II International Meeting on Lorentzian Geometry

Murcia, Spain

November 12–14, 2003

Edited by: Luis J. Alías Linares
Angel Ferrández Izquierdo
María Angeles Hernández Cifre
Pascual Lucas Saorín
José Antonio Pastor González

Organizing Committee

Luis J. Alías Linares (ljalias@um.es)
Angel Ferrández Izquierdo (aferr@um.es)
María Angeles Hernández Cifre (mhcifre@um.es)
Pascual Lucas Saorín (plucas@um.es)
José Antonio Pastor González (jpastor@um.es)

L.J. Alías, A. Ferrández, M.A. Hernández, P. Lucas, J.A. Pastor, editores
II International Meeting on Lorentzian Geometry (Murcia, 2003)

ISBN: 84-933610-5-4
Depósito legal: MU-2109-2004

Impreso en España - Printed in Spain

Preface

This volume contains the proceedings of the meeting «Geometría de Lorentz, Murcia 2003» on Lorentzian Geometry and its Applications. It was held on November 12th, 13th and 14th, at the University of Murcia.

This is the second meeting of a series to which we wish a long life. We take this opportunity to recognize the success of the idea which yielded to organize the meeting of Benalmádena (Málaga). There it was born a very nice atmosphere of friendship and good purposes which we have intended to continue and, as far as we could, to increase. We have done our best to encourage young researchers that year after year joint to the Lorentz adventure. We trust on them as depositaries of a splendid future of the Spanish Lorentzian Geometry.

The local organizing committee of this second meeting accepted the challenge with the aim of pulling up the bar. To do that we have received a generous collaboration from all participants, who helped us to simplify the daily difficulties.

The next meeting will be organized by the Department of Geometry and Topology of the University of Valencia, to whom we yield up the baton wishing them the best.

The organizers would like to thank all participants, specially the invited speakers, for their great efforts in teaching us. We also would like to thank to the referees which have contributed to increase the quality of this volume.

We would like to thank Department of Mathematics of the University of Murcia for all facilities and help given to the organizers. We also would like to thank the financial support of the University of Murcia, Fundación Séneca-Agencia Regional de Ciencia y Tecnología, Ministerio de Ciencia y Tecnología, Fundación Cajamurcia and Caja de Ahorros del Mediterráneo-Obra Social.

We really appreciate the financial support of the Royal Spanish Mathematical Society, which was employed for grants to young researchers attending the meeting as well as for the publication of these proceedings in the series *Publicaciones de la Real Sociedad Matemática Española.*

Luis J. Alías Linares
Angel Ferrández Izquierdo
María Angeles Hernández Cifre
Pascual Lucas Saorín
José Antonio Pastor González

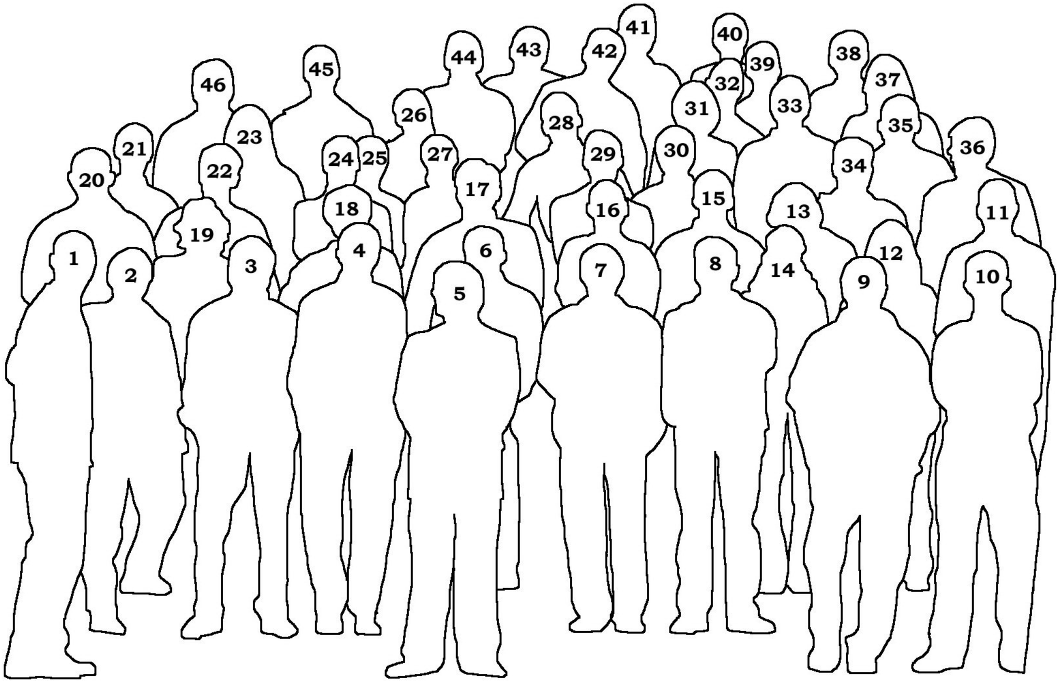
Organizers of the meeting
and editors of the proceedings

List of participants

Aledo Sánchez, Juan Ángel	juanangel.aledo@uclm.es
Alejo Plana, Miguel Ángel	miguel.alejo@terra.es
Alías Linares, Luis José	ljalias@um.es
Barros Díaz, Manuel	mbarros@ugr.es
Brasil Junior, Aldir Chaves	aldir@um.es
Brozos Vázquez, Miguel	mbrozos@usc.es
Caballero Campos, Magdalena	chocolatecafeymenta@hotmail.com
Cabrerizo Jaraíz, José Luis	jaraiz@us.es
Candela, Anna Maria	candela@dm.uniba.it
Caponio, Erasmo	caponio@poliba.it
Díaz Ramos, José Carlos	xtjosec@usc.es
Espinar García, José María	josespinargarc@hotmail.com
Fernández Andrés, Manuel	mafernan@us.es
Fernández Delgado, Isabel	isafer@ugr.es
Fernández Mateos, Víctor Gonzalo	victorfernandez_mateos@hotmail.com
Ferrández Izquierdo, Ángel	aferr@um.es
Ferry Ruiz, José	piensoentucorreo@latinmail.com
Flores Joaquín, Ana Belén	abfj@alu.ua.es
Gálvez López, José Antonio	jagalvez@ugr.es
García Río, Eduardo	xtedugr@usc.es
Guerrero García, Julio	juguerre@um.es
Gutiérrez López, Manuel	mgl@agt.cie.uma.es
Hernández Cifre, Mª Ángeles	mhcifre@um.es
Herrero Piñeyro, Pedro José	pherrero@um.es
Hurtado Cortegana, Ana	ana.maria.hurtado@uv.es

Javaloyes Victoria, Miguel Ángel	majava@um.es
Keller, Jaime	keller@cms.tuwien.ac.at
León Guzmán, María Amelia	mariamelia81@hotmail.com
de León Rodríguez, Manuel	mdeleon@fresno.csic.es
López Rodríguez, Manuel	mlopez@um.es
Lucas Saorín, Pascual	plucas@um.es
Mars Lloret, Marc	marc@usal.es
Mira Carrillo, Pablo	pablo.mira@upct.es
Muñoz Velázquez, Vicente	vicente.munoz@uam.es
Olea Andrades, Benjamín	benji@agt.cie.uma.es
Ortega Titos, Miguel	miortega@ugr.es
Pastor González, José Antonio	jpastor@um.es
Pérez Álvarez, Javier	yaj@eresmas.com
Ristow Montes, Rodrigo	rrm@ime.usp.br
Rodríguez Pérez, M. Magdalena	magdarp@ugr.es
Romero Fuster, María del Carmen	carmen.romero@uv.es
Romero Sarabia, Alfonso	aromero@ugr.es
Salvatore, Addolorata	salvator@dm.uniba.it
Sanabria Codesal, Esther	esanabri@mat.upv.es
Sánchez Caja, Miguel	sanchezm@ugr.es
Sánchez Rodríguez, Ignacio	ignacios@ugr.es
Sánchez Villaseñor, Eduardo Jesús	eduardo@imaff.cfmac.csic.es
Santamaría Merino, Aitor	aitors@imaff.cfmac.csic.es
Saorín Gómez, Eugenia	cottonbutterfly@hotmail.com
Vacaru, Sergiu	vacaru@fisica.ist.utl.pt
Vázquez Lorenzo, Ramón	ravazlor@edu.xunta.es


II Encuentro Internacional
sobre Geometría de Lorentz
Murcia, 12–14 de noviembre de 2003
Departamento de Matemáticas
Universidad de Murcia
CONFERENCIANTES INVITADOS
M. Barros (U. Granada)
E. García-Río (U. Santiago)
M. de León (IMAFF-CSIC)
M. Mars (U. Salamanca)
V. Muñoz (UAM)
A. Romero (U. Granada)
M. Sánchez (U. Granada)
PATROCINADORES Y COLABORADORES



1: José Antonio Pastor González. 2: Anna Maria Candela. 3: Javier Pérez Álvarez. 4: Ángel Ferrández Izquierdo. 5: Pascual Lucas Saorín. 6: Luis José Alías Linares. 7: Manuel Fernández Andrés. 8: Eduardo García Río. 9: Ramón Vázquez Lorenzo. 10: José Carlos Díaz Ramos. 11: Miguel Brozos Vázquez. 12: Magdalena Caballero Campos. 13: Isabel Fernández Delgado. 14: Ana Hurtado Cortegana. 15: Miguel Ortega Titos. 16: Benjamín Olea Andrades. 17: Jaime Keller. 18: Addolorata Salvatore. 19: María del Carmen Romero Fuster. 20: Manuel de León Rodríguez. 21: Vicente Muñoz Velázquez. 22: Aitor Santamaría Merino. 23: Marc Mars Lloret. 24: Miguel Ángel Alejo Plana. 25: María Amelia León Guzmán. 26: María de los Ángeles Hernández Cifre. 27: Eugenia Saorín Gómez. 28: Juan Ángel Aledo Sánchez. 29: Manuel Gutiérrez López. 30: Ignacio Sánchez Rodríguez. 31: María Magdalena Rodríguez Pérez. 32: José María Espinar García. 33: Pablo Mira Carrillo. 34: Julio Guerrero García. 35: Sergiu Vacaru. 36: Eduardo Jesús Sánchez Villaseñor. 37: Víctor Gonzalo Fernández Mateos. 38: Miguel Sánchez Caja. 39: José Antonio Gálvez López. 40: José Luis Cabrerizo Jaraíz. 41: Alfonso Romero Sarabia. 42: Manuel López Rodríguez. 43: Manuel Barros Díaz. 44: Erasmo Caponio. 45: Miguel Ángel Javaloyes Victoria. 46: Aldir Chaves Brasil Junior.

List of contributions

Invited lectures

Communications

Posters

Invited lectures

Proceedings of the II International
Meeting on Lorentzian Geometry
Murcia, November 12–14, 2003
Publ. de la RSME, Vol. 8 (2004), 3–14

Smooth globally hyperbolic splittings and temporal functions

Antonio N. Bernal and Miguel Sánchez

*Depto. Geometría y Topología, Universidad de Granada,
Facultad de Ciencias, Avda. Fuentenueva s/n, E-18071 Granada, Spain*

Abstract

Geroch's theorem about the splitting of globally hyperbolic spacetimes is a central result in global Lorentzian Geometry. Nevertheless, this result was obtained at a topological level, and the possibility to obtain a metric (or, at least, smooth) version has been controversial since its publication in 1970. In fact, this problem has remained open until a definitive proof, recently provided by the authors. Our purpose is to summarize the history of the problem, explain the smooth and metric splitting results (including smoothability of time functions in stably causal spacetimes), and sketch the ideas of the solution.

Keywords: Lorentzian manifold, globally hyperbolic, Cauchy hypersurface, smooth splitting, Geroch's theorem, stably causal spacetime, time function

2000 Mathematics Subject Classification: 53C50, 83C05

1. Introduction

Geroch's theorem [13] is a cut result in Lorentzian Geometry which, essentially, establishes the equivalence for a spacetime (M, g) between: (A) global hyperbolicity, i.e., strong causality plus the compactness of $J^+(p) \cap J^-(q)$ for all $p, q \in M$, and (B) the existence of a Cauchy hypersurface S, i.e. S is an achronal subset which is crossed exactly once by any inextendible timelike curve[1]. Even more, the proof is carried out by finding two elements with interest in its own right: (1) an onto (global) time function $t : M \to \mathbb{R}$ (i.e. the onto function t is continuous and increases strictly on any causal curve) such that each level $t^{-1}(t_0)$, $t_0 \in \mathbb{R}$, is a Cauchy hypersurface and, then, (2) a global topological splitting $M \equiv \mathbb{R} \times S$ such that each slice $\{t_0\} \times S$ is a Cauchy hypersurface. Recall also that the existence of a time function t characterizes stably causal spacetimes (those causal spacetimes which remain causal under C^0 perturbations of the metric).

The possibility to smooth these topological results or continuous elements, have remained as an open folk question since its publication. In fact, Sachs and Wu claimed in their survey on General Relativity in 1977 [20, p. 1155]:

> This is one of the folk theorems of the subject. It is not difficult to prove that every Cauchy surface is in fact a Lipschitzian hypersurface in M [19]. However, to our knowledge, an elegant proof that his Lipschitzian submanifold can be smoothed out [to such an smooth Cauchy hypersurface] is still missing.

Recall that here only the necessity to prove the smoothness of some Cauchy hypersurface S is claimed but, obviously, this would be regarded as a first step towards a fully satisfactory solution of the problem, among the following three:

(i) To ensure the existence of a (smooth) *spacelike* S (necessarily, such an S will be crossed exactly once by any inextendible causal curve).

(ii) To find not only a time function but also a "temporal" one, i.e., smooth with timelike gradient (even for any stably causal spacetime).

(iii) To prove that any globally hyperbolic spacetime admits a smooth splitting $M = \mathbb{R} \times S$ with Cauchy hypersurfaces slices $\{t_0\} \times S$ orthogonal to ∇t (and, thus, with a metric without cross terms between $\mathbb{R}$ and S).

Among the concrete applications of (i), recall, for example: (a) Cauchy hypersurfaces are the natural regions to pose (smooth!) initial conditions for hyperbolic equations, as Einstein's ones, or (b) differentiable achronal hypersurfaces (as those with prescribed mean curvature [9]) can be regarded as differentiable graphs on any smooth Cauchy hypersurface. The smoothness of a time function t claimed in (ii), would yield the possibility to use its gradient, which can be used to split any stably causal spacetime, as in [12]. The applications of the full smooth splitting result (iii) include topics such as Morse

[1]In particular, S is a topological hypersurface (without boundary), and it is also crossed at some point -perhaps even along a segment- by any lightlike curve.

Theory for lightlike geodesics [22], quantization [10] or the possibility to use variational methods [15, Chapter 8]; it also opens the possibility to strengthen other topological splitting results [16] into smooth ones.

Recently, we have given a full solution to these three questions (i)—(iii) [3, 4]. Our purpose in this talk is, first, to summarize the history of the problem and previous attemps (Section 2) as well as the background results (Section 3). In the two following sections, our main results are stated and the ideas of the proofs sketched[2]. Concretely, Section 4 is devoted to the construction of a smooth and spacelike S, following [3], and Section 5 to the full splitting of globally hyperbolic spacetimes, plus the existence of temporal functions in stably causal ones, following [4]. The reader is referred to the original references [3, 4] for detailed proofs and further discussions.

2. A brief history of time functions

As far as we know, the history of the smoothing splitting theorem can be summarized as follows.

1. Geroch published his result in 1970 [13], stating clearly all the results at a topological level. Penrose cites directly Geroch's paper in his book (1972), and regards explicitly the result as topological [19, Theorems 5.25, 5.26]. A subtle detail about his statement of the splitting result [19, Theorem 5.26] is the following. It is said there that, fixed $x \in S$, the curve $t \to \gamma(t) = (t, x) \in \mathbb{R} \times S$ is timelike. Recall that this curve does satisfy $t < t' \Rightarrow \gamma(t) << \gamma(t')$, but it is not necessarily smooth (its parameter is only continuous). In Penrose's definition [19, pp. 2, 3] timelike curves are assumed smooth, but later on [19, p. 17] the definition of causal curves is extended; in general, they will be just Lipschitzian, as previous curve γ.

2. Seifert's thesis (1968) develops a smoothing regularization procedure for time functions, which would yield the global splitting (Theorem 5.1 below). His smoothing results were published in 1977, [21], but the paper contains major gaps which could not be filled later.

3. In Hawking, Ellis' book (1973), the equivalence between stable causality and the existence of a (continuous) time function is achieved, by following a modification of Geroch's technique. Nevertheless, they assert [14,

[2]The results of Section 5 has been obtained just some weeks ago, and they were not known at the time of the meeting in November 2003, but they are sketched here because of their obvious interest for our problem.

Proposition 6.4.9] that stable causality holds if and only if a (smooth) function with timelike gradient exists. Unfortunately, they refer for the details of the smoothing result to Seifert's thesis. Even more, in [14, Proposition 6.6.8], Geroch's result is stated at a topological level, but they refer to the possibility to smooth the result at the end of the proof. Nevertheless, again, the cited technique is the same for time functions.

4. In 1976, Budic and Sachs carried out a smoothing for *deterministic* spacetimes. One year later Sachs and Wu [20] posed the smoothing problem as a folk topic in General Relativity, in the above quoted paragraph.

5. The prestige and fast propagation of some of the previous references, made even the strongest splitting result be cited as proved in many references, including new influential references or books (for example, [10, 15, 22, 23]). But this is not the case for most references in pure Lorentzian Geometry, as O'Neill's book [18] (or, for example, [5, 9, 11, 16]). Even more, in Beem, Ehrlich's book (1981) Sachs and Wu's claim is referred explicitly [1, p. 31].

6. In 1988, Dieckmann claimed to prove the "folk question"; nevertheless, he cited Seifert's at the crucial step [7, Proof of Theorem 1]. More precisely, his study (see [8]) clarified other point in Geroch's proof, concerning the existence of an appropiate finite measure on the manifold. Even though the straightforward way to construct this measure in Hawking-Ellis' book [14, proof of Proposition 6.4.9] is correct, neither these authors nor Geroch considered the necessary abstract properties that such a measure must fulfill (in particular, the measure of the boundaries $\partial I^+(p)$ must be 0). Under this approach, on one hand, the admissible measures for the proof of Geroch's theorem are characterized and, on the other, a striking relationship between continuity of volume functions and reflectivity is obtained.

   In the 2nd edition of Beem-Ehrlich's book, in collaboration with Easley (1996), these improvements by Dieckmann are stressed, but Geroch's result is regarded as topological, and the reference to Sachs and Wu's claim is maintained [2, p. 65].

In general, continuous functions can be approximated by smooth functions. Thus, a natural way to proceed would be to approximate the continuous time function provided in Geroch's result, by a smooth one. Nevertheless, this intuitive idea has difficulties to be formalized. Thus, our approach has been different. First, we managed to smooth a Cauchy hypersurface [3] and, then, we constructed the full time function with the required properties [4].

3. Setup and previous results

Detailed proofs of the fact that the existence of a Cauchy hypersurface implies global hyperbolicity can be found, for example, in [13, 14, 18]. We will be interested in the converse, and then Geroch's results can be summarized in Theorem 3.2, plus Lemma 3.1 and Corollary 3.3.

Lemma 3.1 *Let M be a (C^k-)spacetime which admits a C^r-Cauchy hypersurface S, $r \in \{0, 1, \ldots k\}$. Then M is C^r-diffeomorphic to $\mathbb{R} \times S$ and all the C^r-Cauchy hypersurfaces are C^r diffeomorphic.* This lemma is proven by moving S with the flow Φ_t of any complete timelike vector field. Thus, the (differentiable) hypersurfaces at constant $t \in \mathbb{R}$ are not necessarily Cauchy nor even spacelike, except for $t = 0$.

Theorem 3.2 *Assume that the spacetime M is globally hyperbolic. Then there exists a continuous and onto map $t : M \to \mathbb{R}$ satisfying:*

(1) $S_a := t^{-1}(a)$ is a Cauchy hypersurface, for all $a \in \mathbb{R}$.

(2) t is strictly increasing on any causal curve. Function t is constructed as

$$t(z) = \ln\left(\mathrm{vol}(J^-(z))/\mathrm{vol}(J^+(z))\right)$$

for a (suitable) finite measure on M and, thus, global hyperbolicity implies just its continuity. Finally, combining both previous results,

Corollary 3.3 *Let M be a globally hyperbolic spacetime. Then there exists a homeomorphism*

$$\Psi : M \to \mathbb{R} \times S_0, \quad z \to (t(z), \rho(z)), \tag{1}$$

which satisfies:

(a) Each level hypersurface $S_t = \{z \in M : t(z) = t\}$ is a Cauchy hypersurface.

(b) Let $\gamma_x : \mathbb{R} \to M$ be the curve in M characterized by:

$$\Psi(\gamma_x(t)) = (t, x), \qquad \forall t \in \mathbb{R}.$$

Then the continuous curve γ_x is timelike in the following sense:

$$t < t' \Rightarrow \gamma_x(t) << \gamma_x(t').$$

Remark 3.4 If function t in Corollary 3.3 were smooth with timelike gradient, then the spacetime (M, g) would be isometric to $\mathbb{R} \times \mathcal{S}, \langle \cdot, \cdot \rangle = -\beta\, dt^2 + \bar{g}$, where $\bar{g}$ is a (positive definite) Riemannian metric on each slice $t =$ constant.

The splitting is then obtained projecting M to a fixed level hypersurface by means of the flow of ∇t. In what follows, a smooth function $\mathcal{T}$ with past-pointing timelike gradient $\nabla\mathcal{T}$ will be called *temporal* (and it is necessarily a time function). For the smoothing procedure, some properties of Cauchy hypersurfaces will be needed. Concretely, by using a result on intersection theory, the following one can be proven [3, Section 3], [11, Corollary 2]:

Proposition 3.5 *Let S_1, S_2 be two Cauchy hypersurfaces of a globally hyperbolic spacetime with $S_1 << S_2$ (i.e., $S_2 \subset I^+(S_1)$), and S be a connected closed spacelike hypersurface (without boundary):*

(A) If $S_1 << S$ then S is achronal, and a graph on all S_1 for the descompositions in Corollaries 3.1, 3.3.

(B) If $S_1 << S << S_2$ then S is a Cauchy hypersurface.

4. Smooth spacelike Cauchy hypersurfaces

In this section, we sketch the proof of:

Theorem 4.1 *Any globally hyperbolic spacetime admits a smooth spacelike Cauchy hypersurface S.* (In what follows, "smooth" will mean with the same order of differentiability of the spacetime).

