-free currents, b) conservation laws in propagation of discontinuities of fields, c) the causal propagation of fields, d) null-cone preserving maps, e) generalized Rainich-like conditions, f) causal relations and transformations, and g) generalized symmetries. Among many others.

1 Causal tensors

In this contribution[1] V will denote a differentiable N-dimensional manifold V endowed with a metric g of Lorentzian signature $N-2$. The solid Lorentzian cone at x will be denoted by $\Theta_x = \Theta_x^+ \cup \Theta_x^-$, where $\Theta_x^\pm \subset T_x(V)$ are the future (+) and past (−) half-cones. The null cone

[1]It is worthwhile to check also my joint contribution with García-Parrado, as well as that of Bergqvist's, in this volume, with related results. Notice that signature convention here is opposite to those contributions and to [1].

$\partial\Theta_x$ is the boundary of Θ_x and its elements are the null vectors at x. An arbitrary point $x \in V$ is usually taken, but all definitions and results translate immediately to tensor fields if a time orientation has been chosen. The x-subscript is then dropped.

Definition 1.1 [1] *A rank-r tensor T has the* <u>*dominant property*</u> *at $x \in V$ if*

$$T(\vec{u}_1, \dots, \vec{u}_r) \geq 0 \qquad \forall \vec{u}_1, \dots, \vec{u}_r \in \Theta_x^+.$$

The set of rank-r tensor (fields) with the dominant property will be denoted by $\mathcal{DP}_r^+$. We also put $\mathcal{DP}_r^- \equiv \{T : -T \in \mathcal{DP}_r^+\}$, $\mathcal{DP}_r \equiv \mathcal{DP}_r^+ \cup \mathcal{DP}_r^-$, $\mathcal{DP}^\pm \equiv \bigcup_r \mathcal{DP}_r^\pm$, $\mathcal{DP} \equiv \mathcal{DP}^+ \cup \mathcal{DP}^-$.

By a natural extension $\mathbb{R}^+ = \mathcal{DP}_0^+ \subset \mathcal{DP}^+$. Rank-1 tensors in $\mathcal{DP}^+$ are simply the past-pointing causal 1-forms (while those in $\mathcal{DP}_1^-$ are the future-directed ones). For rank-2 tensors, the dominant property was introduced by Plebański [2] in General Relativity and is usually called the dominant energy condition [3] because it is a requirement for physically acceptable energy-momentum tensors. The elements of $\mathcal{DP}$ will be called "<u>causal tensors</u>". As in the case of past- and future-pointing vectors, any statement concerning $\mathcal{DP}^+$ has its counterpart concerning $\mathcal{DP}^-$, and they will be generally omitted. Trivially one has

Property 1.2 *If $T^{(i)} \in \mathcal{DP}_r^+$ and $\alpha_i \in \mathbb{R}^+$ $(i = 1, ..., n)$ then $\sum_{i=1}^{n} \alpha_i T^{(i)} \in \mathcal{DP}_r^+$. Moreover, if $T^{(1)}$, $T^{(2)} \in \mathcal{DP}^+$ then $T^{(1)} \otimes T^{(2)} \in \mathcal{DP}^+$.*

This tells us that $\mathcal{DP}^+$ is a graded algebra of cones. For later use, let us introduce the following notation

$$T^{(1)}\ {}_i\!\times_j\ T^{(2)} \equiv C_i^1 C_{r+j}^2 \left(g^{-1} \otimes T^{(1)} \otimes T^{(2)}\right)$$

that is to say, the contraction (via the metric) of the i^{th} entry of the first tensor (which has rank r) with the j^{th} of the second. There are of course many different products ${}_i\!\times_j$ depending on where the contraction is made.

Several characterizations of $\mathcal{DP}^+$ can be found. For instance [1, 4]

Proposition 1.3 *The following conditions are equivalent:*

1. *$T \in \mathcal{DP}_r^+$.*
2. *$T(\vec{k}_1, \dots, \vec{k}_r) \geq 0 \ \ \forall \vec{k}_1, \dots, \vec{k}_r \in \partial\Theta_x^+$.*

3. $T(\vec{u}_1, \dots, \vec{u}_r) > 0 \;\; \forall \vec{u}_1, \dots, \vec{u}_r \in \mathrm{int}\Theta^+_x$, *$(T \neq 0)$.*

4. $T(\vec{e}_0, \dots, \vec{e}_0) \geq |T(\vec{e}_{\alpha_1}, ..., \vec{e}_{\alpha_r})| \; \forall \alpha_1, ..., \alpha_r \in \{0, 1, ..., N-1\}$, *in* all *orthonormal bases* $\{\vec{e}_0, ..., \vec{e}_{N-1}\}$ *with a future-pointing timelike* $\vec{e}_0$.

5. $T\,{}_i\times_j \;\; T \in \mathcal{DP}^+_{r+s-2}, \;\; \forall\, T \in \mathcal{DP}^-_s, \; \forall i = 1, \dots, r, \; \forall j = 1, \dots, s.$

Proposition 1.4 *Similarly, some characterizations of* $\mathcal{DP}$ *are [1]*

1. $0 \neq T_i \times_i T \in \mathcal{DP}^-$ *for some* $i \Longrightarrow T_i \times_j T \in \mathcal{DP}^-$ *for* all $i, j \Longrightarrow T \in \mathcal{DP}$.

2. $T_i \times_i T = 0$ *for* all $i \iff T = k_1 \otimes \cdots \otimes k_r$ *where* k_i *are null* $\Longrightarrow T \in \mathcal{DP}$.

2 Superenergy tensors

In this section the questions of how general is the class $\mathcal{DP}$ and how one can build causal tensors are faced. The main result is that:

Given an arbitrary *tensor A, there is a canonical procedure (unique up to permutations) to construct a* <u>*causal*</u> *tensor quadratic in A.*

This procedure was introduced in [5] and extensively considered in [4], and the causal tensors thus built are called "super-energy tensors". The whole thing is based in the following

Remark 2.1 *Given any rank-m tensor A, there is a* minimum *value* $r \in \mathbb{N}$, $r \leq m$ *and a unique set of r numbers* $n_1, \dots, n_r \in \mathbb{N}$, *with* $\sum_{i=1}^r n_i = m$, *such that A is a linear map on* $\Lambda^{n_1} \times \cdots \times \Lambda^{n_r}$.

Here Λ^p stands for the vector space of "contravariant p-forms" at any $x \in V$. In other words, $\exists$ a minimum r such that $A \in \Lambda_{n_1} \otimes \cdots \otimes \Lambda_{n_r}$, where A is the appropriate permutted version of A which selects the natural order for the $n_1, \dots, n_r$ entries. Tensors seen in this way are called r-fold $(n_1, \dots, n_r)$-forms. Some simple examples are: any p-form Ω is trivially a single (that is, 1-fold) p-form, while $\nabla\Omega$ is a double $(1, p)$-form; the Riemann tensor R is a double (2,2)-form which is symmetric (the pairs can be interchanged), while ∇R is a triple (1,2,2)-form; the Ricci tensor Ric is a double symmetric (1,1)-form and, in general, any completely symmetric r-tensor is an r-fold (1,1,...,1)-form. A 3-tensor A with the property $A(\vec{x}, \vec{y}, \vec{z}) = -A(\vec{z}, \vec{y}, \vec{x})$ is a double (2,1)-form and the corresponding A is clearly given by $A(\vec{x}, \vec{y}, \vec{z}) \equiv A(\vec{x}, \vec{z}, \vec{y})$, $\forall \vec{x}, \vec{y}, \vec{z}$.