From Proposition 3.5, given two Cauchy hypersurfaces $S_1 << S_2$ as in Theorem 3.2 (with $S_{t_i} \equiv S_i$; $t_1 < t_2$), it is enough to contruct a connected closed spacelike hypersurface S with $S_1 << S << S_2$. And, in order to prove this, it suffices:

Proposition 4.2 *Let M be a globally hyperbolic spacetime with topological splitting $\mathbb{R} \times S$ as in Corollary 3.3, and fix $S_1 << S_2$. Then there exists a smooth function*

$$h : M \to [0, \infty)$$

which satisfies:

1. *$h(t, x) = 0$ if $t \leq t_1$.*
2. *$h(t, x) > 1/2$ if $t = t_2$.*
3. *The gradient of h is timelike and past-pointing on the open subset $V = h^{-1}((0, 1/2)) \cap I^-(S_2)$.*

In fact, recall that, given such a function h, each $s \in (0, 1/2)$ is a regular value of the restriction of h to $J^-(S_2)$. Thus, $S_s^h := h^{-1}(s) \cap J^-(S_2)$ is a closed smooth spacelike hypersurface with $S_1 << S_s^h << S_2$ and, then, a Cauchy hypersurface (in principle, Proposition 3.5(B) can be applied to any connected component of S_s^h, but one can check that, indeed, S_s^h is connected).

The construction of function h is carried out in two closely related steps. The first one is a local step: to construct, around each $p\,(\in S_2)$, a function h_p with the suitable properties, stated in Lemma 4.3. The second step is to construct (global) function h from the h_p's. This function will be constructed directly as a sum[3] $h = \sum h_i$ for suitable $h_i \equiv h_{p_i}$. This fact must be taken into account for the properties of the h_p's in the first step:

Lemma 4.3 *Fix $p \in S_2$, and a convex neighborhood of p, $\mathcal{C}_p \subset I^+(S_1)$ (that is, $\mathcal{C}_p$ is a normal starshaped neighborhood of any of its points).*

Then there exists a smooth function

$$h_p : M \to [0, \infty)$$

which satisfies:

(i) $h_p(p) = 1$.

(ii) The support of h_p (i.e., the closure of $h_p^{-1}(0,\infty)$) is compact and included in $\mathcal{C}_p \cap I^+(S_1)$.

(iii) If $q \in J^-(S_2)$ and $h_p(q) \neq 0$ then $\nabla h_p(q)$ is timelike and past-pointing. *Sketch of proof.* Function h_p is taken in a neighbourhood of $\mathcal{C}_p \cap J^-(S_2)$ as:

$$h_p(q) = e^{d(p',p)^{-2}} \cdot e^{-d(p',q)^{-2}},$$

where d is the time-separation (Lorentzian distance) on $\mathcal{C}_p$, and p' is a fixed suitably chosen point in the past of p. ■

Now, the second step is carried out by taking advantage directly of the paracompactness of the manifold. Concretely, function $h = \sum_i h_i$ is obtained by choosing the h_i's from the following lemma, with the W_α's equal to $h_p^{-1}(1/2, \infty)$, and $p \in S_2$ (see [3] for details):

Lemma 4.4 *Let d_R be the distance on M associated to any auxiliary complete Riemannian metric g_R. Let S_2 be a closed subset of M and $\mathcal{W} = \{W_\alpha, \alpha \in \mathcal{I}\}$ a collection of open subsets of M which cover S_2. Assume that each W_α is included in an open subset $\mathcal{C}_\alpha$ and the d_R-diameter of each $\mathcal{C}_\alpha$ is smaller than 1. Then there exist a subcollection $\mathcal{W}' = \{W_j : j \in \mathbb{N}\} \subset \mathcal{W}$ which covers*

[3]In Riemannian Geometry, global objects are constructed frequently from local ones by using partitions of the unity. Nevertheless, the causal character of the gradient of functions in the partition are, in principle, uncontrolled. Then, the underlying idea to construct h is to use the paracompactness of M (which is implied by the existence of a Lorentzian –or semi-Riemannian– metric) avoiding to use a partition of the unity.

S_2 and is locally finite (i.e., for each $p \in \cup_j W_j$ there exists a neighborhood V such that $V \cap W_j = \emptyset$ for all j but a finite set of indexes). Moreover, the collection $\{\mathcal{C}_j : j \in \mathbb{N}\}$ (where each $W_j \in \mathcal{W}'$ is included in the corresponding $\mathcal{C}_j$) is locally finite too.

5. Temporal functions and the full splitting

Now, our aim is to sketch the proof of the following theorem.

Theorem 5.1 *Let (M, g) be a globally hyperbolic spacetime. Then, it is isometric to the smooth product manifold*

$$\mathbb{R} \times \mathcal{S}, \quad \langle \cdot, \cdot \rangle = -\beta \, d\mathcal{T}^2 + \bar{g}$$

where $\mathcal{S}$ is a smooth spacelike Cauchy hypersurface, $\mathcal{T} : \mathbb{R} \times \mathcal{S} \to \mathbb{R}$ is the natural projection, $\beta : \mathbb{R} \times \mathcal{S} \to (0, \infty)$ a smooth function, and $\bar{g}$ a 2-covariant symmetric tensor field on $\mathbb{R} \times \mathcal{S}$, satisfying:

1. *$\nabla \mathcal{T}$ is timelike and past-pointing on all M, that is, function $\mathcal{T}$ is temporal.*

2. *Each hypersurface $\mathcal{S}_\mathcal{T}$ at constant $\mathcal{T}$ is a Cauchy hypersurface, and the restriction $\bar{g}_\mathcal{T}$ of $\bar{g}$ to such a $\mathcal{S}_\mathcal{T}$ is a Riemannian metric (i.e. $\mathcal{S}_\mathcal{T}$ is spacelike).*

3. *The radical of $\bar{g}$ at each $w \in \mathbb{R} \times \mathcal{S}$ is Span$\nabla \mathcal{T}$ (=Span $\partial_\mathcal{T}$) at w.*

Essentially, it is enough for the proof to obtain a temporal function $\mathcal{T} : M \to \mathbb{R}$ such that each level hypersurface is Cauchy, see Remark 3.4. The existence of such a $\mathcal{T}$ is carried out in three steps.

Step 1: temporal step functions would solve the problem. Let $t \equiv t(z)$ be a continuous time function as in Geroch's Theorem 3.2. Fixed $t_- < t \in \mathbb{R}$, we have proven in Section 4 the existence of a smooth Cauchy hypersurface $\mathcal{S}$ contained in $t^{-1}(t_-, t)$; this hypersurface is obtained as the regular value of certain function $h \equiv h_t$ with timelike gradient on $t^{-1}(t_-, t]$. As t_- approaches t, $\mathcal{S}$ can be seen as a smoothing of S_t; nevertheless $\mathcal{S}$ always lies in $I^-(S_t)$. Now, we claim that the required splitting of the spacetime would be obtained if we could strengthen the requirements on this function h_t, ensuring the existence of a *temporal step function* τ_t around each S_t. Essentially such a τ_t is a function with timelike gradient on a neighborhood of S_t (and 0 outside) with level Cauchy hypersurfaces which cover a rectangular neighbourhood of S_t:

Lemma 5.2 *All the conclusions of Theorem 5.1 will hold if the globally hyperbolic spacetime M admits, around each Cauchy hypersurface S_t, $t \in \mathbb{R}$, a (temporal step) function $\tau_t : M \to \mathbb{R}$ which satisfies:*

1. *$\nabla\tau_t$ is timelike and past-pointing where it does not vanish, that is, in the interior of its support $V_t := \mathrm{Int}(\mathrm{Supp}(\nabla\tau_t))$.*
2. *$-1 \leq \tau_t \leq 1$.*
3. *$\tau_t(J^+(S_{t+2})) \equiv 1$, $\tau_t(J^-(S_{t-2})) \equiv -1$.*
4. *$S_{t'} \subset V_t$, for all $t' \in (t-1, t+1)$; that is, the gradient of τ_t does not vanish in the rectangular neighborhood of S, $t^{-1}(t-1, t+1) \equiv (t-1, t+1) \times S$.*

Sketch of proof. Consider such a function τ_k for $k \in \mathbb{Z}$, and define the (locally finite) sum $\mathcal{T} = \tau_0 + \sum_{k=1}^{\infty}(\tau_{-k} + \tau_k)$. One can check that $\mathcal{T}$ fulfills the required properties in Remark 3.4, in fact: (a) $\mathcal{T}$ is temporal because subsets $V_{t=k}, k \in \mathbb{Z}$ cover all M (and the timelike cones are convex), and (b) the level hypersurfaces of $\mathcal{T}$ are Cauchy because, for each inextendible timelike curve $\gamma : \mathbb{R} \to M$ parameterized with $\mathcal{T}$, $\lim_{s\to\pm\infty}(\mathcal{T}(\gamma(s))) = \pm\infty$. ■

Step 2: constructing a weakening of a temporal step function. Lemma 5.2 reduces the problem to the construction of a temporal step function τ_t for each t. We will start by constructing a function $\hat{\tau}_t$ which satisfies all the conditions in that lemma but the last one, which is replaced by:

$\hat{4}$. $S_t \subset V_t$.

The idea for the construction of such a $\hat{\tau}_t$ is the following. Consider function h in Lemma 4.2 for $t_1 = t - 1, t_2 = t$. From its explicit construction, it is straightforward to check that h can be also assumed to satisfy: ∇h is timelike and past-pointing on a neighborhood $U' \subset I^-(S_{t+1})$ of S_t. Thus, putting $U = U' \cup I^-(S_t)$ (U satisfies $J^-(S_{t-1}) \subset U \subset I^-(S_{t+1})$), we find a function $h^+ : M \to \mathbb{R}$ which satisfies:

(i$^+$) $h^+ \geq 0$ on U, with $h^+ \equiv 0$ on $I^-(S_{t-1})$.

(ii$^+$) If $p \in U$ with $h^+(p) > 0$ then $\nabla h^+(p)$ is timelike and past-pointing.

(iii$^+$) $h^+ > 1/2$ (and, thus, its gradient is timelike past-pointing) on $J^+(S_t) \cap U$.

Even more, a similar reasoning yields a function $h^- : M \to \mathbb{R}$ for this same U which satisfies:

(i$^-$) $h^- \leq 0$, with $h^- \equiv 0$ on $M \backslash U$.

(ii$^-$) If $\nabla h^-(p) \neq 0$ at $p\,(\in U)$ then $\nabla h^-(p)$ is timelike past-pointing.

(iii$^-$) $h^- \equiv -1$ on $J^-(S_t)$.

Now, as $h^+ - h^- > 0$ on all U, we can define:

$$\hat{\tau}_t = 2\,\frac{h^+}{h^+ - h^-} - 1$$

on U, and constantly equal to 1 on $M\backslash U$. A simple computation shows that $\nabla\hat{\tau}_t$ does not vanish wherever either $h^-\nabla h^+$ or $h^+\nabla h^-$ does not vanish (in particular, on S_t) and, then, it fulfills all the required conditions.

Step 3: construction of a true temporal step function. Now, our aim is to obtain a function $\tau(\equiv \tau_t)$ which satisfies not only the requirements of previous $\hat{\tau} \equiv \hat{\tau}_t$ but also the stronger condition 4 in Lemma 5.2. Fix any compact subset $K \subset t^{-1}([t-1, t+1])$. From the construction of $\hat{\tau}$, it is easy to check that $\hat{\tau}$ can be chosen with $\nabla\hat{\tau}$ non-vanishing on K. Now, choose a sequence $\{G_j : j \in \mathbb{N}\}$ of open subsets such that:

$$\overline{G_j} \text{ is compact, } \overline{G_j} \subset G_{j+1} \quad M = \cup_{j=1}^{\infty} G_j,$$

and let $\hat{\tau}[j]$ be the corresponding sequence of functions type $\hat{\tau}$ with gradients non-vanishing on:

$$K_j = \overline{G_j} \cap t^{-1}([t-1, t+1]).$$

Essentially, the required temporal step function is:

$$\tau = \sum_{j=1}^{\infty} \frac{1}{2^j A_j} \hat{\tau}[j],$$

where each A_j is chosen to make τ smooth (fixed a locally finite atlas on M, each A_j bounds on $\overline{G_j}$ each function $\hat{\tau}[j]$ and its partial derivatives up to order j in the charts of the atlas which intersect G_j). Then, the gradient of τ is timelike wherever one of the gradients $\nabla\hat{\tau}[j]$ does not vanish (in particular, on $t^{-1}([t-1, t+1])$). Moreover, τ is equal to constants (which can be rescaled to ± 1) on $t^{-1}((-\infty, t-2]), t^{-1}([t+2, \infty))$, as required. ■

Finally, it is worth pointing out that similar arguments work to find a smooth time function on any spacetime (even non-globally hyperbolic) which admits a continuous time function t.

Theorem 5.3 *Any spacetime M which admits a (continuous) time function (i.e., is stably causal) also admits a temporal function $\mathcal{T}$.*

Sketch of proof. Notice that each level continuous hypersurface S_t is a Cauchy hypersurface in its Cauchy development $D(S_t)$. Moreover, any temporal step function τ_t around S_t in $D(S_t)$ can be extended to all M (making τ_t equal to 1 on $I^+(S_t) \cap (M\backslash D(S))$, and to -1 on $I^+(S_t) \cap (M\backslash D(S))$). Then, sum suitable temporal step functions as in Step 3 above. ■

Acknowledgments

The second-named author has been partially supported by a MCyT-FEDER Grant BFM2001-2871-C04-01.

References

[1] J.K. Beem and P.E. Ehrlich, *Global Lorentzian geometry*, Monographs and Textbooks in Pure and Applied Math., **67**, Marcel Dekker, Inc., New York (1981).

[2] J.K. Beem, P.E. Ehrlich and K.L. Easley, *Global Lorentzian geometry*, Monographs Textbooks Pure Appl. Math. **202**, Dekker Inc., New York (1996).

[3] A. N. Bernal and M. Sánchez, On Smooth Cauchy Hypersurfaces and Geroch's Splitting Theorem, *Commun. Math. Phys.*, **243** (2003) 461-470.

[4] A. N. Bernal and M. Sánchez, Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes, preprint, gr-qc/0401112

[5] R. Budic, J. Isenberg, L. Lindblom and P.B. Yasskin, On determination of Cauchy surfaces from intrinsic properties. *Comm. Math. Phys.* **61** (1978), no. 1, 87–95.

[6] R. Budic and R.K. Sachs, Scalar time functions: differentiability. Differential geometry and relativity, pp. 215–224, *Mathematical Phys. and Appl. Math.*, Vol. 3, Reidel, Dordrecht, 1976.

[7] J. Dieckmann, Volume functions in general relativity, Gen. Relativity Gravitation **20** (1988), no. 9, 859–867.

[8] J. Dieckmann, Cauchy surfaces in a globally hyperbolic space-time, *J. Math. Phys.* **29** (1988), no. 3, 578–579.

[9] K. Ecker and G. Huisken, Parabolic methods for the construction of spacelike slices of prescribed mean curvature in cosmological spacetimes, *Comm. Math. Phys.* **135** (1991), no. 3, 595–613.

[10] E.P. Furlani, Quantization of massive vector fields in curved spacetime, *J. Math. Phys.* **40** (1999), no. 6, 2611–2626.

[11] G.J. GALLOWAY, Some results on Cauchy surface criteria in Lorentzian geometry, *Illinois J. Math.*, **29** (1985), no. 1, 1–10.

[12] E. GARCÍA-RÍO AND D.N. KUPELI, Some splitting theorems for stably causal spacetimes. *Gen. Relativity Gravitation* **30** (1998), no. 1, 35–44.

[13] R. GEROCH, Domain of dependence, *J. Math. Phys.* **11** (1970) 437–449.

[14] S.W. HAWKING AND G.F.R. ELLIS, *The large scale structure of space-time,* Cambridge Monographs on Mathematical Physics, No. 1. Cambridge University Press, London-New York (1973).

[15] A. MASIELLO, *Variational methods in Lorentzian geometry*, Pitman Res. Notes Math. Ser. **309**, Longman Sci. Tech., Harlow (1994).

[16] R.P.A.C. NEWMAN, The global structure of simple space-times, *Comm. Math. Phys.* **123** (1989), no. 1, 17–52.

[17] K. NOMIZU AND H. OZEKI, The existence of complete Riemannian metrics *Proc. Amer. Math. Soc.* **12** (1961) 889–891.

[18] B. O'NEILL, *Semi-Riemannian geometry with applications to Relativity*, Academic Press Inc., New York (1983).

[19] R. PENROSE, *Techniques of Differential Topology in Relativity*, CBSM-NSF Regional Conference Series in applied Mathematics, SIAM, Philadelphia (1972).

[20] R.K. SACHS AND H. WU, General relativity and cosmology, *Bull. Amer. Math. Soc.* **83** (1977), no. 6, 1101–1164.

[21] H.-J. SEIFERT, Smoothing and extending cosmic time functions, *Gen. Relativity Gravitation* **8** (1977), no. 10, 815–831.

[22] K. UHLENBECK, A Morse theory for geodesics on a Lorentz manifold. *Topology* **14** (1975), 69–90.

[23] R.M. WALD, *General Relativity*, Univ. Chicago Press, 1984.

Proceedings of the II International
Meeting on Lorentzian Geometry
Murcia, November 12–14, 2003
Publ. de la RSME, Vol. 8 (2004), 15–27

On some volume comparison results in Lorentzian geometry

J. C. Díaz-Ramos, E. García-Río and L. Hervella

Department of Geometry and Topology, Faculty of Mathematics, Univ. of Santiago de Compostela, 15782 Santiago de Compostela (Spain)

emails: xtjosec@usc.es, xtedugr@usc.es, hervella@zmat.usc.es

Abstract

Volume comparison results in Lorentzian geometry are reviewed, with special attention to the behavior of geodesic celestial spheres.

Keywords: Volume comparison, truncated light cones, distance wedges, SCLV-sets, geodesic celestial spheres

2000 Mathematics Subject Classification: 53C50

1. Introduction

In the study of geometric properties of a semi-Riemannian manifold (M, g) it is often useful to consider geometric objects naturally associated to (M, g). These can be special hypersurfaces like small geodesic spheres and tubes in Riemannian geometry, bundles with (M, g) as base manifold, families of transformations reflecting symmetry properties of (M, g), or natural operators defined by the curvature tensor of (M, g). The existence of a relation between the curvature of the manifold and the properties of those objects led to the following question.

[0]Work partially supported by project BFM2003-02949, Spain.

> To what extent is the curvature (or the geometry) of a given semi-Riemannian manifold (M, g) influenced, or even determined, by the properties of certain naturally defined families of geometric objects in M?

This problem seems very difficult to handle in such a generality. However, when one looks at manifolds with a high degree of symmetry (e.g., two-point homogeneous spaces), these geometric objects have nice properties and one may expect to obtain characterizations of those spaces by means of such properties. For instance, the Riemannian two-point homogeneous spaces may be characterized by using the spectrum of their geodesic spheres [6] or in most cases by the L^2-norm of the curvature tensor of geodesic spheres [7]. Lorentzian manifolds of constant sectional curvature are characterized by the Osserman property on their Jacobi operators [13], [14].

The purpose of this paper is twofold. Firstly to review some contributions to the study of the general problem above in the framework of Riemannian and Lorentzian geometry by focusing on volume comparison properties of geodesic spheres and some special subsets of a Lorentzian manifold. Secondly we consider a new family of objects in Lorentzian geometry, namely geodesic celestial spheres associated to an observer field and state some comparison results for the volume of such objects.

2. Some remarks on the Riemannian framework

Any Riemannian manifold (M^{n+1}, g) carries a Riemannian distance function which has a very nice behavior with respect to the underlying structure of the manifold. Therefore, a natural family of subregions of a Riemannian manifold to be considered is that defined by the level sets of the Riemannian distance function with respect to a basepoint (i.e., geodesic spheres centered at the basepoint) or with respect to some topologically embedded submanifolds (i.e., tubes around the submanifold).

For sufficiently small radii $r > 0$, geodesic spheres $S_m(r)$ are obtained by projecting the Euclidean spheres $S_0^n(r)$ centered at $0 \in T_mM$ in the tangent space T_mM of the manifold via the exponential map. Therefore, they are a nice family of hypersurfaces and moreover their volume can be calculated as

$$vol(S_m(r)) = r^n \int_{S_0^n(1)} \theta_m(exp_m(ru))du, \tag{1}$$

where u varies along $S_0^n(1) \subset T_mM$ and θ_m denotes the volume density function of exp_m with respect to normal coordinates; $\theta_m = (det(g))^{1/2}$. A fundamental observation for the purposes of volume comparison is that the volume density function satisfies

$$\frac{\theta_m'(r)}{\theta_m(r)} = -\left(\frac{n}{r} + trS(r)\right) \tag{2}$$

where $S(r)$ represents the shape operator of the hypersurface $S_m(r)$ and furthermore, the operators $S(r)$ and $S'(r)$ are symmetric and satisfy a matrix Riccati differential equation

$$S'(r) = S(r)^2 + R(r) \tag{3}$$

where $R(r)$ is the Jacobi operator associated to the vector field defined by the gradient of the distance function with respect to the center m. Now, the basic idea behind the Bishop-Günter and Gromov comparison theorems (see for example [15]) is that under suitable curvature conditions the Riccati differential equation (3) becomes an inequality and its solutions give upper or lower bounds for the volume density function θ_m in terms of the corresponding function in the model space via (2). Finally, an integration process from (1) leads to

Theorem 2.1 [3], [17] *Let (M^{n+1}, g) be a complete Riemannian manifold and assume that r is not greater than the distance between m and its cut locus. Let K^M denote the sectional curvature of (M, g).*

(i) If $K^M \geq \lambda$, then $vol^M(S_m(r)) \leq vol^{M(\lambda)}(S_{\overline{m}}(r))$

(ii) If $K^M \leq \lambda$, then $vol^M(S_m(r)) \geq vol^{M(\lambda)}(S_{\overline{m}}(r))$

where $M(\lambda)$ is a model space of constant sectional curvature λ.

Moreover, equalities hold for (i) or (ii) and some radii if and only if $S_m(r)$ is isometric to the corresponding geodesic sphere in the model space.

A sharper result involving the Ricci curvature instead of the sectional curvature was proved by Bishop as:

Theorem 2.2 [3] *Let (M^{n+1}, g) be a complete Riemannian manifold. Assume that r is not greater than the distance between m and its cut locus and the Ricci curvature ρ^M of (M, g) satisfies $\rho^M(v, v) \geq n\lambda$ for all vectors v.*

Then $vol^M(S_m(r)) \leq vol^{M(\lambda)}(S_{\overline{m}}(r))$, where $M(\lambda)$ is a model space of constant sectional curvature λ, and equality holds if and only if $S_m(r)$ is isometric to the corresponding geodesic sphere in the model space.

A further generalization of Theorem 2.2 was obtained by Gromov as follows.

Theorem 2.3 [16] *Let (M^{n+1}, g) be a complete Riemannian manifold. Assume that r is not greater than the distance between m and its cut locus and the Ricci curvature ρ^M of (M, g) satisfies $\rho^M(v, v) \geq n\lambda$ for all vectors v. Then*

$$r \mapsto \frac{vol^M(S_m(r))}{vol^{M(\lambda)}(S_{\overline{m}}(r))}$$

is nonincreasing, where $M(\lambda)$ is a space of constant sectional curvature λ.

3. Moving to the Lorentzian framework

When the attention is turned from Riemannian manifolds to spacetimes, various difficulties emerge. For example, conditions on bounds for the sectional curvature (resp., the Ricci tensor) easily produce manifolds of constant sectional curvature (resp., Einstein) [2], [19]. This demands a revision of such conditions [1] (see §4). However, a more difficult task is related to the consideration of the regions under investigation. This is mainly due to the fact that when dealing with general semi-Riemannian manifolds there is no "semi-Riemannian distance function". In fact, a distance-like function is only defined for spacetimes, but even in this case its properties are completely different from those in the Riemannian setting (cf. [2]). For instance, level sets of the Lorentzian distance function with respect to a given point are not compact and they do not seem to be adequate for the investigation of volume properties. Therefore, different families of objects have been considered in Lorentzian geometry for the purpose of investigating their volume properties. Among those, truncated light cones, compact distance wedges in the chronological future of some point, and more generally some neighborhoods covered by timelike geodesics emanating from a given point have been investigated. Next we will review some known results on the geometry of those families.

3.1. Truncated light cones

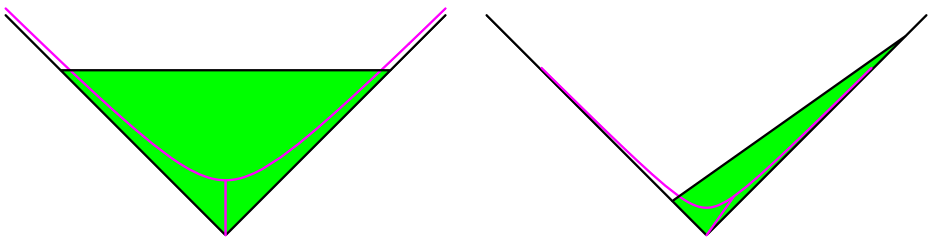
Truncated light cones were defined in [11], [12] were the authors studied the link between the volume of the light cones and the curvature of a Lorentzian manifold.

Definition 3.1 [11], [12] Let ξ be an instantaneous observer. The *truncated light cone of* (sufficiently small) *height* T and *axis* ξ is the set

$$L_\xi(T) = \left\{ exp_m(u) \,/\, \langle u, u\rangle < 0, \;\; 0 \leq -\langle u, \xi\rangle \leq T \right\}$$

It is easy to see that the volume of a truncated light cone in the four-dimensional Minkowski spacetime is given by $vol(L_\xi(T)) = \frac{1}{3}\pi T^4$. The investigation of whether this volume property is characteristic of the Minkowskian space motivated further work by R. Schimming [20], [21], who proved the following

Theorem 3.2 [12], [20] *Let (M, g) be a Lorentzian manifold such that every truncated light cone has the same volume as in the Minkowskian spacetime. Then (M, g) is locally flat.*



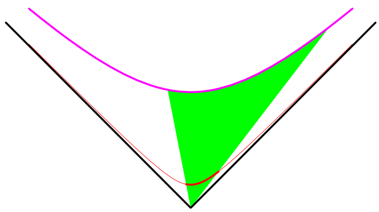
Truncated light cones in $\mathbb{R}^2_1$ of height $T = 3$ and axis $\xi_1 = (0, 1)$ and $\xi_2 = (1, \sqrt{2})$.

3.2. Compact distance wedges

Let E denote the set of future pointing timelike unit vectors in T_mM such that the exponential map is well defined. Let K be a compact subset of E and put $\mathfrak{K} = exp_m(t_0K)$, which is a compact subset of the level set $d_m^{-1}(t_0)$ of the Lorentzian distance function with respect to $m \in M$, and is well defined for sufficiently small t_0.

Definition 3.3 [9] The $\mathfrak{K}$-*distance wedge* $B_m^{\mathfrak{K}}(t_0)$ is defined by

$$B_m^{\mathfrak{K}}(t_0) = \{exp_m(tv) \,/\, v \in K, \;\; 0 \le t \le t_0\}$$



In order to study volume comparison results with model spaces, one needs a method to relate distance wedges on M and the model space. One proceeds as follows. Choose a point $\overline{m}$ in the model space of constant sectional curvature $M(-\lambda)$ and define a differentiable map Ψ by $\Psi = exp_{\overline{m}}^{M(-\lambda)} \circ \psi \circ (exp_m^M)^{-1}$, where $\psi : T_mM \to T_{\overline{m}}M(-\lambda)$ is a linear isometry. Then given a distance wedge $B_m^{\mathfrak{K}}(t_0)$ and using the timelike vectors $\psi(K)$ in $T_{\overline{m}}M(-\lambda)$ to construct the corresponding wedge $B_{\overline{m}}^{\Psi(\mathfrak{K})}(t_0)$ in $M(-\lambda)$, we have $B_{\overline{m}}^{\Psi(\mathfrak{K})}(t_0) = \Psi(B_m^{\mathfrak{K}}(t_0))$ for sufficiently small t_0.

By making use of the Riccati equation and comparison of the Jacobi equations, the following volume comparison results for compact distance wedges have been obtained by P. Ehrlich, Y.-T. Jung and S.-B. Kim as an analogous of Günter-Bishop and Gromov in §2.

Theorem 3.4 [9] *Let (M^{n+1}, g) be a globally hyperbolic spacetime satisfying*

$$\rho^M(v, v) \geq n\lambda > 0 \tag{4}$$

for all timelike unit vectors v. Then for all $0 \leq r_0 \leq inj_{K(m)}$,

$$vol^M(B_m^{\mathfrak{K}}(r_0)) \leq vol^{M(-\lambda)}(B_{\psi(m)}^{\Psi(\mathfrak{K})}(r_0))$$

and equality holds at some $0 < r_0$ if and only if $B_m^{\mathfrak{K}}(r)$ and $B_{\psi(m)}^{\Psi(\mathfrak{K})}(r)$ are isometric for all $0 < r \leq r_0$.

Theorem 3.5 [9] *Let (M^{n+1}, g) be a globally hyperbolic spacetime satisfying*

$$\rho^M(v, v) \geq n\lambda > 0 \tag{5}$$

for all timelike unit vectors v. Then for all $0 \leq r_0 < r_1 \leq inj_{\overline{K}(m)}$,

$$\frac{vol^M(B_m^{\mathfrak{K}}(r_0))}{vol^{M(-\lambda)}(B_{\psi(m)}^{\Psi(\mathfrak{K})}(r_0))} \geq \frac{vol^M(B_m^{\mathfrak{K}}(r_1))}{vol^{M(-\lambda)}(B_{\psi(m)}^{\Psi(\mathfrak{K})}(r_1))}.$$

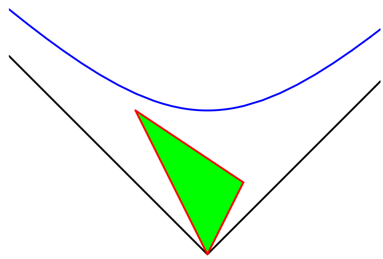
Moreover, equality holds for some $0 \leq r_0 < r_1 \leq inj_{\overline{K}(m)}$ if and only if $B_m^{\mathfrak{K}}(r)$ and $B_{\psi(m)}^{\Psi(\mathfrak{K})}(r))$ are isometric for all $0 < r \leq r_1$.

Note that, when comparing with the corresponding results in §2, inequalities in the previous theorems are with respect to a space of constant sectional curvature $-\lambda$. This is due to the fact that the assumption on the Ricci tensor in theorems 3.4 and 3.5 gives reversed inequalities when considering the equations (2) and (3).

3.3. SCLV sets

A further generalization of the distance wedges is obtained in [10], where the authors considered a more general family of subsets of a Lorentzian manifold as follows. Let $m \in M$ and take $U \subset T_mM$ an open subset in the causal future of the origin, $U \subset J^+(0)$ such that U is starshaped from the origin and the exponential map $\exp_m|_U$ is a diffeomorphism onto its image $\mathfrak{U} = \exp_m U$. Further assume that the closure of U is compact. Then one has

Definition 3.6 [10] A subset $\mathfrak{U}$ as above is called *standard for comparison of Lorentzian volumes* (SCLV-set) at the basepoint $m \in M$.



In order to state some comparison results with spaces of constant sectional curvature $M(\lambda)$ as model spaces, a transplantation process is needed as previously pointed out in §3.2. Let $\psi : T_m M \to T_{\overline{m}} M(\lambda)$ be a linear isometry, and define the transplantation map Ψ on a sufficiently small open set as $\Psi = \exp_{\overline{m}}^{M(\lambda)} \circ \psi \circ (\exp_m^M)^{-1}$. Further, for any $U \subset T_m M$ put $U_\lambda = \psi(U)$ and $\mathfrak{U}_\lambda = \exp_{\overline{m}}^{M(\lambda)}(U_\lambda) = \Psi(\mathfrak{U})$ which makes possible a volume comparison between SCLV-sets in M and $M(\lambda)$.

Theorem 3.7 [10] *Let (M, g) be a $(n+1)$-dimensional Lorentzian manifold and assume that*

$$\rho^M(v, v) \geq n\lambda g(v, v) \tag{6}$$

for all timelike vector fields $v = \frac{d}{dt} exp_m(tv_m) \mid_{t=t_0}$ tangent to $\mathfrak{U}$ at $m \in M$. If $\mathfrak{U}$ is a SCLV-set at m, then

$$vol^M(\mathfrak{U}) \leq vol^{M(\lambda)}(\mathfrak{U}_\lambda)$$

and the equality holds if and only if $\Psi : \mathfrak{U} \to \mathfrak{U}_\lambda$ is an isometry.

A comparison result in the spirit of Bishop-Gromov Theorem can also be stated for SCLV-sets, but it requires some previous conventions. For each $r > 0$ put $U(r) = r \cdot U = \{ru/u \in U\}$, $U_\lambda(r) = r \cdot U_\lambda$, $\mathfrak{U}(r) = exp_m^M(U(r))$, $\mathfrak{U}_\lambda(r) = exp_{\overline{m}}^{M(\lambda)}(U_\lambda(r))$. Note that the starshaped form of SCLV-sets ensures the possibility of constructing the above sets for $r > 0$ sufficiently small.

Theorem 3.8 [10] *Let (M^{n+1}, g) be a Lorentz manifold such that*

$$Ric(v, v) \geq n\lambda g(v, v) \tag{7}$$

for all timelike vector fields $v = \frac{d}{dt} exp_m(tv_m) \mid_{t=t_0}$ tangent to a SCLV-set $\mathfrak{U}$ based at $m \in M$. If one of the following two conditions hold:

(i) $c = 0$;

(ii) The cut function $c_{\mathfrak{U}}$ of $\mathfrak{U}$ is constant;

then the function $r \mapsto vol^M(\mathfrak{U}(r))/vol^{M(\lambda)}(\mathfrak{U}_\lambda(r))$ *is non-increasing. Moreover if there exists* $r_1 < r_2$ *such that*

$$vol^M(\mathfrak{U}(r_1))/vol^{M(\lambda)}(\mathfrak{U}_\lambda(r_1)) = vol^M(\mathfrak{U}(r_2))/vol^{M(\lambda)}(\mathfrak{U}_\lambda(r_2))$$

then $\mathfrak{U}(r)$ *and* $\mathfrak{U}_\lambda(r)$ *are isometric.*

4. Boundedness conditions on the curvature tensor

It is well known that the sectional curvature of a semi-Riemannian manifold is bounded from above or from below if and only if it is constant [2], [19]. Therefore it seemed natural to impose such curvature bounds on the curvature tensor itself rather than on the sectional curvature. Following [1], we will say that $R \geq \lambda$ or $R \leq \lambda$ if and only if for all X, Y,

$$R(X,Y,X,Y) \geq \lambda \left(\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2 \right), \tag{8}$$

or

$$R(X,Y,X,Y) \leq \lambda \left(\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2 \right), \tag{9}$$

respectively. Note that condition (8) (resp., (9)) is equivalent to the sectional curvature be bounded from below (resp., from above) on planes of signature $(++)$ and from above (resp., from below) on planes of signature $(+-)$.

Examples of semi-Riemannian manifolds whose curvature tensor is bounded as in (8), (9) can easily be produced as follows:

- Let (M_1, g_1), (M_2, g_2) be two Riemannian manifolds with nonnegative $K^{M_1} \geq 0$ and nonpositive $K^{M_2} \leq 0$ sectional curvature respectively. Then the product manifold $(M_1 \times M_2, g_1 - g_2)$ is a semi-Riemannian manifold whose curvature tensor satisfies (8) for $\lambda = 0$. (See [1] for related examples).

- A more general construction of Lorentzian manifolds with bounded curvature is as follows. Let (M, g) be a conformally flat Lorentz manifold whose Ricci tensor is diagonalizable, $\rho = diag[\mu_0, \mu_1, \dots, \mu_n]$, where the distinguished eigenvalue μ_0 corresponds to a timelike eigenspace. If $\mu_0 \geq max\{\mu_1, \dots, \mu_n\}$ (resp., $\mu_0 \leq min\{\mu_1, \dots, \mu_n\}$) then $R \leq \lambda$ (resp., $R \geq \lambda$) for some constant λ.

  Note that the previous construction applies to Robertson-Walker spacetimes as well as to locally conformally flat static spacetimes whose restspaces are of constant sectional curvature [4].

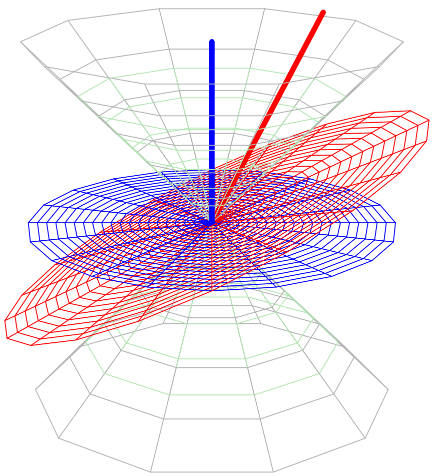
Finally note that, although it is not possible to obtain direct information of the Ricci tensor from conditions (8) and (9), an important observation for the purpose of studying volume properties of celestial geodesic spheres in §5 is the following. Let ξ be an instantaneous observer at $m \in M$ and complete it to an orthonormal basis $\{\xi, e_1, \dots, e_n\}$ of T_mM. Then $\tau^M + 2\rho^M_{\xi\xi} = \sum_{i,j=1}^n R_{ijij}$ where τ^M is the scalar curvature of (M, g) at m. Hence by assuming (8) (resp., (9)) holds, we have $\tau^M + 2\rho^M_{\xi\xi} \geq n(n-1)\lambda$ (resp., $\tau^M + 2\rho^M_{\xi\xi} \leq n(n-1)\lambda$).

5. Geodesic celestial spheres

Next we consider a family of geometric objects different than those in §3, namely the geodesic celestial spheres. Roughly speaking, they are the set of points reached after a fixed time travelling along radial geodesics emanating from a point m which are orthogonal to a given timelike direction. In Relativity, a unit timelike vector $\xi \in T_mM$ is called an *instantaneous observer*, and $\xi^\perp$ is called the *infinitesimal restspace* of ξ, that is, the infinitesimal Newtonian universe where the observer perceives particles as Newtonian particles relative to his rest position.

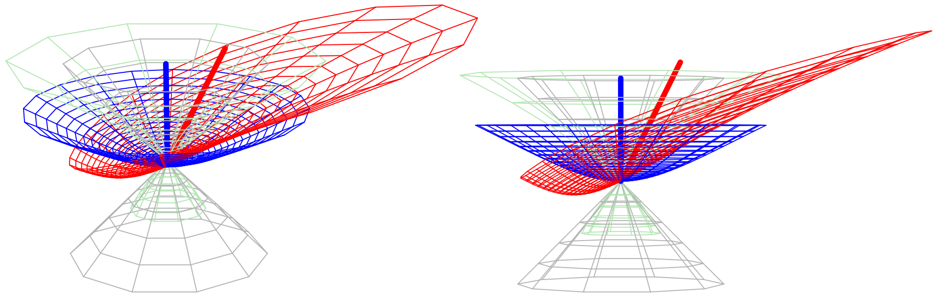
The *celestial sphere of radius* r of ξ is defined by $\mathcal{S}^\xi(r) = \{x \in \xi^\perp; \|x\| = r\}$ (c.f. [22]). If $\mathfrak{U}$ is a sufficiently small neighborhood of the origin in T_mM, $\widetilde{M} = \exp_m(\mathfrak{U} \cap \xi^\perp)$ is an embedded Riemannian submanifold of M. We define the *geodesic celestial sphere of radius* r *associated to* ξ as [8]:

$$S^\xi_m(r) = exp_m\left(\left\{x \in \xi^\perp; \|x\| = r\right\}\right) = exp_m(\mathcal{S}^\xi(r)). \qquad (10)$$



Geodesic celestial spheres in $\mathbb{R}^3_1$ centered at the origin and associated to different instantaneous observers

For r sufficiently small, $S^\xi_m(r)$ is a compact submanifold of $\widetilde{M}$. Therefore, by studying the volumes of geodesic celestial spheres in comparison to the volumes of the corresponding celestial spheres one obtains a measure of how the exponential map distorts volumes on spacelike directions.



Geodesic celestial spheres in the warped product $(\mathbb{R}^1_1 \setminus \{0\}) \times_{\frac{1}{t}} \mathbb{R}^2$ centered at $(1,1,1)$ and associated to the instantaneous observers $\xi_1 = (1,0,0)$ and $\xi_2 = (\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0)$.

As an immediate observation, note that for a given radius, the volume of geodesic celestial spheres depends both on the observer field $\xi \in T_m M$ and the center point $m \in M$. However, if (M, g) is assumed to be of constant sectional curvature, then the volumes depend only on the radii, since Lorentzian space forms are locally isotropic. Conversely

Theorem 5.1 [8] *Let $\left(M^{n+1}, g\right)$ be a Lorentzian manifold. If the volume of the geodesic celestial spheres $S^{\xi}_m(r)$ is independent of the observer field $\xi \in TM$, then M has constant sectional curvature.*

Comparison results in the spirit of Bishop-Günther-Gromov theorems can be obtained for the volumes of geodesic celestial spheres as follows.

Theorem 5.2 [8] *Let (M^{n+1}, g) be a $n+1$-dimensional Lorentzian manifold and $M^{n+1}(\lambda)$ a Lorentzian manifold of constant sectional curvature λ. If $S(r)$ denotes a geodesic celestial sphere of radius r centered at any point $m \in M^{n+1}(\lambda)$ and associated to any instantaneous observer $\xi_\lambda \in T_m M^{n+1}(\lambda)$ then the following statements hold:*

(i) If $R^M \geq \lambda$ then

$$vol^M_{n-1}\left(S^{\xi}_m(r)\right) \leq vol^{M(\lambda)}_{n-1}\left(S(r)\right)$$

for all sufficiently small r and all instantaneous observer $\xi \in T_m M$.

(ii) If $R^M \leq \lambda$ then

$$vol^M_{n-1}\left(S^{\xi}_m(r)\right) \geq vol^{M(\lambda)}_{n-1}\left(S(r)\right)$$

for all sufficiently small r and all instantaneous observer $\xi \in T_m M$.

Moreover, the equality holds at (i) or (ii) for all $\xi \in T_mM$ if and only if M has constant sectional curvature λ at m.

Sketch of the proof.
Note that geodesic celestial spheres are level sets of the Riemannian distance function on $\widetilde{M} = \exp_m(\mathfrak{U} \cap \xi^\perp)$. Since the submanifold $\widetilde{M}$ may fail to be Riemannian when moving far from the basepoint m, geodesic celestial spheres are locally defined objects in M. This fact, together with the difficulties in understanding the geometry of $\widetilde{M}$ motivated an approach to theorems 5.2 and 5.3 different from the one discussed in Section 2 based on the use of Riccati equation. Therefore, one studies the deviation of the volume of geodesic celestial spheres from the Euclidean spheres by looking at the power series expansion of the function $r \mapsto vol_{n-1}\left(S_m^\xi(r)\right)$. Then, after some calculations, one obtains the first terms in such expansion as

$$vol_{n-1}\left(S_m^\xi(r)\right) = c_{n-1}r^{n-1}\left(1 - \frac{(\tau^M + 2\rho_{\xi\xi}^M)}{6n} r^2 + O(r^4)\right)$$

where c_{n-1} denotes the volume of the $(n-1)$-dimensional Euclidean sphere of radius one. Now, considering the coefficient of degree two in the previous expansion the result is obtained just comparing with the corresponding expansion in the model space. $\square$

Proceeding in an analogous way, one has the following Gromov type comparison result.

Theorem 5.3 [8] *Let (M^{n+1}, g) be a $n+1$-dimensional Lorentzian manifold, $M^{n+1}(\lambda)$ a Lorentzian manifold of constant sectional curvature λ and $S(r)$ a geodesic celestial sphere of radius r centered at any point $m \in M^{n+1}(\lambda)$ and associated to any instantaneous observer $\xi_\lambda \in T_mM^{n+1}(\lambda)$.*

If $R^M \geq \lambda$ (resp., $R^M \leq \lambda$) then

$$r \mapsto \frac{vol_{n-1}^M\left(S_m^\xi(r)\right)}{vol_{n-1}^{M(\lambda)}\left(S(r)\right)}$$

is nonincreasing (resp., nondecreasing) for sufficiently small r and all instantaneous observer $\xi \in T_mM$.

References

[1] L. Andersson, R. Howard; Comparison and rigidity theorems in semi-Riemannian geometry, *Comm. Anal. Geom.* **6** (1998), 819–877.

[2] J. K. Beem, P. Ehrlich, K. L. Easley; *Global Lorentzian Geometry* (2nd Ed.), Marcel Dekker, New York, 1996.

[3] R. L. Bishop, R. Crittenden; *Geometry of Manifolds*, Academic Press, New York, 1964.

[4] M. Brozos-Vázquez, E. García-Río, R. Vázquez-Lorenzo; Some remarks on locally conformally flat static spacetimes, to appear.

[5] I. Chavel; *Riemannian Geometry: A Modern Introduction*, Cambridge Tracts in Math. **108**, Cambridge Univ. Press, Cambridge, 1993.

[6] B.-Y. Chen, L. Vanhecke; Differential geometry of geodesic spheres, *J. Reine Angew. Math.*, **325**, (1981), 28 – 67.

[7] J.C. Díaz-Ramos, E. García-Río, L. Hervella; Total curvatures of geodesic spheres associated to quadratic curvature invariants, *Ann. Mat. Pur. Appl.* to appear

[8] J.C. Díaz-Ramos, E. García-Río, L. Hervella; Comparison results for the volume of geodesic celestial spheres in Lorentzian manifolds, to appear

[9] P. E. Ehrlich, Y. T. Jung, S. B. Kim; Volume comparison theorems for Lorentzian manifolds, *Geom. Dedicata* **73** (1998), 39–56.

[10] P. E. Ehrlich, M. Sánchez; Some semi-Riemannian volume comparison theorems, *Tohoku Math. J.* **(2) 52** (2000), 331–348.