For r-fold forms, the interior contraction can be generalized in the obvious way $i_{\vec{x}_1,\dots,\vec{x}_r} : \Lambda_{n_1} \otimes \cdots \otimes \Lambda_{n_r} \longrightarrow \Lambda_{n_1-1} \otimes \cdots \otimes \Lambda_{n_r-1}$ by means of

$$i_{\vec{x}_1,\dots,\vec{x}_r} A = C_1^1 C_{n_1+1}^2 \dots C_{n_1+\dots+n_{r-1}+1}^r \left(\vec{x}_1 \otimes \vec{x}_2 \otimes \cdots \otimes \vec{x}_r \otimes \ A\right)$$

which is simply the interior contraction of each vector with each antysymmetric block. Similarly, by using the canonical volume element of (V, g) one can define the multiple Hodge duals as follows:

$$*_{\mathcal{P}} : \Lambda_{n_1} \otimes \cdots \otimes \Lambda_{n_r} \longrightarrow \Lambda_{n_1+\varepsilon_1(N-2n_1)} \otimes \cdots \otimes \Lambda_{n_r+\varepsilon_r(N-2n_r)}$$

where $\varepsilon_i \in \{0, 1\}\, \forall i = 1, \dots, r$ and the convention is taken that $\varepsilon_i = 1$ if the i^{th} antysymmetric block is dualized and $\varepsilon_i = 0$ otherwise, and where $\mathcal{P} = 1, \dots, 2^r$ is defined by $\mathcal{P} = 1 + \sum_{i=1}^{r} 2^{i-1}\varepsilon_i$. Thus, there are 2^r different Hodge duals for any r-fold form A and they can be adequately written as $A_{\mathcal{P}} \equiv *_{\mathcal{P}} A$. One also needs a product $\odot$ of A by itself resulting in a $2r$-covariant tensor, given by

$$(A \odot A)\,(\vec{x}_1, \vec{y}_1, \dots, \vec{x}_r, \vec{y}_r) \equiv \left(\prod_{i=1}^{r} \frac{1}{(n_i - 1)!}\right) g\left(i_{\vec{x}_1,\dots,\vec{x}_r} A, i_{\vec{y}_1,\dots,\vec{y}_r} A\right)$$

where for any tensor B we write $g\,(B, B) \in \mathbb{R}$ for the complete contraction in all indices in order.

Definition 2.2 [4, 5] *The basic* <u>*superenergy tensor*</u> *of A is defined to be*

$$T\{A\} = \frac{1}{2} \sum_{\mathcal{P}=1}^{2^r} A_{\mathcal{P}} \odot A_{\mathcal{P}}.$$

Here the word basic is used because linear combinations of $T\{A\}$ with its permutted versions maintain most of its properties; however, the completely symmetric part is unique (up to a factor of proportionality) [4]. It is remarkable that one can provide an explicit expression for $T\{A\}$ which is *independent* of the dimension N, see [4]. In the case of a general p-form Ω, its rank-2 superenergy tensor becomes

$$T\{\Omega\}\,(\vec{x}, \vec{y}) = \frac{1}{(p-1)!} \left[g\,(i_{\vec{x}}\Omega, i_{\vec{y}}\Omega) - \frac{1}{2p} g\,(\Omega, \Omega)\, g\,(\vec{x}, \vec{y}) \right]. \qquad (2.1)$$

In Definition 2.2 we implicitly assumed that the r-fold form A has no antysymmetric blocks of maximum degree N. Nevertheless, the above expression (2.1) is perfectly well defined for N-forms: if $\Omega = f\eta$ where η is the canonical volume form and f a scalar, then (2.1) gives $T\{\Omega\} = -\frac{1}{2}f^2 g$. Using this the Definition 2.2 is naturally extended to include N-blocks, see [1] for details.

In $N = 4$, the superenergy tensor of a 2-form F is its Maxwell energy-momentum tensor, and the superenergy tensor of an exact 1-form $d\phi$ has the form of the energy-momentum tensor for a massless scalar field ϕ. If we compute the superenergy tensor of R we get the so-called Bel tensor [6]. The superenergy tensor of the Weyl curvature tensor is the well-known Bel-Robinson tensor [7, 8, 9]. The main properties of $T\{A\}$ are [4]:

Property 2.3 *If A is an r-fold form, then $T\{A\}$ is a $2r$-covariant tensor.*

Property 2.4 $T\{A\}$ *is symmetric on each pair of entries, that is, for all $i = 1, \dots, r$ one has*

$$T\{A\}\left(\vec{x}_1, \dots, \vec{x}_{2i-1}, \vec{x}_{2i}, \dots, \vec{x}_{2r}\right) = T\{A\}\left(\vec{x}_1, \dots, \vec{x}_{2i}, \vec{x}_{2i-1}, \dots, \vec{x}_{2r}\right).$$

Property 2.5 $T\{A\} = T\{A_{\mathcal{P}}\}\ \ \forall \mathcal{P} = 1, \dots, 2^r$.

Property 2.6 $T\{A\} = T\{-A\}; \quad T\{A\} = 0 \Longleftrightarrow A = 0.$

Property 2.7 $T\{A \otimes B\} = T\{A\} \otimes T\{B\}$.

Property 2.8 $T\{A\} \in \mathcal{DP}^+$.

Observe that property 2.8 is what we were seeking, so that $T\{A\}$ is the "positive square" of A in the causal sense.

Property 2.9 $T\{A\}(\vec{e}_0, \dots, \vec{e}_0) = \dfrac{1}{2} \displaystyle\sum_{\alpha_1,\dots,\alpha_m=0}^{N-1} \left(A(\vec{e}_{\alpha_1}, ..., \vec{e}_{\alpha_m})\right)^2$ *in orthonormal bases* $\{\vec{e}_0, ..., \vec{e}_{N-1}\}$.

The set of superenergy tensors somehow build up the class $\mathcal{DP}$; in fact, in many occasions the rank-2 superenergy tensors (that is, those for single p-forms) are the basic building blocks of the whole $\mathcal{DP}$ [1]. This can be seen as follows.