[11] F. Gackstatter, B. Gackstatter; Über Volumendefekte und Krümmung in Riemannschen Mannigfaltigkeiten mit Anwendungen in der Relativitätstheorie, *Ann. Physik* **(7) 41** (1984), 35–44.

[12] F. Gackstatter; Über Volumendefekte und Krümmung bei der Robertson-Walker-Metrik und Konstruktion kosmologischer Modelle, *Ann. Physik* **(7) 44** (1987), 423–439.

[13] E. García-Río, D.N. Kupeli, R. Vázquez-Lorenzo; *Osserman manifolds in semi-Riemannian geometry*, Lecture Notes in Math., **1777**, Springer-Verlag, Berlin, 2002.

[14] P. B. Gilkey; *Geometric properties of natural operators defined by the Riemann curvature tensor*, World Scientific Publ. Co., Inc., River Edge, NJ, 2001.

[15] A. Gray; *Tubes*, Addison–Wesley, Redwood City, 1990.

[16] M. Gromov; Isoperimetric inequalities in Riemannian manifolds. In *Asymptotic Theory of Finite Dimensional Normed Spaces*, Lecture Notes Math. **1263**, 171–227, Springer-Verlag, Berlin, 1987.

[17] P. Günther; Einige Sätze über das Volumenelement eines Riemannschen Raumes, *Publ. Math. Debrecen* **7** (1960), 78–93.

[18] O. Kowalski, L. Vanhecke; Ball-homogeneous and disk-homogeneous Riemannian manifolds, *Math. Z.*, **180** (1982), 429–444.

[19] B. O'Neill; *Semi-Riemannian Geometry With Applications to Relativity*, Academic Press, New York, 1983.

[20] R. Schimming; Lorentzian geometry as determined by the volumes of small truncated light cones, *Arch. Math. (Brno)* **24** (1988), 5–15.

[21] R. Schimming, D. Matel-Kaminska; The volume problem for pseudo-Riemannian manifolds, *Z. Anal. Anwendungen* **9** (1990), 3–14.

[22] R. K. Sachs, H. Wu; *General Relativity for Mathematicians*, Springer–Verlag, New York, 1977.

Proceedings of the II International
Meeting on Lorentzian Geometry
Murcia, November 12–14, 2003
Publ. de la RSME, Vol. 8 (2004), 28–40

Uniqueness of static Einstein-Maxwell-Dilaton black holes

Marc Mars[1]

[1] *Área de Física Teórica, Universidad de Salamanca, Plaza de la Merced s/n, 37008 Salamanca, Spain*

Abstract

We describe the method of Bunting and Masood-ul-Alam to prove uniqueness of static black holes. The method is applied to restrict the possible conformal factors when the field equations correspond to a coupled harmonic map. Finally the particular case of Einstein-Maxwell-Dilaton (for vanishing magnetic field and arbitrary coupling constant) is considered.

Keywords: General Relativity, Black holes, Uniqueness Theorems, Positive Mass theorem, Harmonic maps.

2000 Mathematics Subject Classification: 83C57, 35A05, 35J55.

1. Introduction

In General Relativity the gravitational field is described as a four dimensional manifold $\mathcal{M}$ endowed with a Lorentzian metric g. Among the most relevant spacetimes are the so-called black holes, which roughly speaking are spacetimes containing regions (the so-called black hole regions) from which no observer or light ray can escape to infinity. Obviously this needs a concept of infinity, which is taken in most cases to mean a region where the gravitational field becomes asymptotically weak and the metric approaches Minkowski in a precise way, (the so-called asymptotically flat spacetimes). Black hole physics is a vast discipline within gravitational physics which cannot be summarized in a few lines. The reader is advised to look at [1] for a comprehensive introduction to this topic. Stationary black holes (i.e. those which are in equilibrium and therefore do not evolve in time) play a specially relevant role. This is because they are believed to be the end-states of any configuration with enough concentration of matter-energy so that gravitational collapse, and therefore singularity formation [2], is unavoidable. Although there is no proof of this fact, there are good physical reasons to believe that a collapsing configuration will reach an equilibrium state. The so-called cosmic censorship conjecture [3] states that all singularities formed in a collapse will be hidden behind an event horizon (i.e. a black hole will form). We are not yet even close to proving the cosmic censorship conjecture, which is probably the most important open issue in classical gravitational theory. Nevertheless, it seems likely that many end-states of gravitational collapse will be described by black holes in equilibrium. It is, therefore, important to classify those spacetimes. In General Relativity the theory dealing with these issues has been generically called *black hole uniqueness theorems*. Again, this is a large area of research, see [4] for an account. Equilibrium situations may be divided into stationary and static, the latter being not only time invariant but also time symmetric, so that future and past are indistinguishable. In geometrical terms, $(\mathcal{M}, g)$ admits a Killing vector $\vec{\xi}$ which is timelike sufficiently close to infinity. Moreover, in the static case $\vec{\xi}$ is integrable, i.e. $\boldsymbol{\xi} \wedge \mathrm{d}\boldsymbol{\xi} = 0$ ($\boldsymbol{\xi}$ stands for $g(\vec{\xi}, \cdot)$). So far, the methods for proving uniqueness are very different for static or merely stationary black holes (the static case being much more developed than the stationary one).

Black holes in equilibrium are also organized depending on the geometry of its Killing horizon. Let us recall that a Killing horizon is a null hypersurface $\mathcal{H}$ where a Killing vector $\vec{\eta}$ is null, nowhere zero and tangent to $\mathcal{H}$. In stationary black holes, the boundary of the black hole region is always a Killing horizon [5] and, in the static case, the horizon Killing vector $\vec{\eta}$ coincides with the static Killing $\vec{\xi}$. A connected component $\mathcal{H}_\alpha$ of the Killing horizon is called degenerate or non-degenerate depending on whether the acceleration of $\vec{\eta}$, $\nabla_{\vec{\eta}} \vec{\eta}$, is zero or non-zero on $\mathcal{H}_\alpha$ (it may be proven that this property is independent of the point $x \in \mathcal{H}_\alpha$ so that it becomes a property of $\mathcal{H}_\alpha$ itself). Degenerate horizons turn out to be much more difficult to analyze and few results are known so far (see, however, [6] for the vacuum, static case).

Besides the stationary/static and degenerate/non-degenerate distinction, black holes also depend, obviously, on the energy-momentum tensor T of the gravitating fields present outside the black hole. In General Relativity this imposes conditions on the Ricci tensor of g via the field equations $Ric(g) = T - \frac{1}{2} g \mathrm{Tr}(T)$, where physical units have been chosen appropriately. Uniqueness theorems for static non-degenerate black holes have been obtained for vacuum

($T = 0$), for electrovacuum (T is generated by an electromagnetic field without sources) and for the so-called Einstein-Maxwell-dilaton case. The latter was first analyzed by Masood-ul-Alam [7] for a particular value $\alpha = 1$ of the so-called coupling constant, and for vanishing magnetic field. Walter Simon and myself [8] generalized this result to the case of arbitrary coupling constant (still for vanishing electric or magnetic field) and for general electromagnetic field in the case $\alpha = 1$.

In this contribution I shall review some of these results. The emphasis will be rather on explaining the method than showing the most general results (which are discussed in [8]). I will also present a different argument to find the appropriate conformal factor, which complements the one used in [8].

2. Assumptions on the spacetime.

In this contribution, spacetimes $(\mathcal{M}, g)$ are assumed to be smooth. We are interested in studying static non-degenerate black holes. The precise conditions we shall impose are

A.1 $(\mathcal{M}, g)$ admits an integrable Killing vector $\vec{\xi}$ with a non-degenerate Killing horizon $\mathcal{H}$.

A.2 The horizon $\mathcal{H}$ is of bifurcate type, i.e. the closure $\overline{\mathcal{H}}$ of $\mathcal{H}$ contains points where the Killing vector $\vec{\xi}$ vanishes.

A.3 $(\mathcal{M}, g)$ admits a spacelike asymptotically flat hypersurface Σ which is orthogonal to the Killing vector $\vec{\xi}$ and such that $\partial\Sigma \subset \overline{\mathcal{H}}$.

Condition A.2 can be shown to follow from A.1 under rather mild global requirements [9]. However, we prefer to include it and relax our global requirements as much as possible. In fact, our only global condition is contained in A.3, and more concretely, on the definition of asymptotic flatness.

Definition 2.1 *A spacelike hypersurface $(\Sigma, \hat{g})$ of $(\mathcal{M}, g)$, $\hat{g}$ being the induced metric, is called asymptotically flat iff*

(1) The "end" $\Sigma^\infty = \overline{\Sigma} \setminus \{a \text{ compact set}\}$ is diffeomorphic to $\mathbb{R}^3 \setminus B$, where B is a ball.

(2) On Σ^∞ the metric satisfies (in the flat coordinates defined by the diffeomorphism above and with $r = \sqrt{\sum (x^i)^2}$)

$$\hat{g}_{ij} - \delta_{ij} = O^2(r^{-\delta}) \text{ for some } \delta > 0.$$

(A function $f(x^i)$ is said to be $O^k(r^\alpha)$, $k \in \mathbb{N}$, if $f(x^i) = O(r^\alpha)$, $\partial_j f(x^i) = O(r^{\alpha-1})$ and so on for all derivatives up to and including the kth ones).

Remark In this definition $\overline{\Sigma}$ is the topological closure of Σ. Notice that our definition implies, in particular, that $(\overline{\Sigma}, \hat{g})$ is complete in the metric sense.

Let q be a fixed point of $\vec{\xi}$ on $\overline{\mathcal{H}}$ (i.e. $q \in \overline{\mathcal{H}}$ and $\vec{\xi}(q) = 0$), which exists by assumption A.2. Then, the connected component of the set $\{p;\ \vec{\xi}(p) = 0\}$ containing q is a smooth, embedded, spacelike, two-dimensional submanifold of $\mathcal{M}$ (see Boyer [10]). Each one of these connected components is called a bifurcation surface. Let us consider any connected component $(\partial\Sigma)_\alpha$ of the topological boundary of Σ. By assumption A.3 $(\partial\Sigma)_\alpha$ is contained in the closure of the Killing horizon. Thus, (section 5 in [9]), $(\partial\Sigma)_\alpha$ must be completely contained in one of the bifurcation surfaces of $\vec{\xi}$. Furthermore the induced metric $\hat{g}$ on the hypersurface Σ can be smoothly extended to $\Sigma \cup (\partial\Sigma)_\alpha$ (see Proposition 3.3 in [6]). Hence $(\overline{\Sigma}, \hat{g})$ is a smooth Riemannian manifold with boundary. We finish the section with the following definition

Definition 2.2 *A smooth spacetime $(\mathcal{M}, g)$ is called a static non-degenerate black hole iff conditions A.1, A.2 and A.3 are satisfied.*

3. Method of Bunting and Masood-ul-Alam.

The key idea introduced by Bunting and Masood-ul-Alam [11] to prove uniqueness of static black holes is as follows. The aim is to show that the only possible static black holes are spherically symmetric. These spacetimes have the property that the hypersurface $(\Sigma, \hat{g})$ orthogonal to the static Killing vector $\vec{\xi}$ is conformally flat. So, there exists a suitable, spherically symmetric, conformal factor which brings this 3-metric into the flat metric. Furthermore, all matter fields and the norm of the static Killing vector are also spherically symmetric. Thus, the conformal factor can be considered as a function of these fields. Conversely, if a static black hole is such that the hypersurface orthogonal to the static Killing vector is conformally flat with a conformal factor depending only on the matter fields and the norm of the Killing, then spherical symmetry is usually easy to imply. The problem is then how to prove that some metric is conformally flat. A powerful method to show that an asymptotically flat Riemannian space is in fact *flat* is the rigidity part of the positive mass theorem [12]. This requires dealing with asymptotically flat Riemannian manifolds which are (i) complete and without boundary, (ii) with a non-negative Ricci scalar and (iii) with vanishing total mass. The strategy is to choose carefully a conformal factor Ξ, so that $\Xi^2 \hat{g}$ has zero mass and non-negative Ricci scalar, $R(\Xi^2 \hat{g}) \geq 0$. However, in general this still gives a manifold with boundary and the positive mass theorem cannot be applied yet. This is solved with a remarkably elegant idea: choose *two* different conformal factors $\Xi_\pm$ so that (i) $(\overline{\Sigma}, \Xi_+^2 \hat{g})$ has vanishing mass, (ii) $(\overline{\Sigma}, \Xi_-^2 \hat{g})$ admits a one-point compactification of infinity (thus giving a complete Riemannian manifold with boundary) and (iii) such that both spaces can be glued together to produce a manifold with continuous metric across the common boundary $\partial\Sigma$. The resulting manifold is complete and the positive mass theorem can be applied. Thus, the key idea is to glue together two copies of the same space with two conformally related metrics and then apply the rigidity part of the positive mass theorem to the total space.

It is clear that finding the appropriate conformal factors is the crucial technical part of the proof. Indeed, $\Xi_\pm$ must be not only positive definite, but also transform the mass of $(\Sigma, \hat{g})$ as required, provide Riemannian metrics with non-negative Ricci scalar, and have a suitable behaviour on $\partial\Sigma$ in order to allow for a C^1 matching. By far, the most difficult conditions to fulfill are positivity of the conformal factors and non-negativity of the Ricci scalars.

On the hypersurface Σ we can define the scalar field $V \geq 0$ as the norm of the static Killing vector $V^2 = -(\vec{\xi}, \vec{\xi})$. Moreover, the energy-momentum contents of the spacetime defines further fields on Σ. In order to have some control on the conformal factors, it is natural to assume that $\Xi_\pm(x)$ depends on the space point $x \in \Sigma$ through the values of V and the rest of matter fields on x. In vacuum or for Einstein-Maxwell black holes, the number of fields is small and finding a suitable conformal factor is not-too-hard. For Einstein-Maxwell-dilaton, there are already four fields present and the problem becomes more involved. It is clear that the higher the number of fields, the more difficult is to find a suitable conformal factor. In the following section we will find necessary restrictions on the conformal factors whenever the matter model is a so-called coupled harmonic map.

4. Restrictions on the conformal factor in the coupled harmonic map case.

In many cases of interest (including vacuum, Einstein-Maxwell, Einstein-Maxwell-dilaton, and many others [13]) the Einstein field equations in the static case can be rewritten on Σ as follows: First of all, the matter fields together with V organize themselves into a (pseudo)-Riemannian manifold $(\mathcal{V}, \gamma)$ called the *target space*. Moreover, after defining a conformal metric $h \equiv V^2 \hat{g}$ on Σ, the map defining these fields, $\Psi : (\Sigma, h) \to (\mathcal{V}, \gamma)$ must be a *harmonic map*, i.e. a C^2 map satisfying the field equations

$$D_i D^i \Psi^a(x) + \Gamma^a_{bc}(\Psi(x)) D_i \Psi^b(x) D^i \Psi^c(x) = 0, \tag{1}$$

where D is the covariant derivative of (Σ, h), Ψ^a is the expression of Ψ in local coordinates of $\mathcal{V}$ and Γ^a_{bc} are the Christoffel symbols of $(\mathcal{V}, \gamma)$. The space (Σ, h) is called *domain space*. Finally, the metric h itself satisfies Einstein-like equations with sources involving Ψ. Explicitly,

$$Ric(h)_{ij}(x) = \gamma_{ab}(\Psi(x)) D_i \Psi^a(x) D_j \Psi^b(x). \tag{2}$$

A pair (h, Ψ), with h positive definite, satisfying the field equations (1) and (2) is called a *coupled harmonic map*.

For static black holes such that the Einstein field equations become a coupled harmonic map on Σ, the set of conditions on the conformal factor required for the method of Bunting and Masood-ul-Alam to work are strong enough so that the possible conformal factor are fixed *uniquely and explicitly* on a certain subset $\mathcal{V}_{BH} \subset \mathcal{V}$ (which sometimes coincides with the whole target space). In other words, there are *unique* candidates on $\mathcal{V}_{BH}$ for which the method has a chance to work. This does not mean, of course, that those explicit conformal factor will in fact work (extra conditions like positivity and others still need to be satisfied), but it does restrict strongly the set of functions for which the conditions need to be analyzed. This is a great advantage because the most difficult part of the Bunting and Masood-ul-Alam method is, in fact, to guess the appropriate conformal factors.

Let us now define the subset $\mathcal{V}_{BH}$ and describe briefly why the possible conformal factors are restricted on this set. The uniqueness results for static black holes assert, in general, that spherically symmetric static black holes are the only possible static non-degenerate black holes. So, coupled harmonic maps where both (Σ, h) and Ψ are spherically symmetric play a privileged role in the black hole case. Any coupled harmonic map with these properties must be (see e.g. [13]) of the form $\Psi = \zeta \circ \lambda$, where $\zeta : I \subset \mathbb{R} \to \mathcal{V}$ is an affinely parametrized geodesic of $(\mathcal{V}, \gamma)$ and $\lambda : \Sigma \to \mathbb{R}$ is a spherically symmetric harmonic function on Σ. Thus, spherically symmetric solutions are described by geodesics in the target space. However, not all geodesics of the target space correspond to a spherically symmetric, non-degenerate black hole solution. Let us define $\mathcal{V}_{BH} \subset \mathcal{V}$ as follows: a point x belongs to $\mathcal{V}_{BH}$ if and only if there exists a spherically symmetric, non-degenerate black hole spacetime such that the affinely parametrized geodesic ζ_x in $\mathcal{V}$ defining this solution passes through x. This geodesic will be assumed, without loss of generality, to satisfy $\zeta_x(0) = p$, $\zeta_x(1) = x$, where p is the value of Ψ at infinity in Σ^∞. Notice that this condition restricts the harmonic function λ appearing in $\Psi = \zeta_x \circ \lambda$ to satisfy $\lambda = 0$ at infinity in Σ^∞. It is clear that any conformal factor Ω for which the Bunting and Masood-ul-Alam method has a chance to work, must have the following property: For any spherically symmetric black static black hole, the conformal rescaling with Ω of the domain space metric h must be locally flat. This allows us to prove the following result [8].

Lemma 4.1 *Let $\Psi : \Sigma^\infty \to \mathcal{V}$ be a coupled harmonic map between (Σ^∞, h) and $(\mathcal{V}, \gamma)$. Assume that (Σ^∞, h) has vanishing mass. Let $\Omega^\pm$ be positive, C^2 functions $\Omega^\pm : \mathcal{V} \to \mathbb{R}$ with the following properties*

(1) For any spherically symmetric static black hole (Σ_{sph}, h_{sph}) the metric $(\Omega^\pm)^2 h_{sph}$ is locally flat.

(2) $(\Sigma^\infty_{sph}, (\Omega^+)^2 h_{sph})$ is asymptotically flat and $(\Sigma^\infty_{sph}, (\Omega^-)^2 h_{sph})$ admits a one-point compactification of infinity.

Then, $\Omega^\pm$ must take the following form on $x \in \mathcal{V}_{BH}$.

$$\Omega^+(x) = \cosh^2\left(\sqrt{\frac{\dot{\zeta}^a(x)\dot{\zeta}_a(x)}{8}}\right), \qquad \Omega^-(x) = \sinh^2\left(\sqrt{\frac{\dot{\zeta}^a(x)\dot{\zeta}_a(x)}{8}}\right),$$

where $\vec{\dot{\zeta}}(x)$ is the tangent vector at x of the geodesic $\zeta_x(s)$ in $(\mathcal{V}, \gamma)$ defining the spherically symmetric black hole (Σ_{sph}, h_{sph}).

Remark. In order to simplify notation we shall use the same symbol for a scalar function on the target space and for its pull-back on Σ via Ψ. The meaning should become clear from the context.

5. Uniqueness theorem for static Einstein-Maxwell-Dilaton black holes.

Einstein-Maxwell-Dilaton (EMD) spacetimes are spacetimes $(\mathcal{M}, g)$ with a scalar function τ (the dilaton), a closed two-form $F_{\alpha\beta}$ (the electromagnetic field) and a coupling constant α between these two fields, which we take to be non-zero. We shall describe the field equations only in the static case and, for simplicity we shall assume that the magnetic field vanishes (the arbitrary case is discussed in [8] although it must be remarked that the uniqueness result has not been obtained in full generality yet). Assuming further that the spacetime $(\mathcal{M}, g)$ is simply connected, the EMD field equations take the form of a coupled harmonic map between (Σ, h) and the target space $\mathcal{V} = \{(V, \kappa, \phi) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}\}$ endowed with the metric

$$ds^2 = \gamma_{AB} dx^A dx^B = \frac{2dV^2}{V^2} + \frac{2d\kappa^2}{\alpha^2\kappa^2} - \frac{2d\phi^2}{V^2\kappa^2}, \tag{3}$$

where $\kappa = e^{\alpha\tau}$ and the electric field is $\vec{E} = \vec{\nabla}\phi$. Asymptotic flatness of $(\Sigma, \hat{g})$ implies the following behaviour of the fields near infinity [8]

$$V = 1 - \frac{M}{r} + \mathrm{O}(\frac{1}{r^2}), \quad \kappa = 1 + \frac{\alpha D}{r} + \mathrm{O}(\frac{1}{r^2}), \quad \phi = \frac{Q}{r} + \mathrm{O}(\frac{1}{r^2}), \tag{4}$$

$$h_{ij} = \delta_{ij} + \mathrm{O}(\frac{1}{r^2}), \tag{5}$$

where M, D, Q are constants (called charges) satisfying the inequalities

$$\sqrt{1+\alpha^2}\,|Q| \leq M - \alpha D, \qquad \alpha M + D \geq 0 \tag{6}$$

In order to show uniqueness for EMD static black holes, we shall need to impose additionally that these inequalities are strict. In the spherically symmetric case, the black holes with charges satisfying equality correspond precisely to the degenerate horizon case. It seems therefore plausible that the condition of being non-degenerate should imply, in general, that the inequalities above are necessarily strict. We have been able to prove this fact whenever Σ^∞ admits an *analytic* compactification near spatial infinity, but not in the general case. It would be of interest to settle this issue.

Having discussed the behaviour at infinity we turn to the conditions on the horizon $\partial\Sigma$. In order to do that the induced metric $\hat{g}$ (and its corresponding covariant derivative $\hat{\nabla}$) must be used because $\hat{g}$ admits a smooth extension to $\partial\Sigma$, while h degenerates there.