Definition 2.10 *An r-fold form A is said to be* decomposable *if there are r forms Ω_i $(i = 1, \dots, r)$ such that $A = \Omega_1 \otimes \cdots \otimes \Omega_r$.*

From this and Property 2.7 one derives

Corollary 2.11 *If A is decomposable, then $T\{A\} = T\{\Omega_1\} \otimes \cdots \otimes T\{\Omega_r\}$.*

Notice that each of the $T\{\Omega_i\}$ on the righthand side is a rank-2 tensor. We now have

Theorem 2.12 *Any symmetric $T \in \mathcal{DP}_2^+$ can be decomposed as*

$$T = \sum_{p=1}^{N} T\{\Omega_p\}$$

where Ω_p are simple *p-forms such that for $p > 1$ they have the structure $\Omega_p = k_1 \wedge \cdots \wedge k_p$ where $k_1, \dots, k_p$ are appropriate* null *1-forms and $\Omega_1 \in \mathcal{DP}_1$.*

See [1] for the detailed structure of the above decomposition and for the relation between T and the null 1-forms. From this one obtains[2]

Theorem 2.13 *A symmetric rank-2 tensor T satisfies $T^2 = g$ if and only if $T = \pm T\{\Omega_p\}$, i.e., if T is up to sign the superenergy tensor of a simple p-form Ω_p. Moreover, the rank p of the p-form is given by $\pm \mathrm{tr} T = 2p - N$.*

This important theorem allows to classify all Lorentz transformations and, in more generality, all maps which preserve the null cone $\partial\Theta_x$ at $x \in V$, see [1].

3 Applications

In this section several applications of superenergy and causal tensors are presented. They include both mathematical and physical ones.

[2]From this point on I shall use the standard notation T^2 instead of $T_1 \times_1 T = T_2 \times_2 T = T_2 \times_1 T = T_1 \times_2 T$ for the case of rank-2 symmetric tensors T. T^2 is symmetric.

3.1 Rainich's conditions

The classical Rainich conditions [10, 11] are necessary and sufficient conditions for a spacetime to originate via Einstein's equations in a Maxwell electromagnetic field. They are of two kinds: algebraic and differential. Here I am only concerned with the algebraic part which nowadays are presented as follows (see, e.g., [12]):

(Classical Rainich's conditions) *The Einstein tensor* $G = Ric - \frac{1}{2}S\,g$ *of a 4-dimensional spacetime is proportional to the energy-momentum tensor of a Maxwell field (a 2-form) if and only if* $G^2 \propto g$, $\mathrm{tr}G = 0$ *and* $G \in DP_2^+$.

In fact, from theorem 2.13 one can immediately improve a little this classical result

Corollary 3.1 *In* 4 *dimensions,* G *is* algebraically *up to sign proportional to the energy-momentum tensor of a Maxwell field if and only if* $G^2 \propto g$, $\mathrm{tr}G = 0$. *Furthermore,* G *is proportional to the energy-momentum tensor of (possibly another) Maxwell 2-form which is* simple.

The last part of this corollary is related to the so-called duality rotations of the electromagnetic field [12]. Observe that this is clearly a way of determining physics from geometry because, given a particular spacetime, one only has to compute its Einstein tensor and check the above simple conditions. If they hold, then the energy-momentum tensor is that of a 2-form (and for a complete result the Rainich differential conditions will then be needed).

The classical Rainich conditions are based on a dimensionally-dependent identity, see [13], valid only for $N = 4$. However, theorem 2.13 has universal validity and can be applied to obtain the generalization of Rainich's conditions in many cases. For instance, we were able to derive the following results [1].

Corollary 3.2 *In* N *dimensions, a rank-2 symmetric tensor* T *is* algebraically *the energy-momentum tensor of a minimally coupled massless scalar field* ϕ *if and only if* $T^2 \propto g$ *and* $\mathrm{tr}T = \beta\sqrt{\mathrm{tr}T^2/N}$ *where* $\beta = \pm(N-2)$. *Moreover,* $d\phi$ *is spacelike if* $\beta = 2 - N$ *and* $\mathrm{tr}T \neq 0$, *timelike if* $\beta = N - 2$ *and* $\mathrm{tr}T \neq 0$, *and null if* $\mathrm{tr}T = 0 = \mathrm{tr}T^2$.

Corollary 3.3 *In* N *dimensions, a rank-2 symmetric tensor* T *is the energy-momentum tensor of a perfect fluid satisfying the dominant energy* *condition if and only if there exist two positive functions λ, μ such that*

$$T^2 = 2\lambda T + \left(\mu^2 - \lambda^2\right) g, \qquad \mathrm{tr}T = (N-2)\mu - \lambda N.$$

This is an improvement and a generalization to arbitrary N of the conditions in [14] for $N = 4$. In particular, the case of dust can be deduced from the previous one by setting the pressure of the perfect fluid equal to zero.

Corollary 3.4 *In N dimensions, a symmetric tensor T is* algebraically *the energy-momentum tensor of a dust satisfying the dominant energy condition if and only if*

$$T^2 = (\mathrm{tr}T)\, T, \qquad \mathrm{tr}T < 0.$$

3.2 Causal propagation of fields

Following a classical reasoning appearing in [3], the causal propagation of arbitrary fields can be studied by simply using its superenergy tensor, see [15]. Let ζ be any closed achronal set in V and $D(\zeta)$ its total Cauchy development (an overbar over a set denotes its closure, see [3, 16] for definitions and notation).

Theorem 3.5 *If the tensor $T\{A\}$ satisfies the following divergence inequality*

$$\mathrm{div}T\,\{A\}\,(\vec{x},\ldots,\vec{x}) \leq f\, T\,\{A\}\,(\vec{x},\ldots,\vec{x})$$

where f is a continuous function and $\vec{x} = g^{-1}(\ ,-d\tau)$ is any timelike vector foliating $D(\zeta)$ with hypersurfaces $\tau =$const., then

$$A|_{\zeta} = 0 \qquad \Longrightarrow \qquad A|_{\overline{D(\zeta)}} = 0.$$

This theorem proves the causal propagation of the field A because if $A \neq 0$ at a point $x \notin \overline{D(\zeta)}$ arbitrarily close to $\overline{D(\zeta)}$, then A will propagate in time from x according to its field equations, but it will never be able to enter into $\overline{D(\zeta)}$, showing that A cannot travel faster than light.

The divergence condition in the theorem, being an inequality, is very mild and it is very easy to check whether or not is valid for a given field satysfying field equations. In general, of course, it will work for linear field equations, and for many other cases too. It has been used to prove the causal propagation of gravity in vacuum [17] or in general N-dimensional Lorentzian manifolds conformally related to Einstein spaces [15], and also for the massless spin-$n/2$ fields in General Relativity [15]. It must be stressed that in many occasions the standard energy-momentum tensor of the field does not allow to prove the same result, so that the universality of the superenergy construction reveals itself as essential in this application.

3.3 Propagation of discontinuities: conserved quantities

Several ways to derive conserved quantities and exchange of superenergy properties have been pursued. One of them is the construction of divergence-free vector fields, called currents. This has been succesfully achieved in the case of a minimally coupled scalar field if the Einstein-Klein-Gordon field equations hold, see [4, 18]. In this subsection the propagation of discontinuities of the electromagnetic and gravitational fields will be analyzed from the superenergy point of view. This will be enough to prove the interchange of superenergy quantities between these two physical fields and some conservation laws arising naturally when the field has a 'wave-front', see [4, 18].