Lemma 5.1 [8] *For static Einstein-Maxwell-dilaton non-degenerate black holes, the following relations on the boundary $\partial\Sigma$ hold,*

$$\hat{g}^{ij}\hat{\nabla}_i V \hat{\nabla}_j V\Big|_{\partial\Sigma} = W^2 > 0, \;\; \hat{g}^{ij}\hat{\nabla}_i V \hat{\nabla}_j \kappa\Big|_{\partial\Sigma} = 0, \;\; \hat{g}^{ij}\hat{\nabla}_i V \hat{\nabla}_j \phi\Big|_{\partial\Sigma} = 0. \tag{7}$$

In order to apply Lemma 4.1 we must find the explicit form of the geodesics starting at the point $p = \{V = 1, \phi = 0, \kappa = 1\}$, corresponding to the values of the fields at infinity. In general this is not a trivial task even for spaces admitting Killing vectors. In our case, however, the target space turns out to be a symmetric space, as it may be easily proven. This fact, and, in particular the existence of an isotropy group at p, simplifies greatly the calculations, as we describe next. The isotropy algebra of $(\mathcal{V}, \gamma)$ is easily seen to be one-dimensional and generated by the Killing vector

$$\vec{\eta} = -\phi V \frac{\partial}{\partial V} + \frac{1}{2}\left(1 - \kappa^2 V^2 - \left(\alpha^2+1\right)\phi^2\right)\frac{\partial}{\partial \phi} - \alpha^2 \phi \kappa \frac{\partial}{\partial \kappa}.$$

Isotropies are useful because geodesics passing through p project into geodesics in the quotient space defined by the isotropy orbits (whenever this is a manifold), the latter being lower dimensional and hence simpler to deal with. In order to determine the structure of the set of orbits of $\vec{\eta}$, we solve $\vec{\eta}(f) = 0$. Only two functionally independent solutions are needed. One of them is obvious and reads $F_1 = V^{-1}\kappa^{1/\alpha^2}$. The other one follows from the fact that the norm of a Killing vector is constant along the Killing vector. Thus,

$$-\eta^A \eta_A = \frac{1}{2\kappa^2 V^2} \left[(1-\kappa V)^2 - (\alpha^2+1)\phi^2 \right] \left[(1+\kappa V)^2 - (\alpha^2+1)\phi^2 \right] \qquad (8)$$

solves $\vec{\eta}(f) = 0$, as desired. This expression shows that $\vec{\eta}$ changes causal character in different regions of $\mathcal{V}$. It follows that the canonical metric of the orbit space cannot be well-defined everywhere (because it changes signature). However, for any static EMD non-degenerate black hole, $\vec{\eta}$ turns out to be timelike everywhere on the relevant subset $\Psi(\Sigma)$. This is proven in the following lemma.

Lemma 5.2 *$\Psi(\Sigma)$ is contained in the open subset $U \equiv \{(1-\kappa V)^2 - (\alpha^2 + 1)\phi^2 > 0\} \subset \mathcal{V}$.*

Proof. We only need to show that $|1 - V\kappa| > \sqrt{\alpha^2+1}\,|\phi|$ on $\Psi(\Sigma)$. Let us define $K_\pm = 1 - V\kappa \pm \sqrt{\alpha^2+1}\phi$. A simple calculation yields $\nabla_A \nabla_B K_\pm = V^{-1}\kappa^{-1}\nabla_A K_\pm \nabla_B K_\pm$. Then, the coupled harmonic map equations and the conformal rescaling $\hat{g} = V^{-2}h$ imply

$$\hat{\nabla}_i \hat{\nabla}^i K_\pm + \hat{\nabla}^i K_\pm \left(\frac{\hat{\nabla}_i \kappa}{\kappa} \pm \sqrt{\alpha^2+1}\frac{\hat{\nabla}_i \phi}{V} \right) = 0.$$

The maximum principle applied to this elliptic equation tells us that the minimum is attained at infinity provided $n^i \hat{\nabla}_i K_\pm|_{\partial\Sigma} \geq 0$, where n^i is an outer normal to $\partial\Sigma$. Lemma (5.1) shows that $n_i = -\hat{\nabla}_i V$ is such an outer normal and that $n^i \hat{\nabla}_i K_\pm|_{\partial\Sigma} = W^2\kappa > 0$. Since $K_\pm$ vanish at infinity, the lemma follows $\square$.

Let us define the two scalar functions on $\mathcal{V}$,

$$X = \frac{\kappa^{1/\alpha^2}}{V}, \qquad Y = \frac{(1-\kappa V)^2 - (\alpha^2+1)\phi^2}{4\kappa V}, \qquad (9)$$

which satisfy $X > 0$ and $Y > 0$ on U. Since $-\eta^A\eta_A = 8Y(Y+1)$ we see that Y is constant along $\vec{\eta}$. Furthermore, a simple calculation shows that $\partial_V Y < 0$ wherever $Y > 0$. This implies that $dY \neq 0$ in this region. Similarly $dX \neq 0$ on the whole of $\mathcal{V}$. Furthermore, a direct calculation implies that $dX \wedge dY \neq 0$

on U. Hence, $\{X, Y\}$ is a global coordinate system on $U/\vec{\eta}$, i.e. the set of orbits of $\vec{\eta}$ in U. In particular, this space is a differentiable manifold. Thus, we can consider the induced metric on this quotient, which turns out to be

$$ds^2\big|_{(U/\vec{\eta})} = \frac{2}{1+\alpha^2}\left(\alpha^2 \frac{dX^2}{X^2} + \frac{dY^2}{Y(Y+1)}\right). \tag{10}$$

This is a flat metric and its geodesics are trivial to solve. We are interested in the geodesics passing through p, i.e. geodesics in the quotient space starting at the boundary point $X(p) = 1, Y(p) = 0$. They are easily found to be $X(s) = \exp(\beta_1 s)$ and $Y(s) = \sinh^2(\beta_2 s)$, $s \in \mathbb{R}^+$, $\beta_1, \beta_2 \in \mathbb{R}$ and $\beta_2 \geq 0$ without loss of generality. Each one of these geodesics gives rise to a spherically symmetric solution of the harmonic map equations. However, not all of them correspond to black holes. Only those reaching $V = 0$ with the rest of fields remaining finite have this property. This implies that Y/X must also remain finite at $\partial\Sigma$ and this is only achieved whenever $\beta_1 = 2\beta_2$. Hence the subset V_{BH} of $\mathcal{V}$ corresponding to the black holes solutions is defined by $4XY - (X-1)^2 = 0$. We can now apply Lemma 4.1 easily. The function $N(x) = \dot{\zeta}^a(x)\dot{\zeta}_a(x)$ appearing there is the squared geodesic distance from $(X, Y = (X-1)^2/(4X))$ to $X = 1, Y = 0$. Its explicit expression is $N(X) = 2(\ln X)^2$ and hence Lemma 4.1 forces the conformal factors to satisfy

$$\Omega^{\pm}(X,Y)\big|_{Y=(X-1)^2/4X} = \frac{(1 \pm X)^2}{4X}, \tag{11}$$

Since $\mathcal{V}_{BH}$ does not cover the whole target space we still need to extend these expressions off $\mathcal{V}_{BH}$. This involves some guesswork. Perhaps the simplest try is to impose that the conformal factor depends only on X or only on Y. This would give us two pairs of conformal factors instead of one pair, as we would need. Let us nevertheless analyze the outcome (we shall see that an appropriate combination of both pairs gives us the desired pair of conformal factors). According to (11) we define $\Omega_1^{\pm} = Y + 1/2 \pm 1/2$ and $\Omega_2^{\pm} = (1 \pm X)^2/(4X)$, or in terms of the coordinates of $\mathcal{V}$,

$$\Omega_1^{\pm} = \frac{1}{4\kappa V}\left[(1 \pm V\kappa)^2 - \left(\alpha^2+1\right)\phi^2\right], \qquad \Omega_2^{\pm} = \frac{\left(\kappa^{1/\alpha^2} \pm V\right)^2}{4V\kappa^{1/\alpha^2}}. \tag{12}$$

Let us next check whether the conditions required to prove uniqueness are satisfied by either of the conformal factors $\Omega_A^{\pm}$ $(A = 1, 2)$. First of all, we check positivity on Σ. From the definition, $\Omega_A^+ \geq \Omega_A^-$. Furthermore $\Omega_1^-|_\Sigma > 0$ directly from Lemma 5.2 and Ω_2^+ is manifestly positive. So, only $\Omega_A^-|_\Sigma > 0$, or equivalently $F \equiv \kappa^{1/\alpha^2} V^{-1} > 1$, remains to be proven. The Killing equations of $(\mathcal{V},\gamma)$ show that $\boldsymbol{\xi} = F^{-1}dF$ is, after rising its indices, a Killing vector of the target space. This implies, from general properties of coupled harmonic maps, that $D^i(F^{-1}D_iF) = 0$ on Σ. Applying the maximum principle to this elliptic equation and noticing that F diverges to $+\infty$ on $\partial\Sigma$, we find that F takes up its minimum at infinity where $F = 1$, this proves the desired inequality.

Next we need to check that $(\Sigma, (\Omega^+_A)^2 h)$ has vanishing mass and that $(\Sigma, (\Omega^-_A)^2 h)$ admits a one-point compactification of infinity. The first claim follows from the fact that (Σ, h) has vanishing mass (see Lemma 5.1) and that $\Omega^+_A = 1 + \mathrm{O}(r^{-2})$, which follows directly from (5). The second claim follows because the asymptotic behaviour of Ω^-_A is, after using (5)

$$\Omega^-_1 = \frac{(M-\alpha D)^2 - \left(1+\alpha^2\right)Q^2}{4r^2} + \mathrm{O}(r^{-3}), \quad \Omega^-_2 = \frac{(\alpha M + D)^2}{4\alpha^2 r^2} + \mathrm{O}(r^{-3}).$$

from which the existence of a one-point compactification of $(\Sigma, (\Omega^-_2)^2 h)$ follows from a standard argument, *provided* the strict inequalities in (6) hold (here is where this condition is used). The third step is to glue together the two copies of $\overline{\Sigma}$ with conformal factors $\Omega^{\pm}_A$. First of all, the explicit expressions for these functions show that $V\Omega^{\pm}_A$ are C^2 up to and including $\partial\Sigma$, and the same is true for $\hat{g} = V^{-2}h$, as discussed in Section 2. It only remains to check that the usual matching conditions (see e.g. [14]) are satisfied. This follows easily from the fact that $\partial\Sigma$ is defined by $V = 0$ and that $\partial\Sigma$ is a totally geodesic submanifold of $(\Sigma, \hat{g})$. Thus, after the gluing, we have two pairs of Riemannian manifolds $(\mathcal{N}, h_A)$ which are $C^{1,1}$, complete and with vanishing mass. Before using the rigidity part of the positive mass theorem we need to check whether the Ricci scalar of any of these metrics is non-negative. A straightforward, if somewhat long, calculation shows that *none* has a sign. Nevertheless, a suitable combination of them *is* non-negative everywhere on $\mathcal{N}$, namely $\Omega_1^2 R(h_1) + \alpha^2\Omega_2^2 R(h_2) > 0$. From the transformation law of Ricci scalars under conformal rescalings (see [8]) for details) it follows that the metric $\hat{h} = \Omega_1^{\frac{2}{1+\alpha^2}}\Omega_2^{\frac{2\alpha^2}{1+\alpha^2}} h)$ has non-negative Ricci scalar, while still keeping the property of being complete, $C^{1,1}$ and with vanishing mass. Hence, the positive mass theorem can be applied to conclude that $\hat{h}$ is flat. Consequently, the induced metric $\hat{g}$ on Σ is conformally flat. Showing spherical symmetry from conformal flatness is then straightforward. This argument shows, after filling in all the details, the following uniqueness theorem

Theorem 5.3 (Mars & Simon, 02) *Let $(\mathcal{M}, g)$ be a simply connected, static, non-degenerate Einstein-Maxwell-dilaton black hole with non-zero coupling constant α and vanishing magnetic field. Assume that the charges M, D, Q*

satisfy the strict inequalities in (6). Then $(\mathcal{M}, g)$ must be spherically symmetric.

References

[1] V.Frolov and I.Novikov, "Black Hole Physics: Basic Concepts and New Developments" (Kluwer Academic Publishers, 1998)

[2] S.W.Hawking, G.F.R.Ellis, "The large scale structure of space-time" (Cambridge University Press, Cambridge 1973).

[3] R.Penrose, "Gravitational collapse: the role of general relativity" *Nuovo Cimento* **1** 252-276 (1969).

[4] M.Heusler, "Black Hole Uniqueness Theorems", Lecture Notes in Physics **6** (Cambridge University Press, Cambridge, 1996).

[5] B.Carter, *Killing horizons and orthogonally transitive groups in space-time*, J. Math. Phys. **10** 70-81 (1969).

[6] P.T.Chruściel, *The classification of static vacuum spacetimes containing an asymptotically flat spacelike hypersurface with compact interior*, Class. Quant. Grav. **16** 661-687 (1999).

[7] A.K.M.Masood-ul-Alam, "Uniqueness of a static charged dilaton black hole" *Class. Quant. Grav.* **10** 2649-2656 (1993).

[8] M.Mars, W.Simon, "On Uniqueness of static Einstein-Maxwell-Dilaton black holes", *Advances in Theor. and Math. Physics*, **6** 279-306 (2002).

[9] I.Rácz, R.M.Wald, "Extensions of spacetimes with Killing horizons", *Class. Quant. Grav.* **9** 2643-2656 (1992).

[10] R.H.Boyer, "Geodesic Killing orbits and bifurcate Killing horizons", *Proc. Roy. Soc. London A* **311** 245-253 (1969).

[11] G.L.Bunting, K.M.Masood-ul-Alam, "Nonexistence of multiple back holes in Asymptotically Euclidean Static Vacuum Space-Time" *Gen. Rel. Grav.* **19**, 147-154 (1987).

[12] R.Schoen, S.-T.Yau, "Proof of the positivity mass theorem, II" *Commum. Math. Phys.* **79** 231-260 (1981).

[13] P.Breitenlohner, D.Maison, "On Non-linear σ-Models Arising in (Super-)Gravity", *Commun. Math. Phys.* **209** 785-810 (2000).

[14] M.Mars, J.M.M.Senovillla, "Geometry of general hypersurfaces in spacetime, junction conditions", *Class. Quant. Grav.* **10** 1865-97 (1993)

Proceedings of the II International
Meeting on Lorentzian Geometry
Murcia, November 12–14, 2003
Publ. de la RSME, Vol. 8 (2004), 41–53

Examples of symplectic s–formal manifolds

Marisa Fernández[1] and Vicente Muñoz[2]

[1]*Departamento de Matemáticas, Universidad del País Vasco*
[2]*Departamento de Matemáticas, Universidad Autónoma de Madrid*

Abstract

In [3] the authors have introduced the concept of *s–formality* as a weaker version of *formality*. In this note, we use a double iteration of symplectic blow-ups of the complex projective space CP^m along symplectic submanifolds $M \subset CP^m$ to construct examples of compact symplectic manifolds which are s–formal but not $(s+1)$–formal for some values of s.

Keywords: formality, symplectic manifold, blow-up.

2000 Mathematics Subject Classification: Primary: 57R17. Secondary: 55P62, 53D35, 55S30.

1. Introduction

A *symplectic manifold* is a pair (M, ω) where M is an even-dimensional differentiable manifold and ω is a closed non-degenerate 2–form on M. The form ω is called a *symplectic form.* The simplest examples of such manifolds are Kähler manifolds. Topological properties of Kähler manifolds allow to construct examples of symplectic manifolds not admitting Kähler metrics. One of the most interesting topological properties of Kähler manifolds is the *formality.* If M is simply connected (or if M is nilpotent, i.e., $\pi_1(M)$ is nilpotent and acts nilpotently on $\pi_i(M)$ for $i \geq 2$) then formality is equivalent to saying that its real homotopy type is completely determined by its real cohomology algebra. Kähler manifolds are always formal [2], whereas there are symplectic manifolds which are not [12].

In [3] the authors have weakened the condition of formality to that of s–*formality* (see Definition 2.2). As an obvious fact, whenever M is a non-formal manifold, there is some $s \geq 0$ such that M is s–formal but not $(s+1)$–formal.

Also in [3] examples of (*non-symplectic*) manifolds which are s–formal but not $(s+1)$–formal for any $s \geq 0$ are shown. However the question of the existence of *symplectic* examples was left open.

In [4] the authors have studied how the s–formality property behaves under symplectic blow-ups of the complex projective space along a symplectic submanifold. Doing iterated blow-ups, there were produced *symplectic* manifolds which are s–formal but not $(s+1)$–formal, for arbitrarily large values of s. However, the examples of [4] do not cover all the values of s.

The purpose of this note is to study in detail two examples of a double iteration of the symplectic blow-up construction treated in [4]. This allows to find *symplectic* manifolds which are s–formal but not $(s+1)$–formal, for the cases $s = 7, 8$. We must note that examples for the cases $s = 0, 1, 3, 4$ are given in [3, 4]. The ideas behind the examples studied here may be useful to cover the general case with $s \geq 5$. However the problem of finding a symplectic manifold which is 2–formal but not 3–formal is much harder and it is likely that there are no such examples.

2. s–formal manifolds and Massey products

We recall some definitions and results about minimal models and s–formal manifolds [2, 3, 6, 9]. Let (A, d) be a *differential algebra*, that is, A is a graded commutative algebra over the real numbers, with a differential d which is a derivation, i.e. $d(a \cdot b) = (da) \cdot b + (-1)^{\deg(a)} a \cdot (db)$, where $\deg(a)$ is the degree of a. Given a differential algebra (A, d), we denote by $H^*(A)$ its cohomology. A is *connected* if $H^0(A) = \mathbb{R}$. Morphisms between differential algebras are required to be degree preserving algebra maps which commute with the differentials. A morphism of differential algebras inducing an isomorphism on cohomology is called a *quasi-isomorphism*.

A differential algebra (A, d) is said to be *minimal* if:

1. A is a free algebra $\bigwedge V$ over a graded vector space $V = \oplus V^i$, and
2. there exists a collection of generators $\{a_\tau, \tau \in I\}$, for some well ordered index set I, such that $\deg(a_\mu) \leq \deg(a_\tau)$ if $\mu < \tau$ and each da_τ is expressed in terms of preceding a_μ, $\mu < \tau$. This implies that da_τ does not have a linear part, i.e., it lives in $\bigwedge V^{>0} \cdot \bigwedge V^{>0} \subset \bigwedge V$.

A minimal model for the differential algebra (A, d) is a minimal differential algebra $(\mathcal{M}, d)$ and a quasi-isomorphism $\rho\colon (\mathcal{M}, d) \longrightarrow (A, d)$. Halperin [6] proved that any connected differential algebra (A, d) has a minimal model unique up to isomorphism.

A minimal model $(\mathcal{M}, d)$ is said to be *formal* if there is a morphism of differential algebras $\psi\colon (\mathcal{M}, d) \to (H^*(\mathcal{M}), d = 0)$ that induces the identity on cohomology. The formality of a minimal model can be detected as follows.

Theorem 2.1 *[2]. A minimal model $(\mathcal{M}, d)$ is formal if and only if we can write $\mathcal{M} = \bigwedge V$ and the space V decomposes as a direct sum $V = C \oplus N$ with $d(C) = 0$, d is injective on N and such that every closed element in the ideal $I(N)$ generated by N in $\bigwedge V$ is exact.*

A *minimal model* of a connected differentiable manifold M is a minimal model $(\bigwedge V, d)$ for the de Rham complex $(\Omega M, d)$ of differential forms on M. If M is a simply connected manifold, then the dual of the real homotopy vector space $\pi_i(M) \otimes \mathbb{R}$ is isomorphic to V^i for any i. (This relation also happens when $i > 1$ and M is nilpotent.) We shall say that M is *formal* if its minimal model is formal or, equivalently, the differential algebras $(\Omega M, d)$ and $(H^*(M), d = 0)$ have the same minimal model. (For details see [2] for example.)

To detect non-formality, we have Massey products. Let us recall its definition. Let M be a (not necessarily simply connected) manifold and let $a_i \in H^{p_i}(M)$, $1 \leq i \leq 3$, be three cohomology classes such that $a_1 \cup a_2 = 0$ and $a_2 \cup a_3 = 0$. Take forms α_i in M with $a_i = [\alpha_i]$ and write $\alpha_1 \wedge \alpha_2 = d\xi$, $\alpha_2 \wedge \alpha_3 = d\eta$. The Massey product of the classes a_i is defined as

$$\begin{aligned} \langle a_1, a_2, a_3 \rangle &= [\alpha_1 \wedge \eta + (-1)^{p_1+1} \xi \wedge \alpha_3] \\ &\in \frac{H^{p_1+p_2+p_3-1}(M)}{a_1 \cup H^{p_2+p_3-1}(M) + H^{p_1+p_2-1}(M) \cup a_3}. \end{aligned}$$

If M has a non-trivial Massey product then M is non-formal [2, 12].

In [3] we weaken the condition of *formal* manifold to *s–formal* manifold as follows.

Definition 2.2 *Let $(\mathcal{M}, d)$ be a minimal model of a differentiable manifold M. We say that $(\mathcal{M}, d)$ is s–formal, or M is an s–formal manifold ($s \geq 0$) if we can write $\mathcal{M} = \bigwedge V$ such that for each $i \leq s$ the space V^i of generators of degree i decomposes as a direct sum $V^i = C^i \oplus N^i$, where the spaces C^i and N^i satisfy the three following conditions:*

1. *$d(C^i) = 0$,*
2. *the differential map $d\colon N^i \longrightarrow \bigwedge V$ is injective,*

3. *any closed element in the ideal* $I(\bigoplus_{i\leq s} N^i)$, *generated by* $\bigoplus_{i\leq s} N^i$ *in* $\bigwedge(\bigoplus_{i\leq s} V^i)$, *is exact in* $\bigwedge V$.

Lemma 2.3 *[3]. Let M be a manifold with minimal model $(\bigwedge V, d)$. Then M is s–formal if and only if there is a map of differential algebras*

$$\varphi : (\bigwedge V^{\leq s}, d) \longrightarrow (H^*(M), d = 0),$$

such that the map $\varphi^* : H^i(\bigwedge V) \longrightarrow H^i(M)$ induced on cohomology is an s–quasi-isomorphism, that is, φ^* is an isomorphism for $i \leq s$, and a monomorphism for $i = s + 1$. Equivalently, if $(\bigwedge W, d)$ is the minimal model of $(H^*(M), d = 0)$, then $V^i = W^i$ for $i \leq s$.

The relationship between s–formality and Massey products is given by

Lemma 2.4 *[3]. Let M be an s–formal manifold. Suppose that there are cohomology classes $a_i \in H^{p_i}(M)$, $1 \leq i \leq 3$, such that the Massey product $\langle a_1, a_2, a_3\rangle$ is defined. If $p_1 + p_2 \leq s + 1$ and $p_2 + p_3 \leq s + 1$, then $\langle a_1, a_2, a_3\rangle$ vanishes.*

To study in Section 4 the s–formality of our examples, we shall need the following result.

Proposition 2.5 *Let M be a differentiable manifold. Suppose that, for $i \leq s + 1$, there exists a map $\phi : H^i(M) \to \Omega^i(M)$ satisfying the two following conditions:*

1. *for any cohomology class $a \in H^i(M)$, $\phi(a)$ is closed and represents the class a in cohomology;*
2. *ϕ is multiplicative, in the sense that $\phi(a \cup b) = \phi(a) \wedge \phi(b)$ whenever* $\deg(a) + \deg(b) \leq s + 1$.

Then M is s–formal.