To that end, we need to recall some well-known basic properties of the wave-fronts, which are null hypersurfaces. Let σ be such a null hypersurface and n a 1-form normal to σ. Obviously, n is null $g^{-1}(n,n) = 0$ and therefore $\vec{n} \equiv g^{-1}(\ ,n)$ is in fact a vector *tangent* to σ, see e.g. [19], and n cannot be normalized so that it is defined up to a transformation of the form

$$n \longrightarrow \rho\, n\,, \qquad\qquad \rho > 0. \tag{3.1}$$

The null curves tangent to $\vec{n}$ are null geodesics $\nabla_{\vec{n}}\vec{n} = \Psi\,\vec{n}$, called 'bicharacteristics', contained in σ. Let $\bar{g}$ denote the first fundamental form of σ, which is a degenerate metric because $\bar{g}(\vec{n},\ ) = 0$ [3, 19, 16]. The second fundamental form of σ relative to n can be defined as

$$K \equiv \frac{1}{2}\pounds_{\vec{n}}\,\bar{g}$$

where $\pounds_{\vec{n}}$ denotes the Lie derivative with respect to $\vec{n}$ within σ. K is *intrinsic* to the null hypersurface σ and shares the degeneracy with $\bar{g}$: $K(\vec{n},\ ) = 0$ [19]. Because of this, and even though $\bar{g}$ has no inverse, one can define the "trace" of K by contracting with the inverse of the metric induced on the quotient spaces $T(\sigma)/ < \vec{n} >$. This trace will be denoted by ϑ and has the following interpretation: if $s \subset \sigma$ denotes any spacelike cut of σ, that is, a spacelike $(N-2)$-surface orthogonal to n and within σ, then ϑ measures the volume expansion of s along the null generators of σ. In fact, ϑ can be easily related to the derivative along $\vec{n}$ of the $(N-2)$-volume element of s [16].

Let us consider the case when there is an electromagnetic field (a 2-form F) propagating in a background spacetime so that there is a *null* hypersurface of discontinuity σ [20, 21], called a 'characteristic'. Let $[E]_\sigma$ denote the discontinuity of any object E across σ. Using the classical Hadamard results [22, 20, 21], one can prove the existence of a 1-form c on σ such that

$$[F]_\sigma = n \wedge c\,, \quad g^{-1}(n,c) = 0.$$

Observe that c transforms under the freedom (3.1) as $c \longrightarrow c/\rho$. ¿From Maxwell's equations for F considered in a distributional sense one derives a propagation law [21, 18], or 'transport equation' [20],

$$\vec{n}\left(|c|^2\right) + |c^2|(\vartheta + 2\Psi) = 0, \qquad |c|^2 \equiv g^{-1}(c,c) \geq 0.$$

This propagation equation implies that if $c|_x = 0$ at any point $x \in \sigma$, then $c = 0$ along the null geodesic originated at x and tangent to $\vec{n}$. Moreover, for arbitrary conformal Killing vectors $\vec{\zeta}_1, \vec{\zeta}_2$, the above propagation law allows to prove that [20, 18]

$$\int_s |c|^2 n(\vec{\zeta}_1) n(\vec{\zeta}_2)\, \boldsymbol{\omega}|_s \tag{3.2}$$

are conserved quantities along σ, where $\boldsymbol{\omega}|_s$ is the canonical volume element $(N-2)$-form of s, in the sense that the integral is independent of the cut s chosen. Notice also that (3.2) are invariant under the transformation (3.1). These conserved quantities can be easily related to the energy-momentum properties of the electromagnetic field because $T\{[F]_\sigma\} = |c|^2 n \otimes n$, so that the Maxwell tensor of the discontinuity $[F]_\sigma$ contracted with the conformal Killing vectors $\vec{\zeta}_1, \vec{\zeta}_2$ gives the function integrated in (3.2).

However, the integral (3.2) vanishes when $[F]_\sigma = 0 \Longleftrightarrow c = 0$. Using again Hadamard's theory, now there exist a 2-covariant symmetric tensor B and a 1-form f defined only on σ such that [8, 23, 21, 19]

$$[R]_\sigma = n \wedge B \wedge n, \qquad B(\vec{n},\ ) + \mathrm{tr}B\, n = 0,$$
$$[\nabla F]_\sigma = n \otimes (n \wedge f), \qquad g^{-1}(n,f) = 0.$$

These objects transform under (3.1) according to $f, B \longrightarrow f/\rho^2, B/\rho^2$. Assuming that the Einstein-Maxwell equations hold Lichnerowicz deduced the propagation laws for f, B in [21], and in particular he found the following transport equation [21, 4]

$$\vec{n}\left(|B|^2+|f|^2\right)+\left(|B|^2+|f|^2\right)(\vartheta+4\Psi)=0,$$

where $|f|^2 \equiv g^{-1}(f,f) \geq 0, |B|^2 \equiv \mathrm{tr} B^2 \geq 0$. Once again, with the help of any conformal Killing vectors $\vec{\zeta}_1, \ldots, \vec{\zeta}_4$, the following quantities

$$\int_s \left(|B|^2+|f|^2\right) n(\vec{\zeta}_1) n(\vec{\zeta}_2) n(\vec{\zeta}_3) n(\vec{\zeta}_4)\ \boldsymbol{\omega}|_s \tag{3.3}$$

are conserved along σ in the sense that the integral is independent of the spacelike cut s. Two important points can be derived from this relation: first, *both* the electromagnetic and gravitational contributions are necessary, so that neither the integrals involving only $|B|^2$ or only $|f|^2$ are equal for different cuts s in general. Second, the integrand in (3.3) is related to superenergy tensors because $T\{[R]_\sigma\} = 2|B|^2 n\otimes n\otimes n\otimes n$ and $T\{[\nabla F]_\sigma\} = 2|f|^2 n\otimes n\otimes n\otimes n$, and thus the function integrated in (3.3) is simply

$$\frac{1}{2}\left(T\{[R]_\sigma\}+T\{[\nabla F]_\sigma\}\right)(\vec{\zeta}_1,\vec{\zeta}_2,\vec{\zeta}_3,\vec{\zeta}_4)$$

which demonstrates the interplay between superenergy quantities of different fields, in this case the electromagentic and gravitational ones. Observe that the above tensors are completely symmetric in this case, and that they together with the conserved quantity (3.3) are invariant under the transformation (3.1).

3.4 Causal relations

The fact that the tensors in $\mathcal{DP}_2$ can be seen as linear mappings preserving the Lorentz cone leads in a natural way to consider the possibility of relating different Lorentzian manifolds at their corresponding causal levels, even before the metric properties are taking into consideration. To that end, using the standard notation φ' and φ^* for the push-forward and pull-back mappings, respectively, we give the next definition, see [24].

Definition 3.6 *Let $\varphi : V \to W$ be a global diffeomorphism between two Lorentzian manifolds. W is said to be properly causally related with V by*

φ, denoted $V \prec_\varphi W$, if $\forall\, \vec{u} \in \Theta^+(V)$, $\varphi' \vec{u} \in \Theta^+(W)$. W is simply said to be properly causally related with V, denoted by $V \prec W$, if $\exists \varphi$ such that $V \prec_\varphi W$.

In simpler terms, what one demands here is that the solid Lorentz cones at all $x \in V$ are mapped by φ to sets contained in the solid Lorentz cones at $\varphi(x) \in W$ keeping the time orientation: $\varphi'\Theta^+_x \subseteq \Theta^+_{\varphi(x)}$, $\forall x \in V$.

Observe that two Lorentzian manifolds can be properly causally related by some diffeomorphisms but not by others. As a simple example, consider $\mathbb{L}$ with typical Cartesisan coordinates $x^0, \dots, x^{N-1}$ (the 0-index indicating the time coordinate) and let φ_q be the diffeomorphisms

$$\begin{array}{rcl} \varphi_q : \mathbb{L} & \longrightarrow & \mathbb{L} \\ (x^0, \dots, x^{N-1}) & \longrightarrow & (q\, x^0, \dots, x^{N-1}) \end{array}$$

for any constant $q \neq 0$. It is easily checked that φ_q is a proper causal relation for all $q \geq 1$, but is not for all $q < 1$. Thus $\mathbb{L} \prec \mathbb{L}$ but, say, $\mathbb{L} \not\prec_{\varphi_{1/2}} \mathbb{L}$. Notice also that for $q \leq -1$ the diffeomorphisms φ_q change the time orientation of the causal vectors, but still $\varphi'\Theta_x \subseteq \Theta_{\varphi(x)}$ (with $\varphi'\Theta^+_x \subseteq \Theta^-_{\varphi(x)}$.)

Proper causal relations can be easily characterized by some equivalent simple conditions [24].

Theorem 3.7 *The following statements are equivalent:*

1. *$V \prec_\varphi W$.*
2. *$\varphi^*\left(\mathcal{DP}^+_r(W)\right) \subseteq \mathcal{DP}^+_r(V)$ for all $r \in \mathbb{N}$.*
3. *$\varphi^*\left(\mathcal{DP}^+_1(W)\right) \subseteq \mathcal{DP}^+_1(V)$.*
4. *$\varphi^* h \in \mathcal{DP}^-_2(V)$ where h is the metric of W up to time orientation.*

Proof. (See also [24])
$1 \Rightarrow 2$: let $T \in \mathcal{DP}^+_r(W)$, then $(\varphi^*T)(\vec{u}_1, \dots, \vec{u}_r) = T\,(\varphi'\vec{u}_1, \dots, \varphi'\vec{u}_r) \geq 0$ for all $\vec{u}_1, \dots, \vec{u}_r \in \Theta^+(V)$ given that $\varphi'\vec{u}_1, \dots, \varphi'\vec{u}_r \in \Theta^+(W)$ by assumption. Thus $\varphi^*T \in \mathcal{DP}^+_r(V)$.
$2 \Rightarrow 3$: Trivial.
$3 \Rightarrow 4$: For any $\vec{u} \in \Theta^+(V)$ we have that $0 \leq (\varphi^*w)(\vec{u}) = w(\varphi'\vec{u})$ holds for all $w \in \mathcal{DP}^+_1(W)$, so that necessarily $\varphi'\vec{u} \in \Theta^+(W)$ (which already implies *1*). Then, $0 \leq h(\varphi'\vec{u}, \varphi'\vec{v}) = \varphi^*h(\vec{u}, \vec{v})$ for all $\vec{u}, \vec{v} \in \Theta^+(V)$, which proves that $\varphi^*h \in \mathcal{DP}^-_2(V)$.

$4 \Rightarrow 1$: For every $\vec{u} \in \Theta^+(V)$ we have that $(\varphi^* h)(\vec{u},\vec{u}) = h(\varphi'\vec{u},\varphi'\vec{u}) \leq 0$ and hence $\varphi'\vec{u} \in \Theta(W)$. Besides, for any other $\vec{v} \in \Theta^+(V)$, $(\varphi^* h)(\vec{u},\vec{v}) = h(\varphi'\vec{u},\varphi'\vec{v}) \leq 0$ so that every two vectors with the same time orientation are mapped to vectors with the same time orientation. However, it could happen that $\Theta^+(V)$ is actually mapped to $\Theta^-(W)$, and $\Theta^-(V)$ to $\Theta^+(W)$. By changing the time orientation of W, if necessary, the result follows.

□

Condition *4* in this theorem can be replaced by

4'. $\varphi^* h \in \mathcal{DP}_2^-(V)$ and $\varphi'\vec{u} \in \Theta^+(W)$ for at least one $\vec{u} \in \Theta^+(V)$.

Leaving this time-orientation problem aside (in the end, condition *4* just means that W *with one of its time orientations* is properly causally related with V), let us stress that the condition *4* (or *4'*) is very easy to check and thereby extremely valuable in practical problems: first, one only has to work with one tensor field h, and also as we saw in proposition 1.3 there are several simple ways to check whether $\varphi^* h \in \mathcal{DP}_2^-(V)$ or not.

The combination of theorem 2.12 and condition *4* in theorem 3.7 provides a classification of the proper causal relations according to the number of independent null vectors which are mapped by φ to null vectors at each point. The key result here is that

Proposition 3.8 *Let $V \prec_\varphi W$ and $\vec{u} \in \Theta^+_x$, $x \in V$. Then $\varphi'\vec{u}$ is null at $\varphi(x) \in W$ if and only if $\vec{u}$ is a null eigenvector of $\varphi^* h|_x$.*

Proof. See [25, 24]. Recall that $\vec{u}$ is called an "eigenvector" of a 2-covariant tensor T if $T(\ ,\vec{u}) = \lambda g(\ ,\vec{u})$ and λ is then the corresponding eigenvalue.

□

The vectors which remain null under the causal relation φ are called its *canonical null directions*, and there are at most N of them linearly independent. Hence, using theorem 2.12 one can see that there essentially are N different types of proper causal relations, and that the conformal relations are included as the particular case in which *all* null directions are canonical [25].

Clearly $V \prec V$ for all V (by just taking the identity mapping). Moreover

Proposition 3.9 *$V \prec W$ and $W \prec U \Longrightarrow V \prec U$.*

Proof. Consider any $\vec{u} \in \Theta^+(V)$. Since there are φ, ψ such that $V \prec_\varphi W$ and $W \prec_\psi U$, then $\varphi' \vec{u} \in \Theta^+(W)$ and $\psi'[\varphi' \vec{u}] \in \Theta^+(U)$ so that $(\psi \circ \varphi)' \vec{u} \in \Theta^+(U)$ from where $V \prec U$.

$\square$

It follows that the relation $\prec$ is a preorder for the class of all Lorentzian manifolds. This is not a partial order as $V \prec W$ and $W \prec V$ does not imply that $V = W$ and, actually, it does not even imply that V is conformally related to W. The point here is that $V \prec_\varphi W$ and $W \prec_\psi V$ can perfectly happen with $\psi \neq \varphi^{-1}$. In the case that $V \prec_\varphi W$ and $W \prec_{\varphi^{-1}} V$ then necessarily φ is a conformal relation and $\varphi^* h = e^{2f} g$, but we are dealing with more general and basic causal equivalences.