Proof : Let $\psi : (\bigwedge W, d) \longrightarrow (H^*(M), d = 0)$ be the minimal model of $(H^*(M), 0)$. By definition, ψ is a quasi-isomorphism of differential graded algebras. Composing with ϕ, we have maps $W^i \to H^i(M) \to \Omega^i(M)$ for $i \leq s+1$. These maps, together with the zero maps $W^i \to \Omega^i(M)$ for $i > s + 1$, define uniquely a differential graded algebra morphism $\tilde{\psi} : (\bigwedge W, d) \longrightarrow (\Omega^*(M), d)$. Restricting to the elements of degree $\leq (s + 1)$, $\tilde{\psi} : (\bigwedge W)^{\leq(s+1)} \to \Omega^{s+1}(M)$ coincides with $\phi \circ \psi : (\bigwedge W)^{\leq(s+1)} \to \Omega^{s+1}(M)$, by the multiplicativity property for ϕ. Hence $\tilde{\psi}$ is an s–quasi-isomorphism.

So the minimal model $(\bigwedge V, d)$ of $(\Omega^*(M), d)$ coincides with $(\bigwedge W, d)$ up to degree s, i.e., $(\bigwedge V^{\leq s}, d) \cong (\bigwedge W^{\leq s}, d)$. In particular, this implies the existence of an s–quasi-isomorphism $(\bigwedge V^{\leq s}, d) \longrightarrow (H^*(M), d = 0)$. By Lemma 2.3, M is s–formal. QED

3. Symplectic blow-ups

In this section we recollect some results on Massey products for the symplectic blow-up of the complex projective space along a symplectic submanifold.

Let (M, ω) be a compact *symplectic* manifold of dimension $2n$. Without loss of generality we can assume that the symplectic form ω is integral (by perturbing it to make it rational and then rescaling). A theorem of Gromov and Tischler [5, 11] states that there is a symplectic embedding $i\colon (M, \omega) \longrightarrow (CP^m, \omega_0)$, with $m \geq 2n+1$, where ω_0 is the standard Kähler form on CP^m defined by its natural complex structure and the Fubini–Study metric. We take the symplectic blow-up $\widetilde{CP^m}$ of CP^m along the embedding i (see [7]). Then $\widetilde{CP^m}$ is a simply connected compact symplectic manifold.

In [4] the authors studied the s–formality and the formality of $\widetilde{CP^m}$ for an arbitrary compact symplectic manifold (M, ω) and $m \geq 2n+1$, proving that if (M, ω) is s–formal then $\widetilde{CP^m}$ is at least $(s+2)$–formal. Moreover, if M is formal then $\widetilde{CP^m}$ is formal.

Recall that $i^*\omega_0 = \omega$. We will denote also by ω_0 the pull back of ω_0 to $\widetilde{CP^m}$ under the natural projection $\widetilde{CP^m} \to CP^m$. Let $\widetilde{M}$ be the projectivization of the normal bundle of the embedding $M \hookrightarrow CP^m$. Then $\pi\colon \widetilde{M} \longrightarrow M$ is a locally trivial bundle with fiber CP^{m-n-1}. We will denote by ν the Thom form of the submanifold $\widetilde{M} \subset \widetilde{CP^m}$. The class $[\nu]$ is called the Thom class of the blow-up. Then $\widetilde{CP^m}$ has a symplectic form Ω whose cohomology class is $[\Omega] = [\omega_0] + \epsilon\,[\nu]$ for $\epsilon > 0$ small enough.

Let us consider a closed tubular neighborhood $\widetilde{W}$ of $\widetilde{M}$ in $\widetilde{CP^m}$. By the tubular neighborhood theorem we know that the normal bundle of $\widetilde{M} \hookrightarrow \widetilde{CP^m}$ contains a disk subbundle which is diffeomorphic to $\widetilde{W}$. Denote by $p\colon \widetilde{W} \longrightarrow \widetilde{M}$ the natural map. There is a map $q : \Omega^*(M) \to \Omega^{*+2}(\widetilde{CP^m})$ given by pull-back by $\pi : \widetilde{M} \to M$, followed by extending to a neighborhood of $\widetilde{M}$ using $p : \widetilde{W} \to \widetilde{M}$ and then wedging by ν, i.e., $q(\alpha) = p^*\pi^*(\alpha) \wedge \nu$. We shall denote $q(\alpha) = \alpha \wedge \nu$ for short. Note that

$$(\alpha \wedge \nu) \wedge (\beta \wedge \nu) = (\alpha \wedge \beta \wedge \nu) \wedge \nu\,, \tag{1}$$

for $\alpha, \beta \in \Omega^*(M)$. This makes notations of the type $\alpha \wedge \beta \wedge \nu^2$ unambiguous.

Also remark that $[\omega_0 \wedge \nu] = [\omega \wedge \nu]$ although $\omega_0 \wedge \nu \neq \omega \wedge \nu$ as forms. To circumvent this problem, we work as follows. Extend the projection map $p : \widetilde{W} \to \widetilde{M}$ to a map

$$\bar{p} : \widetilde{CP}^m \to \widetilde{CP}^m \tag{2}$$

by pasting p with the identity. The map $\bar{p}$ induces the identity on cohomology. The form $\bar{\omega}_0 = \bar{p}^*(\omega_0)$ is no longer a symplectic form, but $[\bar{\omega}_0] = [\omega_0]$ and $\bar{\omega}_0 \wedge \nu = q(\omega) = \omega \wedge \nu$.

The cohomology of $\widetilde{CP}^m$ was studied by McDuff [7]. There she proved that there is a short exact sequence

$$0 \longrightarrow H^*(CP^m) \longrightarrow H^*(\widetilde{CP}^m) \longrightarrow A^* \longrightarrow 0, \tag{3}$$

where A^* is a free module over $H^*(M)$ generated by $\{[\nu], [\nu^2], \cdots, [\nu^{m-n-1}]\}$.

Regarding to the s–formality of $\widetilde{CP}^m$, we have the following results.

Lemma 3.1 *[4]. Let M be a compact symplectic manifold of dimension $2n \geq 2$. Then the symplectic blow-up $\widetilde{CP}^m$ of CP^m along $M \subset CP^m$ is 3–formal for any $m \geq 2n+1$.*

Lemma 3.2 *[4]. Let M be a compact symplectic manifold of dimension $2n$. Suppose that $m - n \geq 4$ and that M has a non-trivial Massey product $\langle a_1, a_2, a_3 \rangle$, $a_j = [\alpha_j] \in H^{p_j}(M)$, with $p_1 + p_2 = s+1$ and $p_2 + p_3 \leq s+1$. Then $\langle [\alpha_1 \wedge \nu], [\alpha_2 \wedge \nu], [\alpha_3 \wedge \nu] \rangle$ is a non-trivial Massey product in $\widetilde{CP}^m$. Hence $\widetilde{CP}^m$ is not $(s+4)$–formal.*

Lemma 3.3 *[8]. Let (M, ω) be a compact symplectic manifold of dimension $2n$. Suppose that $m - n \geq 3$ and that M has a non-trivial Massey product $\langle a_1, \omega, a_3 \rangle$, $a_j = [\alpha_j] \in H^{p_j}(M)$, with $p_1 + 2 = s+1$ and $2 + p_3 \leq s+1$. Then $\langle [\alpha_1 \wedge \nu], [\omega_0], [\alpha_3 \wedge \nu] \rangle$ is a non-trivial Massey product in $\widetilde{CP}^m$. Hence $\widetilde{CP}^m$ is not $(s+2)$–formal.*

4. Double iteration of symplectic blow-ups

To show examples of compact symplectic manifolds which are s–formal but not $(s+1)$–formal for $s = 7$ and 8, we do a double iteration of symplectic blow-ups.

We begin with the Heisenberg group H, that is, the connected nilpotent Lie group of dimension 3 consisting of matrices of the form

$$a = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z \in \mathbb{R}$. A global system of coordinates $\{x, y, z\}$ for H is given by $x(a) = x$, $y(a) = y$, $z(a) = z$, and a standard calculation shows that a basis for the left invariant 1–forms on H consists of $\{dx, dy, dz - xdy\}$. Let Γ be the discrete subgroup of H consisting of matrices whose entries are integer numbers. Then the quotient space $N = \Gamma\backslash H$ is compact. Hence the forms dx, dy, $dz - xdy$ descend to 1–forms α, β, γ on N such that

$$d\alpha = d\beta = 0, \quad d\gamma = -\alpha \wedge \beta.$$

The manifold $KT = N \times S^1$ is called *Kodaira–Thurston manifold* (cf. [10]). It is symplectic with the symplectic form $\omega = \alpha\wedge\gamma + \beta\wedge\delta$, where δ is the standard invariant 1–form on S^1. Then KT is 0–*formal* (all connected manifolds are 0–formal) but it is not 1–*formal*. For instance $\langle[\alpha], [\alpha], [\beta]\rangle = [\gamma \wedge \alpha]$ is non-zero in $H^2(KT)/(H^1(KT) \cup [\beta] + [\alpha] \cup H^1(KT))$.

By Gromov–Tischler theorem [5, 11] there exists a symplectic embedding of (KT, ω) in the complex projective space (CP^5, ω_0), where ω_0 is the standard symplectic form. Now perform the symplectic blow up $\widetilde{CP}^5$ of CP^5 along KT. This is a symplectic manifold with symplectic form $\Omega = \omega_0 + \epsilon\,\nu$, where ν denotes the Thom form of the blow-up $\widetilde{CP}^5$.

By Lemma 3.1, $\widetilde{CP}^5$ is 3–*formal*. On the other hand, there is a non-zero Massey product given as $\langle[\alpha], [\omega], [\alpha]\rangle = 2[\gamma \wedge \alpha \wedge \delta]$ in KT. Hence by Lemma 3.3, there is a Massey product of the form $\langle[\alpha\wedge\nu], [\omega_0], [\alpha\wedge\nu]\rangle \neq 0$ in $\widetilde{CP}^5$, so it is not 4–*formal*. (This example already appears in [3, 4] as a *non* 4–*formal* manifold, and in [1, 8] as a non-formal simply connected symplectic manifold.)

Consider the compact symplectic manifold (Y, Ω) where $Y = \widetilde{CP}^5$ and Ω is the symplectic form $\Omega = \omega_0 + \epsilon\,\nu$ for $\epsilon > 0$ small. Embed symplectically (Y, Ω) in (CP^{11}, Ω_0), where Ω_0 is the standard symplectic form. Let $\widetilde{CP}^{11}$ be the symplectic blow-up of CP^{11} along Y. It has symplectic form $\Omega_1 = \Omega_0 + \epsilon'\,\nu_1$, where ν_1 is the Thom form of the blow-up $\widetilde{CP}^{11}$.

In order to show that $\widetilde{CP}^{11}$ is 7–formal and not 8–formal, we describe the cohomology groups $H^i(Y)$ with $i \leq 6$. Using (3) and the notation fixed in Section 3, we have

$$\begin{aligned}
H^0(Y) &= \langle 1\rangle,\\
H^1(Y) &= 0,\\
H^2(Y) &= \langle[\omega_0], [\nu]\rangle,\\
H^3(Y) &= H^1(KT)\cdot[\nu] = \langle[\alpha\wedge\nu], [\beta\wedge\nu], [\delta\wedge\nu]\rangle,\\
H^4(Y) &= \langle[\omega_0^2]\rangle \oplus H^2(KT)\cdot[\nu] \oplus \langle[\nu^2]\rangle =\\
&= \langle[\omega_0^2], [\omega_0\wedge\nu], [\nu^2], [\alpha\wedge\gamma\wedge\nu], [\alpha\wedge\delta\wedge\nu], [\beta\wedge\gamma\wedge\nu]\rangle,\\
H^5(Y) &= H^1(KT)\cdot[\nu^2] \oplus H^3(KT)\cdot[\nu] =
\end{aligned}$$

$$\begin{aligned} &= \langle [\alpha\wedge\nu^2],[\beta\wedge\nu^2],[\delta\wedge\nu^2],[\alpha\wedge\beta\wedge\gamma\wedge\nu],[\alpha\wedge\gamma\wedge\delta\wedge\nu],\\ &\quad [\beta\wedge\gamma\wedge\delta\wedge\nu]\rangle,\\ H^6(Y) &= \langle[\omega_0^3]\rangle\oplus H^2(KT)\cdot[\nu^2]\oplus H^4(KT)\cdot[\nu] =\\ &= \langle[\omega_0^3],[\omega_0^2\wedge\nu],[\omega_0\wedge\nu^2],[\alpha\wedge\gamma\wedge\nu^2],[\alpha\wedge\delta\wedge\nu^2],[\beta\wedge\gamma\wedge\nu^2]\rangle. \end{aligned}$$

Proposition 4.1 *The symplectic blow-up $\widetilde{CP}^{11}$ of CP^{11} along Y is 7–formal but not 8–formal.*

Proof : By Lemma 3.2, $\widetilde{CP}^{11}$ is not 8–formal, since the non-trivial Massey product $\langle[\alpha\wedge\nu],[\omega_0],[\alpha\wedge\nu]\rangle\neq 0$ in $Y=\widetilde{CP}^5$ defines a non-trivial Massey product $\langle[\alpha\wedge\nu\wedge\nu_1],[\omega_0\wedge\nu_1],[\alpha\wedge\nu\wedge\nu_1]\rangle$ in $\widetilde{CP}^{11}$. Let us see that $\widetilde{CP}^{11}$ is 7–formal. For this, we compute its cohomology. Taking into account the cohomology groups of Y, from (3) we get

$$\begin{aligned} H^0(\widetilde{CP}^{11}) &= \langle 1\rangle,\\ H^1(\widetilde{CP}^{11}) &= 0,\\ H^2(\widetilde{CP}^{11}) &= \langle[\Omega_0],[\nu_1]\rangle,\\ H^3(\widetilde{CP}^{11}) &= 0,\\ H^4(\widetilde{CP}^{11}) &= \langle[\Omega_0^2],[\omega_0\wedge\nu_1],[\nu\wedge\nu_1],[\nu_1^2]\rangle,\\ H^5(\widetilde{CP}^{11}) &= \langle[\alpha\wedge\nu\wedge\nu_1],[\beta\wedge\nu\wedge\nu_1],[\delta\wedge\nu\wedge\nu_1]\rangle,\\ H^6(\widetilde{CP}^{11}) &= \langle[\Omega_0^3],[\omega_0^2\wedge\nu_1],[\omega_0\wedge\nu\wedge\nu_1],[\alpha\wedge\gamma\wedge\nu\wedge\nu_1],[\alpha\wedge\delta\wedge\nu\wedge\nu_1],\\ &\quad [\beta\wedge\gamma\wedge\nu\wedge\nu_1],[\nu^2\wedge\nu_1],[\omega_0\wedge\nu_1^2],[\nu\wedge\nu_1^2],[\nu_1^3]\rangle,\\ H^7(\widetilde{CP}^{11}) &= \langle[\alpha\wedge\beta\wedge\gamma\wedge\nu\wedge\nu_1],[\alpha\wedge\gamma\wedge\delta\wedge\nu\wedge\nu_1],[\beta\wedge\gamma\wedge\delta\wedge\nu\wedge\nu_1],\\ &\quad [\alpha\wedge\nu^2\wedge\nu_1],[\beta\wedge\nu^2\wedge\nu_1],[\delta\wedge\nu^2\wedge\nu_1],[\alpha\wedge\nu\wedge\nu_1^2],\\ &\quad [\beta\wedge\nu\wedge\nu_1^2],[\delta\wedge\nu\wedge\nu_1^2]\rangle,\\ H^8(\widetilde{CP}^{11}) &= \langle[\Omega_0^4],[\omega_0^3\wedge\nu_1],[\omega_0^2\wedge\nu\wedge\nu_1],[\omega_0\wedge\nu^2\wedge\nu_1],[\alpha\wedge\gamma\wedge\nu^2\wedge\nu_1],\\ &\quad [\alpha\wedge\delta\wedge\nu^2\wedge\nu_1],[\beta\wedge\gamma\wedge\nu^2\wedge\nu_1],[\omega_0^2\wedge\nu_1^2],[\omega_0\wedge\nu\wedge\nu_1^2],\\ &\quad [\nu^2\wedge\nu_1^2],[\alpha\wedge\gamma\wedge\nu\wedge\nu_1^2],[\alpha\wedge\delta\wedge\nu\wedge\nu_1^2],[\beta\wedge\gamma\wedge\nu\wedge\nu_1^2],\\ &\quad [\omega_0\wedge\nu_1^3],[\nu\wedge\nu_1^3],[\nu_1^4]\rangle. \end{aligned}$$

Now since $\Omega=\omega_0+\epsilon\,\nu$ we have that $[\Omega_0\wedge\nu_1]=[\Omega\wedge\nu_1]=[\omega_0\wedge\nu_1]+\epsilon\,[\nu\wedge\nu_1]$. This implies that $\langle[\omega_0^i\wedge\nu_1],[\omega_0^{i-1}\wedge\nu\wedge\nu_1],\dots,[\nu^i\wedge\nu_1]\rangle=\langle[\Omega_0^i\wedge\nu_1],[\Omega_0^{i-1}\wedge\nu\wedge\nu_1],\dots,[\nu^i\wedge\nu_1]\rangle$, for any $i>0$. Also we have that

$$[\alpha\wedge\nu^2\wedge\nu_1]=\frac{1}{\epsilon}\left([\Omega_0\wedge\alpha\wedge\nu\wedge\nu_1]-[(\omega_0\wedge\alpha)\wedge\nu\wedge\nu_1]\right).$$

Using these two facts, we can rewrite

$$\begin{aligned}
H^2(\widetilde{CP}^{11}) &= \langle[\Omega_0],[\nu_1]\rangle,\\
H^4(\widetilde{CP}^{11}) &= \langle[\Omega_0^2],[\Omega_0\wedge\nu_1],[\nu_1^2],[\nu\wedge\nu_1]\rangle,\\
H^5(\widetilde{CP}^{11}) &= \langle[\alpha\wedge\nu\wedge\nu_1],[\beta\wedge\nu\wedge\nu_1],[\delta\wedge\nu\wedge\nu_1]\rangle,\\
H^6(\widetilde{CP}^{11}) &= \langle[\Omega_0^3],[\Omega_0^2\wedge\nu_1],[\Omega_0\wedge\nu_1^2],[\nu_1^3],[\Omega_0\wedge\nu\wedge\nu_1],[\nu\wedge\nu_1^2],[\nu^2\wedge\nu_1],\\
&\quad [\alpha\wedge\gamma\wedge\nu\wedge\nu_1],[\alpha\wedge\delta\wedge\nu\wedge\nu_1],[\beta\wedge\gamma\wedge\nu\wedge\nu_1]\rangle,\\
H^7(\widetilde{CP}^{11}) &= [\Omega_0\wedge\alpha\wedge\nu\wedge\nu_1],[\Omega_0\wedge\beta\wedge\nu\wedge\nu_1],[\Omega_0\wedge\delta\wedge\nu\wedge\nu_1],\\
&\quad [\alpha\wedge\beta\wedge\gamma\wedge\nu\wedge\nu_1],[\alpha\wedge\gamma\wedge\delta\wedge\nu\wedge\nu_1],[\beta\wedge\gamma\wedge\delta\wedge\nu\wedge\nu_1],\\
&\quad [\alpha\wedge\nu\wedge\nu_1^2],[\beta\wedge\nu\wedge\nu_1^2],[\delta\wedge\nu\wedge\nu_1^2]\rangle,\\
H^8(\widetilde{CP}^{11}) &= \langle[\Omega_0^4],[\Omega_0^3\wedge\nu_1],[\Omega_0^2\wedge\nu_1^2],[\Omega_0\wedge\nu_1^3],[\nu_1^4],[\Omega_0^2\wedge\nu\wedge\nu_1],\\
&\quad [\Omega_0\wedge\nu^2\wedge\nu_1],[\Omega_0\wedge\nu\wedge\nu_1^2],[\nu^2\wedge\nu_1^2],[\nu\wedge\nu_1^3],[\alpha\wedge\gamma\wedge\nu\wedge\nu_1^2],\\
&\quad [\alpha\wedge\delta\wedge\nu\wedge\nu_1^2],[\beta\wedge\gamma\wedge\nu\wedge\nu_1^2],[\Omega_0\wedge\alpha\wedge\gamma\wedge\nu\wedge\nu_1],\\
&\quad [\Omega_0\wedge\alpha\wedge\delta\wedge\nu\wedge\nu_1],[\Omega_0\wedge\beta\wedge\gamma\wedge\nu\wedge\nu_1]\rangle.
\end{aligned}$$

Once that we have written the cohomology in this way, we define a map $\phi : H^i(\widetilde{CP}^{11}) \to \Omega^i(\widetilde{CP}^{11})$, $i \leq 8$, by sending each cohomology class $[z]$ to its representative inside the brackets z. More specifically,

$$\phi([\Omega_0^k\wedge x\wedge\nu^a\wedge\nu_1^b]) = \Omega_0^k\wedge((x\wedge\nu^a)\wedge\nu_1^b),$$

where x is an invariant form on KT.

It is easy to prove that ϕ satisfies $\phi(z_1\cup z_2) = \phi(z_1)\wedge\phi(z_2)$ whenever $\deg(z_1)+\deg(z_2)\leq 8$. For this one uses (1) and note that the possibilities for $(\deg(z_1),\deg(z_2))$ are $(2,2),(2,4),(2,5),(2,6)$ and $(4,4)$. Therefore by Proposition 2.5, $\widetilde{CP}^{11}$ is 7–formal.

QED

Remark 4.2 *It is possible to prove that* $\widetilde{CP}^{11}$ *is* 7*–formal by computing the* 7*–stage of its minimal model. This is the differential graded algebra* $(\mathcal{M}, d)$ *given by* $\mathcal{M} = \bigwedge(a_1,a_2)\otimes\bigwedge(b_1)\otimes\bigwedge(c_1,c_2,c_3)\otimes\bigwedge(e_1,e_2,e_3,e_4)\otimes\bigwedge(f_1,f_2,f_3,f_4)\otimes\bigwedge V^{\geq 8}$ *where the degree of the generators is* $\deg(a_i) = 2$, $\deg(b_1) = 4$, $\deg(c_j) = 5$, $\deg(e_k) = 6$, $\deg(f_k) = 7$, *and all the generators are closed except for* $df_4 = b_1^2 - e_4\cdot a_2$. *The morphism* $\rho\colon \mathcal{M}\to\Omega^*(\widetilde{CP}^{11})$, *inducing a* 7*–quasi-isomorphism is defined by* $\rho(a_1) = \omega_0$, $\rho(a_2) = \nu_1$, $\rho(b_1) = \nu\wedge\nu_1$, $\rho(c_1) = \alpha\wedge\nu\wedge\nu_1$, $\rho(c_2) = \beta\wedge\nu\wedge\nu_1$, $\rho(c_3) = \delta\wedge\nu\wedge\nu_1$, $\rho(e_1) = \alpha\wedge\delta\wedge\nu\wedge\nu_1$, $\rho(e_2) = \alpha\wedge\gamma\wedge\nu\wedge\nu_1$, $\rho(e_3) = \beta\wedge\gamma\wedge\nu\wedge\nu_1$, $\rho(e_4) = \nu^2\wedge\nu_1$, $\rho(f_1) = \alpha\wedge\beta\wedge\gamma\wedge\nu\wedge\nu_1$, $\rho(f_2) = \alpha\wedge\gamma\wedge\delta\wedge\nu\wedge\nu_1$, $\rho(f_3) = \beta\wedge\gamma\wedge\delta\wedge\nu\wedge\nu_1$ *and* $\rho(f_4) = 0$.