Definition 3.10 *Two Lorentzian manifolds V and W are called causally isomorphic, denoted by $V \sim W$, if $V \prec W$ and $W \prec V$.*

If $V \sim W$ then the causal structures in both manifolds are somehow the same. Clearly, $\sim$ is an equivalence relation, and now one can obtain a partial order $\preceq$ for the corresponding classes of equivalence $\mathrm{coset}(V) \equiv \{U \,:\, V \sim U\}$, by setting

$$\mathrm{coset}(V) \preceq \mathrm{coset}(W) \Longleftrightarrow V \prec W\,.$$

This partial order provides chains of (classes of equivalence of) Lorentzian manifolds which keep "improving" the causal properties of the spacetimes. To see this, we need the following (see [3, 16] for definitions)

Proposition 3.11 *Let $V \prec W$. Then, if V violates any of the following*

1. *the chronology condition,*
2. *the causality condition,*
3. *the future-distinguishing condition (or the past one),*
4. *the strong causality condition,*
5. *the stable causality condition,*
6. *the global hyperbolicity condition,*

so does W.

Proof. (see [24] for the details). For *1* to *4*, let γ be a causal curve responsible for the given violation of causality (that is, a closed time-like curve for *1*, or a curve cutting any neighbourhood of a point in a disconnected set for *4*, and so on). Then, $\varphi(\gamma)$ has the corresponding property in W. To prove *5*, if there were a function f in W such that $-df \in \mathcal{DP}_1^+(W)$, from theorem 3.7 point *3* it would follow that $d(\varphi^* f) = \varphi^* df \in \mathcal{DP}_1^-(V)$ so that $\varphi^* f$ would also be a time function. Finally, *6* follows from corollary 3.1 in [25].

$\square$

With this result at hand, we can build the afore-mentioned chains of spacetimes, such as

$$\text{coset}(V) \preceq \cdots \preceq \text{coset}(W) \preceq \cdots \preceq \text{coset}(U) \preceq \cdots \preceq \text{coset}(Z)$$

where the spacetimes satisfying stronger causality properties are to the left, while those violating causality properties appear more and more to the right. This is natural because the light cones "open up" under a causal mapping. The actual properties depend on the particular chain and its length, but an optimal one would start with a V which is globally hyperbolic, and then it could pass through a W which is just causally stable, then U could be causal, say, and the last step Z could be a totally vicious spacetime [16].

Perhaps the above results can be used to give a first fundamental characterization of *asymptotically equivalent spacetimes*, at a level prior to the existence of the metric, which might then be included in a subsequent step. This could be accomplished by means of the following tentative definitions, which may need some refinement.

Definition 3.12 *An open set $\zeta \subset V$ is called a <u>neighbourhood of</u>*

1. *<u>the causal boundary</u> of V if $\zeta \cap \gamma \neq \emptyset$ for all endless causal curves γ;*
2. *<u>a singularity set</u> $\mathcal{S}$ if $\zeta \cap \gamma \neq \emptyset$ for all endless causal curves γ which <u>are incomplete towards</u> $\mathcal{S}$;*
3. *<u>the causal infinity</u> if $\zeta \cap \gamma \neq \emptyset$ for all complete causal curves γ.*

(See [16] for definition of boundaries, singularity sets, etcetera).

Definition 3.13 *W is said to be causally asymptotically V if any two neighbourhoods of the causal infinity $\zeta \subset V$ and $\zeta \subset W$ contain corresponding neighbourhoods $\zeta' \subset \zeta$ and $\zeta' \subset \zeta$ of the causal infinity such that $\zeta' \sim \zeta'$.*

Similar definitions can be given for W having causally the singularity structure of V, or the causal boundary of V, replacing in the given definition the neighbourhoods of the causal infinity by those of the singularity and of the causal boundary, respectively. The usefulness of these investigations is still unclear.

3.5 Causal transformations and generalized symmetries

Here the natural question of whether the causal relations can be used to define a generalization of the group of conformal motions is analyzed. To start with, we need a basic concept.

Definition 3.14 *A transformation $\varphi : V \longrightarrow V$ is called* causal *if $V \prec_\varphi V$. The set of causal transformations of V is written as $\mathcal{C}(V)$.*

$\mathcal{C}(V)$ is a subset of the group of transformations of V. In fact, from the proof of proposition 3.9 follows that $\mathcal{C}(V)$ is closed under the composition of diffeomorphisms. As the identity map is also clearly in $\mathcal{C}(V)$ its algebraic strucuture is that of a submonoid, see e.g. [26], of the group of diffeomorphisms of V. However, $\mathcal{C}(V)$ generically fails to be a subgroup, because (see [25] for the proof):

Proposition 3.15 *Every subgroup of causal transformations is a group of conformal motions.*

From standard results, see [26], one identifies $\mathcal{C}(V) \cap \mathcal{C}(V)^{-1}$ as the group of conformal motions of V and there is no other subgroup of $\mathcal{C}(V)$ containing $\mathcal{C}(V) \cap \mathcal{C}(V)^{-1}$. The transformations in $\mathcal{C}(V) \setminus (\mathcal{C}(V) \cap \mathcal{C}(V)^{-1})$ are called *proper* causal transformations.

Take now a one-parameter *group* of causal transformations $\{\varphi_t\}_{t\in\mathbb{R}}$. From proposition 3.15 it follows that $\{\varphi_t\}$ must be in fact a group of conformal motions, and its infinitesimal generator is a conformal Killing vector, so that nothing new is found here. Nevertheless, one can generalize naturally the conformal Killings by building one-parameter groups of transformations $\{\varphi_t\}$ such that only part of them are causal transformations. Given that the problem arises because both φ_t and $\varphi_{-t} = \varphi_t^{-1}$

belong to the family and thus they would both be conformal if they are *both* causal, one readily realizes that the natural generalization is to assume that $\{\varphi_t\}_{t\in\mathbb{R}}$ is such that either $\{\varphi_t\}_{t\in\mathbb{R}^+}$ or $\{\varphi_t\}_{t\in\mathbb{R}^-}$ is a subset of $\mathcal{C}(V)$, but *only one of the two.* Any group $\{\varphi_t\}_{t\in\mathbb{R}}$ with this property is called a maximal one-parameter submonoid of proper causal transformations. Of course, the one-parameter submonoid can just be a *local* one so that the transformations are defined only for some interval $I=(-\varepsilon,\varepsilon)\in\mathbb{R}$ and only those with $t\in(0,\varepsilon)$ (or $t\in(-\varepsilon,0)$) are proper causal transformations.

Let then $\{\varphi_t\}_{t\in I}$ be a local one-parameter submonoid of proper causal transformations, and assume that $t\geq 0$ provides the subset of proper causal transformations (otherwise, just change the sign of t). The infinitesimal generator of $\{\varphi_t\}_{t\in I}$ is defined as the vector field

$$\vec{\xi}=\left.\frac{d\varphi_t}{dt}\right|_{t=0}$$

so that for every covariant tensor field T one has

$$\left.\frac{d(\varphi_t^* T)}{dt}\right|_{t=0}=\pounds_{\vec{\xi}}T$$

where $\pounds_{\vec{\xi}}$ denotes the Lie derivative with respect to $\vec{\xi}$. As $\{\varphi_t\}_{t\geq 0}$ are proper causal transformations, and using point *2* in theorem 3.7, one gets $\varphi_t^*T\in\mathcal{DP}_r^+$ for $t\geq 0$ and for all tensor fields $T\in\mathcal{DP}_r^+$. In particular,

$$\varphi_t^*T\,(\vec{u}_1,\dots,\vec{u}_r)\geq 0,\ \ \forall\vec{u}_1,\dots,\vec{u}_r\in\Theta^+,\ \forall T\in\mathcal{DP}_r^+,\ t\geq 0, \qquad (3.4)$$

from where we can derive the next result.

Lemma 3.16 *Let $T\in\mathcal{DP}_r^+$ and $\vec{k}\in\Theta^+$ be such that $T(\vec{k},\dots,\vec{k})=0$. If $\varphi_t\in\mathcal{C}(V)$ for $t\in[0,\varepsilon)$, then*

$$(\pounds_{\vec{\xi}}T)(\vec{k},\dots,\vec{k})\geq 0\,.$$

Proof. Under the conditions of the lemma, and due to points *2* and *3* of proposition 1.3, it is necessary that $\vec{k}$ is null, that is, $\vec{k}\in\partial\Theta^+$. From formula (3.4) one obtains $\varphi_t^*T\,(\vec{k},\dots,\vec{k})\geq 0$ for all $t\in[0,\varepsilon)$. But φ_0 is the identity transformation, so $\varphi_0^*T\,(\vec{k},\dots,\vec{k})=T\,(\vec{k},\dots,\vec{k})=0$, from where necessarily follows that $\varphi_t^* T\,(\vec{k},\ldots,\vec{k})$ is a non-decreasing function of t at $t=0$, that is to say, $(d/dt)(\varphi_t^* T\,(\vec{k},\ldots,\vec{k}))|_{t=0} \geq 0$.

□

Corollary 3.17 *Let $\vec{\xi}$ be the infinitesimal generator of a local one-parameter submonoid of proper causal transformations $\{\varphi_t\}_{t\in I}$ and choose the sign of t such that $\{\varphi_t\}_{t\geq 0} \subset \mathcal{C}(V)$. Then*

$$(\pounds_{\vec{\xi}}\, g)(\vec{k},\vec{k}) \leq 0\ , \qquad \forall \vec{k} \in \partial\Theta$$

Proof. Obviously $g(\vec{k},\vec{k})=0$ for all null $\vec{k}$, and also $g \in \mathcal{DP}_2^-$, so lemma 3.16 can be applied to $-g$ and the result follows.

□

This result is a generalization of the condition for conformal Killing vectors ($\pounds_{\vec{\xi}}\, g \propto g$) and can be analyzed in a similar manner. As a matter of fact, the application of the decomposition theorem 2.12 to $\varphi_t^* g \in \mathcal{DP}_2^-$ leads to a much stronger result which allows for a complete characterization of the vector fields $\vec{\xi}$ and their properties.

Theorem 3.18 *Let $\vec{\xi}$ be the infinitesimal generator of a local one-parameter submonoid of proper causal transformations $\{\varphi_t\}_{t\in I}$ and choose the sign of t such that $\{\varphi_t\}_{t\geq 0} \subset \mathcal{C}(V)$. Then there is a function ψ such that*

$$\left[\pounds_{\vec{\xi}}\, g - 2\psi g\right] \in \mathcal{DP}_2^-.$$

Proof. From theorem 2.12 and given that $\varphi_t^* g \in \mathcal{DP}_2^-$ for $t \in [0,\varepsilon)$ one has

$$\varphi_t^* g = -\sum_{p=1}^{N} T_t\{\Omega_p\} = -\sum_{p=1}^{N-1} T_t\{\Omega_p\} + \Psi_t^2\, g$$

where $T_t\{\Omega_p\}$ are superenergy tensors of simple p-forms for all values of $t \in [0,\varepsilon)$ and Ψ_t are functions on V with $\Psi_0 = 1$. Then we have $\varphi_t^* g(\vec{u},\vec{v}) \leq \Psi_t^2\, g(u,v) \leq 0$ for all $\vec{u},\vec{v} \in \Theta^+$, or equivalently,

$$\Psi_t^{-2}\varphi_t^* g(\vec{u},\vec{v}) \leq g(\vec{u},\vec{v}) = \Psi_0^{-2}\varphi_0^* g(\vec{u},\vec{v}) \leq 0$$

from where a reasoning similar to that in lemma 3.16, by taking the derivative with repsect to t at $t = 0$, gives

$$\left[\pounds_{\vec{\xi}}\, g - 2\psi g\right](\vec{u}, \vec{v}) \leq 0, \qquad \forall \vec{u}, \vec{v} \in \Theta^{+}$$

where $\psi \equiv d\Psi_t/dt|_{t=0}$.

□

This set of vector fields generalize the traditional (conformal) symmetries and the previous theorem together with theorem 2.12 provides first a definition of generalized symmetries, and second its full classification because $\pounds_{\vec{\xi}}\, g - 2\psi g$ itself can be written as a sum of superenergy tensors of simple p-forms. The number of independent null eigenvectors of $\pounds_{\vec{\xi}}\, g - 2\psi g$ (ranging from 0 to N) gives the desired classification, where N corresponds to the conformal Killing vectors. This is under current investigation. It must be remarked that the above theorem does not provide a sufficient condition for a vector field to generate locally a one-parameter submonoid of causal transformations.

Several examples of generalized Killing vectors in this sense can be presented. One of them is a particular case of a previous partial generalization of isometries considered in [27] and called Kerr-Schild vector fields. They are vector fields which satisfy $\pounds_{\vec{\xi}}\, g \propto \ell \otimes \ell$ and $\pounds_{\vec{\xi}}\, \ell \propto \ell$ where ℓ is a null 1-form. Obviously, as $\pounds_{\vec{\xi}}\, g \in \mathcal{DP}_2$ this can give rise to a one-parameter submonoid of causal transformations. See Example 4 in [25] for an explicit case of this.

Another interesting example arises by considering the typical Robertson-Walker spacetimes $\mathbb{RW}$, the manifold being $I \times M_{N-1}(\kappa)$ where $I \subseteq \mathbb{R}$ is an open interval of the real line with coordinate x^0 and $M_{N-1}(\kappa)$ is the $(N-1)$-dimensional Riemannian space of constant curvature κ, its canonical positive-definite metric being denoted here by g_κ. The Lorentzian metric in $\mathbb{RW}$ is the warped product

$$g = -dx^0 \otimes dx^0 + a^2(x^0)\, g_\kappa$$

where $a(x^0) > 0$ is a C^2 function on I. Take the diffeomorphisms $\varphi_t : \mathbb{RW} \longrightarrow \mathbb{RW}$ which leave $M_{N-1}(\kappa)$ invariant (they are the identity on $M_{N-1}(\kappa)$) and act on I as $x^0 \to x^0 + t$. It is immediate that

$$\varphi_t^* g = -dx^0 \otimes dx^0 + a^2(x^0 + t)\, g_\kappa$$

so that $\varphi_t^* g \in \mathcal{DP}_2^-(\mathbb{RW})$ if and only if $a(x^0 + t) \leq a(x^0)$, and therefore $\varphi_t^* g \in \mathcal{DP}_2^-(\mathbb{RW})$ for $t \in [0, \varepsilon)$ if and only if a is a non-increasing function. Physically this means that $\{\varphi_t\}_{t \in I}$ is a one-parameter submonoid of proper causal transformations in $\mathbb{RW}$ if and only if the Robertson-Walker spacetime is non-expanding. Naturally, the non-contracting case, perhaps of more physical importance, can be studied analogously by simply changing the sign of t.