According to Definition 2.2, the minimal model of $\widetilde{CP}^{11}$ satisfies $V^i = C^i$ and $N^i = 0$ for $i \leq 6$, thus $\widetilde{CP}^{11}$ is 6–formal. Moreover $N^7 = \langle f_4 \rangle$. The only closed element in the ideal $N^{\leq 7} \cdot (\bigwedge V^{\leq 7})$ is zero. Indeed, suppose that $z = f_4 \cdot y \in N^{\leq 7} \cdot (\bigwedge V^{\leq 7})$ is closed, then $f_4 \cdot dy - df_4 \cdot y = 0$. This implies that $dy = 0$ and so $df_4 \cdot y = 0$. Hence $y = 0$. Therefore $\widetilde{CP}^{11}$ is 7–formal.

To produce an example of a symplectic manifold which is 8–formal but not 9–formal, let us consider the 6–dimensional manifold $M = KT \times T^2$, where T^2 is a 2–torus. Then $\{\alpha, \beta, \gamma, \delta, \eta_1, \eta_2\}$ is a basis for the 1–forms on M, where $\{\eta_1, \eta_2\}$ is a basis of the 1–forms on T^2. The 2–form $\omega = \alpha \wedge \gamma + \beta \wedge \delta + \eta_1 \wedge \eta_2$ is a symplectic form on M.

Take a symplectic embedding of (M, ω) in the complex projective space (CP^7, ω_0). By Proposition 4.3 in [4], the symplectic blow-up $\widetilde{CP}^7$ of CP^7 along M is 4–formal but not 5–formal. In fact, the non-zero Massey product $\langle [\alpha], [\alpha], [\beta] \rangle$ of KT defines a non-zero Massey product $\langle [\alpha], [\alpha], [\beta] \rangle$ for $M = KT \times T^2$. This gives a non-zero Massey product for $\widetilde{CP}^7$ by using Lemma 3.2. Hence $\widetilde{CP}^7$ is not 5–formal.

Now consider the manifold $Z = \widetilde{CP}^7$ with the symplectic form $\Omega = \omega_0 + \epsilon\, \nu$ for $\epsilon > 0$ small, where ω_0 denotes the pull back to $\widetilde{CP}^7$ of the Kähler form of CP^7, and ν is the Thom form of the blow-up $\widetilde{CP}^7$. Using (3), we get the cohomology groups $H^i(Z)$, $i \leq 7$,

$$
\begin{aligned}
H^0(Z) &= \langle 1 \rangle, \\
H^1(Z) &= 0, \\
H^2(Z) &= \langle [\omega_0], [\nu] \rangle, \\
H^3(Z) &= H^1(M) \cdot [\nu], \\
H^4(Z) &= \langle [{\omega_0}^2], [\omega_0 \wedge \nu], [\nu^2] \rangle \oplus \frac{H^2(M)}{[\omega]} \cdot [\nu], \\
H^5(Z) &= H^1(M) \cdot [\nu^2] \oplus H^3(M) \cdot [\nu], \\
H^6(Z) &= \langle [{\omega_0}^3], [\omega_0^2 \wedge \nu], [\omega_0 \wedge \nu^2], [\nu^3] \rangle \oplus \frac{H^2(M)}{[\omega]} \cdot [\nu^2] \oplus \frac{H^4(M)}{[\omega^2]} \cdot [\nu], \\
H^7(Z) &= H^1(M) \cdot [\nu^3] \oplus H^3(M) \cdot [\nu^2] \oplus H^5(M) \cdot [\nu].
\end{aligned}
$$

Embed symplectically (Z, Ω) in (CP^{15}, Ω_0), where Ω_0 is the standard symplectic form. Let $\widetilde{CP}^{15}$ be the symplectic blow-up of CP^{15} along Z. It has symplectic form $\Omega_0 + \epsilon'\, \nu_1$, where ν_1 is the Thom form of the blow-up $\widetilde{CP}^{15}$ and $\epsilon' > 0$ is small enough.

Proposition 4.3 *The symplectic blow-up $\widetilde{CP}^{15}$ of CP^{15} along Z is 8–formal but not 9–formal.*

Proof : By Lemma 3.3, $\widetilde{CP}^{15}$ is not 9–formal, using the non-zero Massey product $\langle[\alpha\wedge\nu],[\alpha\wedge\nu],[\beta\wedge\nu]\rangle$ of $\widetilde{CP}^{7}$. Let us see that it is 8–formal. For this, we compute the cohomology groups $H^i(\widetilde{CP}^{15})$ for $i\leq 9$,

$$\begin{aligned}
H^0(\widetilde{CP}^{15}) &= \langle 1\rangle,\\
H^1(\widetilde{CP}^{15}) &= 0,\\
H^2(\widetilde{CP}^{15}) &= \langle[\Omega_0],[\nu_1]\rangle,\\
H^3(\widetilde{CP}^{15}) &= 0,\\
H^4(\widetilde{CP}^{15}) &= \langle[\Omega_0^2],[\Omega_0\wedge\nu_1],[\nu_1^2],[\nu\wedge\nu_1]\rangle,\\
H^5(\widetilde{CP}^{15}) &= H^1(M)\cdot[\nu\wedge\nu_1],\\
H^6(\widetilde{CP}^{15}) &= \langle[\Omega_0^3],[\Omega_0^2\wedge\nu_1],[\Omega_0\wedge\nu_1^2],[\nu_1^3],[\nu^2\wedge\nu_1],[\nu\wedge\nu_1^2],[\Omega_0\wedge\nu\wedge\nu_1]\rangle\\
&\quad\oplus\frac{H^2(M)}{[\omega]}\cdot[\nu\wedge\nu_1],\\
H^7(\widetilde{CP}^{15}) &= H^3(M)\cdot[\nu\wedge\nu_1]\oplus H^1(M)\cdot\langle[\Omega_0\wedge\nu\wedge\nu_1],[\nu\wedge\nu_1^2]\rangle,\\
H^8(\widetilde{CP}^{15}) &= \langle[\Omega_0^4],[\Omega_0^3\wedge\nu_1],[\Omega_0^2\wedge\nu_1^2],[\Omega_0\wedge\nu_1^3],[\nu_1^4],[\Omega_0\wedge\nu^2\wedge\nu_1],\\
&\quad[\Omega_0\wedge\nu\wedge\nu_1^2],[\Omega_0^2\wedge\nu\wedge\nu_1],[\nu^3\wedge\nu_1],[\nu^2\wedge\nu_1^2],[\nu\wedge\nu_1^3]\rangle\\
&\quad\oplus\frac{H^4(M)}{[\omega^2]}\cdot[\nu\wedge\nu_1]\oplus\frac{H^2(M)}{[\omega]}\cdot\langle[\Omega_0\wedge\nu\wedge\nu_1],[\nu\wedge\nu_1^2]\rangle,\\
H^9(\widetilde{CP}^{15}) &= H^5(M)\cdot[\nu\wedge\nu_1]\oplus H^3(M)\cdot\langle[\Omega_0\wedge\nu\wedge\nu_1],[\nu\wedge\nu_1^2]\rangle\\
&\quad\oplus H^1(M)\cdot\langle[\Omega_0^2\wedge\nu\wedge\nu_1],[\Omega_0\wedge\nu\wedge\nu_1^2],[\nu\wedge\nu_1^3]\rangle.
\end{aligned}$$

Consider the maps $\bar p_1:\widetilde{CP}^7\to\widetilde{CP}^7$ and $\bar p_2:\widetilde{CP}^{15}\to\widetilde{CP}^{15}$ defined in (2). Define the 2–form $\bar\omega_0$ on Y by $\bar\omega_0=\bar p_1^*(\omega_0)$. As $[\omega_0]=[\bar\omega_0]$, there is a 1–form ξ on Y such that $\bar\omega_0-\omega_0=d\xi$. Extend ξ to $\widetilde{CP}^m$ by pulling it back to $\widetilde{Y}$, then to a tubular neighborhood $\widetilde{W}$ and finally multiplying by a cut-off function with value 1 in the support of ν_1. Call $\tilde\xi$ to this extension. Define

$$\bar\Omega_0=p_2^*(\Omega_0+d\tilde\xi).$$

Then we have the following equality of forms: $\bar\Omega_0\wedge\nu\wedge\nu_1=(\Omega+d\xi)\wedge\nu\wedge\nu_1=(\omega_0+\epsilon\,\nu+\bar\omega_0-\omega_0)\wedge\nu\wedge\nu_1=\omega\wedge\nu\wedge\nu_1+\epsilon\,\nu^2\wedge\nu_1$.

Define $\phi:H^i(\widetilde{CP}^{15})\to\Omega^i(\widetilde{CP}^{15})$, $i\leq 9$ in the following way. We start by defining a map $\phi_0:H^i(M)\to\Omega^i(M)$, $i\leq 5$ which sends each class $[z]$ to an invariant closed representative z, $\phi_0([z])=z$, in such a way that it satisfies the condition that $\phi_0([z]\cup[\omega]) = z\wedge\omega$, for $[z]\in H^1(M)$. This is possible since $[\omega] : H^1(M)\to H^3(M)$ is injective. Now set

$$\phi([\Omega_0^k\wedge z\wedge\nu^a\wedge\nu_1^b]) = \bar\Omega_0^k\wedge\phi_0([z])\wedge\nu^a\wedge\nu_1^b\,.$$

Then ϕ is a multiplicative map. The only non-obvious product is

$$\begin{aligned}
\phi([x\wedge\nu\wedge\nu_1]\cup[\nu\wedge\nu_1]) &= \phi([x\wedge\nu^2\wedge\nu_1^2]) = \frac{1}{\epsilon}\phi([x\wedge(\Omega_0-\omega_0)\wedge\nu\wedge\nu_1^2]) = \\
&= \frac{1}{\epsilon}\left(\bar\Omega_0\wedge\phi_0([x])\wedge\nu\wedge\nu_1^2 - \phi_0([x\wedge\omega])\wedge\nu\wedge\nu_1^2\right) = \frac{1}{\epsilon}\big(\omega\wedge x\wedge\nu\wedge\nu_1^2 + \\
&+ \epsilon\, x\wedge\nu^2\wedge\nu_1^2 - (x\wedge\omega)\wedge\nu\wedge\nu_1^2\big) = x\wedge\nu^2\wedge\nu_1^2 = \\
&= \phi([x\wedge\nu\wedge\nu_1])\wedge\phi([\nu\wedge\nu_1]),
\end{aligned}$$

for $[x]\in H^1(M)$. Finally, Proposition 2.5 implies that $\widetilde{CP}^{15}$ is 8–formal.

QED

Acknowledgments

The first author is partially supported by CICYT (Spain) Project BFM2001-3778-C03-02 and UPV 00127.310-E-14813/2002. The second author is supported by CICYT Project BFM2000-0024. Also, both authors are partially supported by The European Contract Human Potential Programme, Research Training Network HPRN-CT-2000-00101.

References

[1] I.K. BABENKO AND I.A. TAIMANOV, On nonformal simply connected symplectic manifolds, *Siberian Math. J.* **41** (2000), 204–217.

[2] P. DELIGNE, P. GRIFFITHS, J. MORGAN AND D. SULLIVAN, Real homotopy theory of Kähler manifolds, *Invent. Math.* **29** (1975), 245–274.

[3] M. FERNÁNDEZ AND V. MUÑOZ, Formality of Donaldson submanifolds, *Math. Zeit.* To appear.

[4] M. FERNÁNDEZ AND V. MUÑOZ, Homotopy properties of symplectic blow-ups, Proc. XII Fall Workshop on Geom. and Physics, Coimbra 2003, *Public. RSME.* To appear.

[5] M. GROMOV, A topological technique for the construction of solutions of differential equations and inequalities, *Actes Congrés Intern. Math. (Nice 1970)*, Gauthier–Villars, Paris, No. 2 (1971), 221–225.

[6] S. Halperin, *Lectures on minimal models*, Mém. Soc. Math. France **230** (1983).

[7] D. McDuff, Examples of symplectic simply connected manifolds with no Kähler structure, *J. Diff. Geom.* **20** (1984), 267–277.

[8] Y. Rudyak and A. Tralle, On Thom spaces, Massey products and nonformal symplectic manifolds, *Internat. Math. Res. Notices* **10** (2000), 495–513.

[9] D. Tanré, *Homotopie rationnelle: Modèles de Chen, Quillen, Sullivan*, Lecture Notes in Math. **1025**, Springer–Verlag (1983).

[10] W.P. Thurston, Some simple examples of symplectic manifolds, *Proc. Amer. Math. Soc.* **55** (1976), 467–468.

[11] D. Tischler, Closed 2–forms and an embedding theorem for symplectic manifolds, *J. Diff. Geom.* **12** (1977), 229–235.

[12] A. Tralle and J. Oprea, *Symplectic manifolds with no Kähler structure*, Lecture Notes in Math. **1661**, Springer–Verlag (1997).

Communications

PROCEEDINGS OF THE II INTERNATIONAL
MEETING ON LORENTZIAN GEOMETRY
MURCIA, NOVEMBER 12–14, 2003
PUBL. DE LA RSME, VOL. 8 (2004), 57–62

Björling Representation for spacelike surfaces with $H = cK$ in $\mathbf{L}^3$

JUAN A. ALEDO[1], JOSÉ A. GÁLVEZ[2] AND PABLO MIRA[3]

[1] *Universidad de Castilla-La Mancha, e-mail: juanangel.aledo@uclm.es*
[2] *Universidad de Granada, e-mail: jagalvez@ugr.es*
[3] *Univiversidad Politécnica de Cartagena, e-mail: pablo.mira@upct.es*

Abstract

In this work we study the Björling problem for linear Weingarten spacelike surfaces of maximal type in the 3-dimensional Lorentz-Minkowski space, i.e. spacelike surfaces whose mean and Gaussian curvature are related by $H = cK$ for some $c \in \mathbf{R}$.

Keywords: Björling problem, linear Weingarten surfaces, maximal surfaces, Lorentz-Minkowski space

2000 Mathematics Subject Classification: 53A10, 53C50

1. Introduction

A linear Weingarten spacelike surface of maximal type (in short, an LWM-spacelike surface) in the 3-dimensional Lorentz-Minkowski space $\mathbf{L}^3$ is a spacelike surface whose mean curvature is proportional to its Gaussian curvature.

In the previous paper [1], the first two authors described a conformal representation for LWM-spacelike surfaces, and used it in order to prove the existence of complete examples and to study its geometric behaviour. The representation actually extends the one for maximal surfaces in $\mathbf{L}^3$, i.e. for the spacelike surfaces with $H = 0$ in $\mathbf{L}^3$, obtained by McNertney and Kobayashi.

In this work we consider an initial value problem for LWM-spacelike surfaces, which consists on the following: given a regular analytic spacelike curve

β in $\mathbf{L}^3$ together with a unit timelike analytic vector field V parallel along β, can one find all LWM-spacelike surfaces in $\mathbf{L}^3$ that span this configuration?

This problem has been motivated by the Björling problem for maximal surfaces in $\mathbf{L}^3$ studied in [2]. Some other research regarding the Björling problem for maximal surfaces can be found in [3, 4]

The goal of the present paper is to characterize when the initial data β, V of the above problem can actually span an LWM-spacelike surface, and to construct in such case the only solution to the Björling problem in terms of the initial data.

2. Preliminaries

Let $\mathbf{L}^3$ be the 3-dimensional *Lorentz-Minkowski space*, that is, the real vector space $\mathbf{R}^3$ endowed with the Lorentzian metric tensor $\langle , \rangle = dx_1^2 + dx_2^2 - dx_3^2$, where (x_1, x_2, x_3) are the canonical coordinates of $\mathbf{R}^3$. An immersion $\psi : M^2 \longrightarrow \mathbf{L}^3$ of a 2-dimensional connected manifold M is said to be a *spacelike surface* if the induced metric via ψ is a Riemannian metric on M, which, as usual, is also denoted by $\langle , \rangle$.

It is well-known that such a surface is orientable. Thus, we can choose a unit timelike normal vector field N globally defined on M. Observe that, up to a symmetry of $\mathbf{L}^3$, we can suppose that the image of N lies on $\mathbf{H}^2_+ = \{x \in \mathbf{H}^2 : x_3 > 0\}$. We shall call N the *unit normal* of ψ. Let us introduce complex coordinates in $\mathbf{H}^2_+$ using the usual stereographic projection $\pi : \mathbf{H}^2_+ \longrightarrow \mathbb{D}$ from the hyperbolic plane $\mathbf{H}^2_+$ onto the unit disk $\mathbb{D}$ given by

$$\pi(x_1, x_2, x_3) = \frac{x_1 - ix_2}{1 + x_3},$$

with inverse map

$$\pi^{-1}(z) = \left(\frac{z + \bar{z}}{1 - |z|^2}, i\, \frac{z - \bar{z}}{1 - |z|^2}, \frac{1 + |z|^2}{1 - |z|^2} \right).$$

We will refer to $g = \pi \circ N$ as the *Gauss map* of the surface.

Let $H = -\text{trace}(A)/2$ and $K = -\det(A)$ denote the mean and Gaussian curvatures of M respectively, where $A : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ stands for the shape operator of M in $\mathbf{L}^3$ associated to N, given by $A = -dN$. Then, we will say that $\psi : M^2 \longrightarrow \mathbf{L}^3$ is a *linear Weingarten spacelike surface of maximal type*, in short, an LWM-spacelike surface, if there exist $c \in \mathbf{R}$ such that $H = cK$. This condition is equivalent to the existence of $a, b \in \mathbf{R}$, $a \neq 0$, satisfying $-2aH + bK = 0$. We will adopt this last notation to follow the one in [1].

As it can be seen in [1], on such a surface the Gaussian curvature is either always negative or always non negative. Moreover, the symmetric tensor σ on M for the immersion ψ

$$\sigma(X,Y) = a\langle X,Y\rangle - b\langle AX,Y\rangle, \qquad X,Y\in\mathfrak{X}(M),$$

is positive definite (reversing orientation if necessary). Hence, we will choose N so that σ is a Riemannian metric.

If we consider M as a Riemann surface with the conformal structure induced by σ, then $g=\pi\circ N$ is a conformal map from M into $\mathbb{D}$, and

$$\triangle^{\sigma}(2a\psi + bN) = 0.$$

These two facts are the basis of a conformal representation for LWM-spacelike surfaces, obtained in [1]:

Theorem 2.1 ([1]) *Let $\psi : M^2 \longrightarrow \mathbf{L}^3$ be an LWM-spacelike surface such that $-2aH + bK = 0$, $a \neq 0$, and let us consider on M the conformal structure induced by σ. Then there exists a function $\phi : M \longrightarrow \mathbf{C}^3$ such that the immersion can be recovered as*

$$\psi = -\frac{b}{2a}\pi^{-1}(g) + \frac{1}{a}\,\mathrm{Re}\int \phi(\zeta)d\zeta. \tag{1}$$

Here $g : M\longrightarrow\mathbb{D}$ is its Gauss map and:

- *If $K \geq 0$ then*

$$\phi = (2a\psi + bN)_z \tag{2}$$

  and both g and ϕ are holomorphic;

- *If $K < 0$ then*

$$\phi = (2a\psi + bN)_{\overline{z}}$$

  and both g and ϕ are anti-holomorphic,

being z a conformal parameter on M and N the unit normal of ψ. We refer the readers to [1] for the details.

3. Björling Representation

Let $\psi : \Omega\subset\mathbf{R}^2\longrightarrow\mathbf{L}^3$ be an LWM-spacelike surface and $z = s + it$ be a local conformal parameter on Ω with respect to σ. First, we will study the case $K \geq 0$.

Since $\langle a\psi_t + bN_t, \psi_s\rangle = \sigma(\psi_t, \psi_s) = 0$, it can be easily obtained that $a\psi_t + bN_t = -aN \times \psi_s$. Using this equality and (2), we get that $\phi + bN_z = a\psi_s + bN_s + iaN \times \psi_s$ and so

$$\phi = \frac{1}{2}\left(a\left(2\psi_s + i(N \times \psi_s - \psi_t)\right) + bN_s\right). \tag{3}$$

On the other hand, as $\langle a\psi_s + bN_s, \psi_t\rangle = \sigma(\psi_s, \psi_t) = 0$ it follows from a straightforward computation that $\psi_t = \psi_s \times N + (b/a)N_s \times N$, which jointly with (3) allow us to obtain

$$\phi = \frac{1}{2}\left(a\left(2\psi_s + i(2N \times \psi_s - \frac{b}{a}N_s \times N)\right) + bN_s\right). \tag{4}$$

Let us define $\beta(s) = \psi(s,0)$, $V(s) = N(s,0)$ on a real interval $I \subset \Omega$. Observe that $\beta(s)$ and $V(s)$ are real analytic functions, because so is ψ. Let us choose any simply connected open set Δ containing I over which we can define holomorphic extensions $\beta(z)$, $V(z)$ of β, V. Then, formula (4) can be written on the curve $\beta(s)$ as

$$\phi(s,0) = \frac{1}{2}\left(a\left(2\beta'(s) + i(2V(s) \times \beta'(s) - \frac{b}{a}V'(s) \times V(s))\right) + bV'(s)\right),$$

and by analytic extension one has

$$\phi(z) = a\left(\beta'(z) + iV(z) \times \beta'(z)\right) + \frac{b}{2}\left(V'(z) + iV(z) \times V'(z)\right).$$

Finally, since

$$N(z) = \pi^{-1}\left(\frac{V_1(z) - iV_2(z)}{1 + V_3(z)}\right)$$

we get from (1) that

$$\begin{aligned}\psi(z) \;&=\; \mathrm{Re}\beta(z) + \frac{b}{2a}\left(\mathrm{Re}V(z) - \pi^{-1}\left(\frac{V_1(z) - iV_2(z)}{1 + V_3(z)}\right)\right) \\ &\quad -\mathrm{Im}\int_{s_o}^{z}\left(V(\omega) \times \beta'(\omega) + \frac{b}{2a}V(\omega) \times V'(\omega)\right)d\omega. \end{aligned} \tag{5}$$

As it can be checked, this formula agrees with the one for maximal surfaces in [2] when we take $a = 1$, $b = 0$.

The case $K < 0$ is analogous, resulting

$$\begin{aligned}\psi(z) \;&=\; \mathrm{Re}\beta(z) + \frac{b}{2a}\left(\mathrm{Re}V(z) - \pi^{-1}\left(\frac{\overline{V_1(z)} - i\overline{V_2(z)}}{1 + \overline{V_3(z)}}\right)\right) \\ &\quad -\mathrm{Im}\int_{s_o}^{z}\left(V(\omega) \times \beta'(\omega) + \frac{b}{2a}V(\omega) \times V'(\omega)\right)d\omega. \end{aligned} \tag{6}$$

The following lemma, whose proof is a simple exercise, shows that the geometry of the surface along the curve $\beta(s)$ can be expressed in terms of $\beta(s)$, $V(s)$.