The infinitesimal generator of this one-parameter group is

$$\vec{\xi} \equiv \left. \frac{d\varphi_t}{dt} \right|_{t=0} = \frac{\partial}{\partial x^0}$$

and the deformation of the metric tensor reads

$$\pounds_{\vec{\xi}}\, g = 2a\dot{a}\ g_\kappa = \frac{2\dot{a}}{a}\ (g + \xi \otimes \xi)$$

where $\dot{a}$ is the derivative of a and $\xi = g(\ ,\vec{\xi}) = dx^0$. Observe that,

$$\pounds_{\vec{\xi}}\, g = \frac{2\dot{a}}{a} T\{\xi\} + \frac{\dot{a}}{a}\, g$$

where $T\{\xi\}$ is the superenergy tensor of ξ. Obviosuly, the sign of $\dot{a}$ is determinant here for $\pounds_{\vec{\xi}}\, g - (\dot{a}/a)\, g$ to be in $\mathcal{DP}_2$, in accordance with the previous reasoning and the theorem 3.18. In fact, in this explicit case, as g_κ is a positive-definite metric, one can prove

$$(\pounds_{\vec{\xi}}\, g)(\vec{x}, \vec{x}) = 2a\dot{a}\ g_\kappa(\vec{x}, \vec{x}), \quad \forall \vec{x} \in T(\mathbb{RW})$$

which has the sign of $\dot{a}$ for *all* vector fields $\vec{x}$. This same property is shared by the Example 4 of [25].

All in all, the deformation $\pounds_{\vec{\xi}}\, g$ produced by one-parameter local submonoids of causal transformations has been shown to be controllable and the generalized symmetries thereby defined can be attacked using traditional techniques.

Acknowledgments

I am grateful to the organizers of the meeting for inviting me to such an exciting exchange of ideas and information between mathematicians and physicists. Some parts of this contribution arise from collaborations with Alfonso García-Parrado and Göran Bergqvist. This work is supported by the research project UPV 172.310-G02/99 of the University of the Basque Country.

References

[1] G. Bergqvist and J.M.M.Senovilla, *Null cone preserving maps, causal tensors and algebraic Rainich theory*, Class. Quantum Grav. **18**, 5299-5325, (2001).

[2] J.F.Plebański, *The algebraic structure of the tensor of matter*, Acta Phys. Pol. **26**, 963-1020 (1964).

[3] S.W.Hawking and G.F.R.Ellis, *The large-scale structure of spacetime*, Cambridge University Press, Cambridge (1973).

[4] J.M.M.Senovilla, *Super-energy tensors*, Class. Quantum Grav. **17**, 2799-2842 (2000).

[5] J.M.M. Senovilla, *Remarks on superenergy tensors*, In *Gravitation and Relativity in General. Proc. Spanish Relativity Meeting in Honour of the 65th Birthday of L Bel* Salamanca 1998, eds. J.Martín, E.Ruiz, F.Atrio and A.Molina, pp.175-182, World Scientific, Singapore (1999).

[6] L. Bel, *Introduction d'un tenseur du quatrième ordre*, C.R. Acad Sci. Paris **248**, 1297-1300 (1959).

[7] L. Bel, *Sur la radiation gravitationnelle*, C.R. Acad Sci. Paris **247**, 1094-1096 (1958)

[8] L. Bel, *Les états de radiation et le problème de l'énergie en relativité générale*, Cahiers de Physique **16**, 59-81 (1962); English translation: Gen. Rel. Grav. **32**, 2047-2078 (2000).

[9] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, Freeman, San Francisco (1970).

[10] G.Y.Rainich, *Electrodynamics in general relativity*, Trans. Amer. Math. Soc. **27**, 106-136 (1925).

[11] C.W.Misner and J.A.Wheeler, *Classical physics as geometry: Gravitation, electromagnetism, unquantized charge, and mass as properties of curved empty space*, Ann.Phys. NY **2**, 525-603 (1957).

[12] R.Penrose and W.Rindler, *Spinors and spacetime*, vol.1-2, Cambridge University Press, Cambridge (1986).

[13] D.LOVELOCK, *Dimensionally dependent identities*, Proc. Camb. Phil. Soc. **68**, 345-350 (1970).

[14] B.COLL AND J.J.FERRANDO, *Thermodynamic perfect fluid. Its Rainich theory*, J. Math. Phys. **30**, 2918-2922 (1989).

[15] G.BERGQVIST AND J.M.M.SENOVILLA, *On the causal propagation of fields*, Class.Quantum Grav. **16**, L55-L61 (1999).

[16] J. M. M. SENOVILLA *Singularity Theorems and their Consequences*, Gen. Rel. Grav. **30**, 701-848, (1998).

[17] M.Á.G.BONILLA AND J.M.M.SENOVILLA *Very simple proof of the causal propagation of gravity in vacuum*, Phys. Rev. Lett. **78**, 783-786 (1997).

[18] J.M.M. SENOVILLA, *(Super)^n-Energy for Arbitrary Fields and its Interchange: Conserved Quantities*, Mod. Phys. Lett. A **15**, 159-165 (2000).

[19] M. MARS AND J.M.M. SENOVILLA, *Geometry of general hypersurfaces in spacetime: junction conditions*, Class. Quantum Grav. **10**, 1865-1897 (1993).

[20] F.G.FRIEDLANDER, *The wave equation on a curved spacetime*, Cambridge University Press, Cambridge (1975).

[21] A. LICHNEROWICZ, *Ondes et radiations électromagnétiques et gravitationnelles en relativité générale*, Ann. di Mat. Pura ed Appl. **50**, 1-95 (1960).

[22] J. HADAMARD, *Leçons sur la propagation des ondes et les équations de l'hydrodynamique*, Hermann, Paris, (1903).

[23] G. GEMELLI, *Gravitational waves and discontinuous motions*, Gen. Rel. Grav. **29** 161-178 (1997).

[24] A. GARCÍA-PARRADO AND J.M.M. SENOVILLA, *Causal relationship: a new tool for the causal characterization of Lorentzian manifolds*, Class. Quantum Grav. **20**, 625-664 (2003).

[25] A. GARCÍA-PARRADO AND J.M.M. SENOVILLA, *Causal relations and their applications*, contribution to this volume.

[26] J. Hilgert, K. H. Hofmann and J. D. Lawson *Lie groups, Convex Cones and Semigroups.* Oxford Sciencie Publications, (1989).

[27] B. Coll, S. R. Hildebrandt and J. M. M. Senovilla. *Kerr Schild Symmetries.* Gen. Rel. Grav. **33**, 649-670, (2001).