Lemma 3.1 *Let $\psi : \Omega \subset \mathbf{R}^2 \longrightarrow \mathbf{L}^3$ be an LWM-spacelike surface and let us consider on Ω the conformal structure induced by σ. Let us define $\beta(s) = \psi(s,0)$, $V(s) = N(s,0)$ on a real interval $I \subset \Omega$. Then*

i) $aD(s) + b\langle \beta', V' \rangle \neq 0$ *for all* $s \in I$.

ii) $$K|_{\beta(s)} = \frac{a}{D(s)} \left(\frac{\det(\beta', V, V')^2 + \langle \beta', V' \rangle^2}{aD(s) + b\langle \beta', V' \rangle} \right)$$

iii) $$H|_{\beta(s)} = \frac{b}{2D(s)} \left(\frac{\det(\beta', V, V')^2 + \langle \beta', V' \rangle^2}{aD(s) + b\langle \beta', V' \rangle} \right)$$

where

$$\begin{aligned} D(s)^2 &= \det \langle , \rangle|_{\beta(s)} \\ &= \langle \beta', \beta' \rangle^2 + \frac{b^2}{a^2} \langle \beta' \times V', \beta' \times V' \rangle + \frac{2b}{a} \langle \beta', \beta' \rangle \langle \beta', V' \rangle. \end{aligned}$$

A pair made up of a regular analytic spacelike curve $\beta(s) : I \longrightarrow \mathbf{L}^3$ and an analytic unit vector field $V(s) : I \longrightarrow \mathbf{H}^2_+$ such that $\langle \beta', V \rangle = 0$ will be called a pair of *Björling data*. The previous Lemma shows that one cannot expect in general for prescribed Björling data the existence of an LWM-spacelike surface spanning such configuration.

Taking this into account, we can now formulate the *Björling problem* for LWM-spacelike surfaces in $\mathbf{L}^3$.

> Let $\beta : I \longrightarrow \mathbf{L}^3$ and $V : I \longrightarrow \mathbf{L}^3$ be a pair of Björling data such that for some $a \neq 0$, $b \in \mathbf{R}$ the condition $aD(s) + b\langle \beta', V' \rangle \neq 0$ holds for all $s \in I$. Determine all LWM-spacelike surfaces with $-2aH + bK = 0$ that contain $\beta(s)$, and whose unit normal along $\beta(s)$ is given by $V(s)$.

Any pair of Björling data in the above conditions will be called *admissible.*

Our main result is the following, where we assure the existence and uniqueness of the solution to Björling problem. Observe that the sign of the Gaussian curvature of the solution is given by the pair of curves β, V and the sign of a, as follows from Lemma 3.1.

Theorem 3.2 *Let $\beta(s), V(s)$ be admissible Björling data. There exists a unique solution to Björling problem for LWM-spacelike surfaces in $\mathbf{L}^3$ with the initial data $\beta(s), V(s)$. This unique solution can be constructed in a neighbourhood of the curve as follows:*

- *if $a(aD(s) + b\langle \beta', V'\rangle) > 0$, the map $\psi : \Omega \longrightarrow \mathbf{L}^3$ given by (5) is the only solution to Björling problem, and has non-negative Gaussian curvature;*
- *if $a(aD(s) + b\langle \beta', V'\rangle) < 0$, the map $\psi : \Omega \longrightarrow \mathbf{L}^3$ given by (6) is the only solution to Björling problem, and has negative Gaussian curvature.*

Here $\Omega \subseteq \mathbf{C}$ is a sufficiently small simply connected open set containing I over which β, V admit holomorphic extensions $\beta(z)$, $V(z)$. *Proof:* The uniqueness result follows from the computations used to derive the formulas (5) and (6). The existence is a straightforward computation.

Acknowledgments

The first author is partially supported by Fundación Séneca, Grant No PI-3/00854/FS/01, and Junta de Comunidades de Castilla-La Mancha, Grant No PAI-02-027. The second author is partially supported by DGICYT Grant No BFM2001-3318. The third author is partially supported by Dirección General de Investigación (MCYT), Grant No. BFM2001-2871 and Fundación Séneca (CARM), Grant No. PI-3/00854/FS/01.

References

[1] J.A. Aledo and J.A. Gálvez, A Weierstrass Representation for Linear Weingarten Spacelike Surfaces of Maximal Type in the Lorentz-Minkowski Space, *J. Math. Anal. Appl.* **283** (2003), 25-45.

[2] L.J. Alías, R.M.B. Chaves and P. Mira, Björling problem for maximal surfaces in Lorentz-Minkowski space, *Math. Proc. Cambridge Philos. Soc.* **134** (2003)289-316.

[3] P. Mira, *Construction of maximal surfaces in the Lorentz-Minkowski space*, Geometry, Integrability and Quantization III, I.M. Mladenov and G. Naber (eds), pp. 337–350. Coral Press (Sofia), 2002.

[4] P. Mira and J.A. Pastor, Helicoidal maximal surfaces in Lorentz-Minkowski space, *Monatsh. Math.* **140** (2003), 315–334.

Proceedings of the II International
Meeting on Lorentzian Geometry
Murcia, November 12–14, 2003
Publ. de la RSME, Vol. 8 (2004), 63–68

Old Bolza problem and its new links to General Relativity

Anna Maria Candela

Dipartimento di Matematica, Università degli Studi di Bari,
Via E. Orabona 4, 70125 Bari, Italy

Abstract

Since the beginning of Calculus of Variations the classical Bolza problem has been widely studied not only in an Euclidean space but also in a Riemannian manifold. Recently, it has been completely solved if the Lagrangian is $\mathcal{L}(s, x, v) = \langle v, v\rangle/2 - V(x, s)$, with potential V which grows at most quadratically at infinity with respect to x, and its solutions have been related to geodesics in gravitational waves.

Keywords: Bolza problem, quadratic behavior, gravitational waves, general plane waves.

2000 Mathematics Subject Classification: 58E05, 58E10, 53C50, 83C35.

1. Old Bolza problem

Since the beginning of Calculus of Variations it has been studied the existence of a curve in $\mathbb{R}^N$ with fixed endpoints x_0, $x_1 \in \mathbb{R}^N$ which minimizes a given integral

$$\int_0^T \mathcal{L}(s, x, \dot{x})ds$$

over all suitable paths $x = x(s)$ joining x_0 to x_1 in a time $T > 0$.

Such a problem was studied since the end of the 17th century (for example, in 1686 Newton found the shape of a shell which would minimize the air resistance, while in 1696 Jakob Bernoulli found the curve which would bring a heavy body from a higher point A to a lower one B 'sliding' with zero initial velocity in the least time) but it is named *Bolza problem* as, at the beginning of the last century, Bolza wrote a book on the fixed endpoints problem which became very popular (see [3]). Since then, many people studied Bolza problem and countless references can be cited, but here we just remark that it was solved in full generality by Tonelli, who invented lower semi–continuity for that purpose (see, e.g., [11]), and was reformulated in a Riemannian manifold only in 1962 by Hermann (cf. [8]).

In this paper we want to point out some results on Bolza problem when the Lagrangian $\mathcal{L} : [0,T] \times \mathcal{M}_0 \times T\mathcal{M}_0 \to \mathbb{R}$ is

$$\mathcal{L}(s,x,v) = \frac{1}{2}\,\langle v,v\rangle - V(x,s), \qquad (s,x,v) \in [0,T] \times \mathcal{M}_0 \times T_x\mathcal{M}_0,$$

where $(\mathcal{M}_0, \langle\cdot,\cdot\rangle)$ is a Riemannian manifold with tangent bundle $T\mathcal{M}_0$ and $V : \mathcal{M}_0 \times [0,T] \to \mathbb{R}$ is a given C^1 function. Thus, fixed x_0, $x_1 \in \mathcal{M}_0$, in this case Bolza problem is minimizing the functional

$$J_T(x) = \frac{1}{2}\int_0^T \langle \dot{x}, \dot{x}\rangle ds - \int_0^T V(x,s)ds \tag{1.1}$$

on a suitable space of curves joining x_0 to x_1 in time T, more precisely

$$\Omega^T(x_0,x_1) = \{x \in H^1([0,T],\mathcal{M}_0) : x(0) = x_0,\ x(T) = x_1\}.$$

Standard arguments allow one to prove that J_T is a C^1 functional and its critical points are solutions of the corresponding Euler equation

$$\begin{cases} D_s\dot{x} + \nabla_x V(x,s) = 0 \\ x(0) = x_0,\ x(T) = x_1 \end{cases} \tag{1.2}$$

(here, $D_s\dot{x}$ is the covariant derivative of the tangent field $\dot{x}$ along x induced by the Levi–Civita connection of $\langle\cdot,\cdot\rangle$, while $\nabla_x V(x,s)$ is the partial derivative of V with respect to x).

In order to prove the existence of a minimum point for functional J_T in $\Omega^T(x_0,x_1)$, we recall the following abstract minimization theorem.

Theorem 1.1 *Let Ω be a complete Riemannian manifold and $J : \Omega \to \mathbb{R}$ a C^1 functional which satisfies Palais–Smale condition, i.e., if $(x_n)_n \subset \Omega$ is such that $(J(x_n))_n$ is bounded and $J'(x_n) \to 0$ then it converges in Ω up to subsequences. So, if J is bounded from below, it attains its infimum.*

With no special assumption on the growth of potential V, some useful propositions can be stated.

Proposition 1.2 *If* $(\mathcal{M}_0, \langle\cdot,\cdot\rangle)$ *is a* C^3 *complete Riemannian manifold then*

- $\mathcal{M}_0$ *is a submanifold of an Euclidean space* $\mathbb{R}^N$ *and* $\langle\cdot,\cdot\rangle$ *is the restriction to* $\mathcal{M}_0$ *of the Euclidean metric of* $\mathbb{R}^N$*;*
- $\Omega^T(x_0, x_1)$ *is a complete Riemannian submanifold of* $H^1([0,T], \mathbb{R}^N)$*.*

(For the proof, see [9, 10]).

Proposition 1.3 *Let* $(\mathcal{M}_0, \langle\cdot,\cdot\rangle)$ *be a* C^3 *complete Riemannian manifold and let potential* $V = V(x,s)$ *be* C^1 *on* $\mathcal{M}_0 \times [0,T]$*. If* $(x_n)_n$ *is a bounded sequence in* $\Omega^T(x_0,x_1)$ *such that* $J'_T(x_n) \to 0$ *then it converges in* $\Omega^T(x_0,x_1)$ *up to subsequences.*

(The proof follows from [2, Lemma 2.1] reasoning as in [4, Lemma 3.5]).

Remark 1.4 Obviously, if functional J_T is coercive in $\Omega^T(x_0,x_1)$, i.e., two constants a_1, $a_2 > 0$ exist such that

$$J_T(x) \;\geq\; a_1 \int_0^T \langle \dot{x}, \dot{x}\rangle ds \;-\; a_2 \qquad \text{for all } x \in \Omega^T(x_0,x_1),$$

then it is bounded from below and a sequence $(x_n)_n$ has to be bounded if $(J_T(x_n))_n$ is bounded.

Thus, by Propositions 1.2, 1.3 and Remark 1.4, previous abstract Theorem 1.1 can apply to functional J_T once its coerciveness on $\Omega^T(x_0,x_1)$ has been proved.

Clearly, it is quite simple to prove it if potential $V = V(x,s)$ grows subquadratically at infinity with respect to x, while it cannot hold if it grows more than quadratically.

Hence, the "critical" growth is the quadratic one, i.e.,

$$V(x,s) \leq \lambda d^2(x,\bar{x}) + k \quad \text{for all } (x,s) \in \mathcal{M} \times [0,T], \tag{1.3}$$

for some $\bar{x} \in \mathcal{M}$ and $\lambda > 0$, $k \geq 0$.

In assumption (1.3) the existence of at least a solution of Euler equation (1.2) has been proved by Clarke and Ekeland in [7] if $\mathcal{M}_0 = \mathbb{R}^N$ and, up to further hypotheses, the "arrival time" T is smaller than $1/\sqrt{\lambda}$.

On the other hand, if $T = \pi/\sqrt{2\lambda}$ the simple example of harmonic oscillator may not have a solution (see [4, Example 3.6]).

So, there is a gap which needs to be overcome if $1/\sqrt{\lambda} \leq T < \pi/\sqrt{2\lambda}$ and it has been done in [4, Theorem 1.1]. In fact, the following lemma holds (it is essentially proved in [4, Lemma 3.4] but in this form it has been stated in [1]).

Lemma 1.5 *Let $(\mathcal{M}, \langle\cdot,\cdot\rangle)$ be a C^3 complete Riemannian manifold and V a C^1 potential such that (1.3) holds. Then, if $T < \pi/\sqrt{2\lambda}$, functional J_T is coercive on $\Omega^T(x_0, x_1)$.*

Whence, Theorem 1.1 implies:

Theorem 1.6 *Let $(\mathcal{M}, \langle\cdot,\cdot\rangle)$ be a C^3 complete Riemannian manifold. Let $V \in C^1(\mathcal{M}\times[0,T], \mathbb{R})$ be such that (1.3) holds. Then if $T < \pi/\sqrt{2\lambda}$, functional J_T is bounded from below and attains its infimum in $\Omega^T(x_0, x_1)$.*

2. New links to General Relativity

In General Relativity widely studied spacetimes are exact gravitational waves, i.e., Lorentzian manifolds $(\mathbb{R}^4, ds^2)$ endowed with the metric

$$ds^2 = dx^2 + 2\ du\ dv + A(u)x\cdot x\ du^2, \qquad \text{with } A(u) = \begin{pmatrix} f(u) & g(u) \\ g(u) & -f(u) \end{pmatrix},$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, dx^2 is the Euclidean metric in $\mathbb{R}^2$ and $f, g \in C^2(\mathbb{R}, \mathbb{R})$ such that $f^2 + g^2 \not\equiv 0$ (for more details, see [5] and references therein).

A generalization of such models can be introduced as follows.

Definition 2.1 *A semi–Riemannian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle_z)$ is a general plane wave, briefly GPW, if there exists a connected finite dimensional Riemannian manifold $(\mathcal{M}_0, \langle\cdot,\cdot\rangle)$ such that $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$ and*

$$\langle\cdot,\cdot\rangle_z = \langle\cdot,\cdot\rangle + 2\ du\ dv + H(x,u)\ du^2,$$

where $x \in \mathcal{M}_0$, the variables (v, u) are the natural coordinates of $\mathbb{R}^2$ and the smooth scalar field $H : \mathcal{M}_0 \times \mathbb{R} \to \mathbb{R}$ is such that $H \not\equiv 0$.

In order to investigate the geodesic connectedness on a GPW we are able to introduce a suitable variational principle which links geodesics joining two given points in a GPW to the critical points of a functional which looks like J_T defined in (1.1) (see [5, Proposition 3.1] or [6, Proposition 2.2] for a variational proof).

Proposition 2.2 *Let $z^* : [0,1] \to \mathcal{M}$, $z^* = (x^*, v^*, u^*)$, be a curve on a GPW $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$ with constant energy $\langle \dot z^*, \dot z^*\rangle_z \equiv E_{z^*}$. Fixed $z_0 = (x_0, v_0, u_0)$ and $z_1 = (x_1, v_1, u_1) \in \mathcal{M}$ then z^* is a geodesic on $\mathcal{M}$ joining z_0 to z_1 if and* *only if* $u^* = u^*(s)$ *is affine, i.e.,* $u^*(s) = u_0 + s(u_1 - u_0)$ *for all* $s \in [0,1]$, $x^* = x^*(s)$ *is a critical point of*

$$J_*(x) = \frac{1}{2} \int_0^1 \langle \dot{x}, \dot{x} \rangle ds - \int_0^1 V_*(x,s) ds \qquad on\ \Omega^1(x_0, x_1),$$

where

$$V_*(x,s) = -\frac{(u_1 - u_0)^2}{2} H(x, u_0 + s(u_1 - u_0)), \tag{2.1}$$

while if $u_0 = u_1$ *it is* $v^*(s) = v_0 + s(v_1 - v_0)$ *for all* $s \in [0,1]$, *otherwise for all* $s \in [0,1]$ *it is*

$$v^*(s) = v_0 + \frac{1}{2(u_1 - u_0)} \int_0^s (E_{z^*} - \langle \dot{x}^*(\sigma), \dot{x}^*(\sigma) \rangle + 2V_*(x^*(\sigma), \sigma))\, d\sigma.$$

Thus, Proposition 2.2 and arguments in Section 1 allow one to prove the following result.

Theorem 2.3 *Let* $(\mathcal{M}, \langle \cdot, \cdot \rangle_z)$, $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$, *be a GPW and fix* $u_0, u_1 \in \mathbb{R}^2$, *with* $u_0 \leq u_1$. *Suppose that* $(\mathcal{M}_0, \langle \cdot, \cdot \rangle)$ *is a* C^3 *complete Riemannian manifold and there exist* $\bar{x} \in \mathcal{M}_0$ *and* R_0, $R_1 \geq 0$ *such that for all* $(x,u) \in \mathcal{M}_0 \times [u_0, u_1]$ *it is*

$$H(x,u) \geq -(R_0\ d^2(x, \bar{x}) + R_1).$$

If $R_0(u_1 - u_0)^2 < \pi^2$, *then for all* x_0, $x_1 \in \mathcal{M}_0$ *and* v_0, $v_1 \in \mathbb{R}$ *there exists at least a geodesic joining* $z_0 = (x_0, v_0, u_0)$ *to* $z_1 = (x_1, v_1, u_1)$ *in* $\mathcal{M}$.

Let us point out that in the physical model of an exact gravitational wave, potential V_* in (2.1) becomes

$$\tilde{V}_*(x,s) = -\frac{(u_1 - u_0)^2}{2} A(u_0 + s(u_1 - u_0)) x \cdot x$$

which has quadratic growth with respect to x, of course.

Hence, Theorem 2.3 in this special setting gives the following result (for more details, see [5, Subsection 4.3]).

Proposition 2.4 *Let* $(\mathbb{R}^4, ds^2)$ *be an exact gravitational plane wave, fix* u_0, $u_1 \in \mathbb{R}$ *and assume*

$$R_0[u_0, u_1] := \max\{(f^2 + g^2)^{1/2}(u) : u \in [u_0, u_1] \cup [u_1, u_0]\}.$$

If it is

$$R_0[u_0, u_1]\ (u_1 - u_0)^2 < \pi^2,$$

then for all x_0, $x_1 \in \mathcal{M}_0$ *and* v_0, $v_1 \in \mathbb{R}$ *the corresponding two points* $z_0 = (x_0, v_0, u_0)$ *and* $z_1 = (x_1, v_1, u_1)$ *are geodesically connected in* $\mathcal{M}$.

Acknowledgments

This work has been supported by M.I.U.R. (research funds ex 40% and 60%).

References

[1] R. Bartolo, A.M. Candela, Quadratic Bolza problems in static space-times with critical asymptotic behavior. *In progress.*

[2] V. Benci, D. Fortunato, On the existence of infinitely many geodesics on space–time manifolds, *Adv. Math.* **105** (1994), 1-25.

[3] O. Bolza, *Lectures on the Calculus of Variations*, University of Chicago Press, Chicago (1904).

[4] A.M. Candela, J.L. Flores, M. Sánchez, A quadratic Bolza–type problem in a Riemannian manifold, *J. Differential Equations* **193** (2003), 196-211.

[5] A.M. Candela, J.L. Flores, M. Sánchez, On general Plane Fronted Waves. Geodesics, *Gen. Relativity Gravitation* **35** (2003), 631-649.

[6] A.M. Candela, J.L. Flores, M. Sánchez, Geodesic connectedness in Plane Fronted Waves. A variational approach. *Preprint.*

[7] F.H. Clarke, I. Ekeland, Nonlinear oscillations and boundary value problems for Hamiltonian systems, *Arch. Rational Mech. Anal.* **78** (1982), 315-333.

[8] R. Hermann, Some differential geometric aspects of the Lagrange variational problem, *Illinois J. Math.* **6** (1962), 634-673.

[9] J. Nash, The imbedding problem for Riemannian manifolds, *Ann. of Math.* **63** (1956), 20-63.

[10] R.S. Palais, Morse Theory on Hilbert manifolds, *Topology* **2** (1963), 299-340.

[11] L. Tonelli, *Fondamenti di Calcolo delle Variazioni*, Vol. I – II, Zanichelli, Bologna (1921–1923).

Proceedings of the II International
Meeting on Lorentzian Geometry
Murcia, November 12–14, 2003
Publ. de la RSME, Vol. 8 (2004), 69–75

Null geodesics for Kaluza-Klein metrics and worldlines of charged particles

Erasmo Caponio

Dipartimento di Matematica
Politecnico di Bari

Abstract

World-lines of classical particles, moving in an electromagnetic field can be obtained as projections, on the space-time M, of null geodesics in the one higher dimensional manifold $M \times \mathbb{R}$, endowed with a Kaluza-Klein metric. We use this fact to prove the existence of a world-line, connecting two chronologically related events on a globally hyperbolic space-time, for any particle whose charge-to-mass ratio is in a suitable neighborhood of 0 in $\mathbb{R}$.

Keywords: Lorentz force equation, charge-to-mass ratio, Kaluza-Klein metric, null geodesics.

2000 Mathematics Subject Classification: 53C50; 83C10; 83C50; 83E15.

1. Introduction

Let (M, g) be a space-time and p_0, p_1 two causally related points on M. Let $C(p_0, p_1)$ be the set of continuous one-parameterized causal curves from p_0 to p_1 and $C'(p_0, p_1) \subset C(p_0, p_1)$ be the subset of piecewise C^1 timelike curves. It is well known that the *Lorentzian length functional*

$$L\colon z \in C'(p_0, p_1) \longmapsto \int_z \sqrt{-g[\dot{z}, \dot{z}]}\mathrm{d}s$$

it is upper semicontinuous on $C'(p_0, p_1)$ endowed with the C^0 topology (cf. [2]). L can be extended to $C(p_0, p_1)$ remaining upper semicontinuous and, whenever M is globally hyperbolic, the following result holds:

Theorem (Avez and Seifert) *Let (M, g) be a globally hyperbolic Lorentzian manifold, $p_0 \in M$ and $J^+(p_0) \subset M$ the subset of points which can be reached from p_0 by future-pointing causal curves. If $p_1 \in J^+(p_0)$, there is a causal future-pointing geodesic connecting p_0 to p_1 and maximizing L.*

Proof. If there is no timelike future pointing curve on $C(p_0, p_1)$ then there is a future pointing null geodesic connecting p_0 and p_1. In either case, there exists a timelike curve $z \in C(p_0, p_1)$ and $L(z) > 0$. Now in a globally hyperbolic space-time $C(p_0, p_1)$ is compact, so being L upper semicontinuous on $C(p_0, p_1)$, L has a maximum point γ such that $L(\gamma) > 0$. γ is a timelike geodesic because timelike geodesics locally maximize length. □

We would like to extend the above result to the action functional of a classical charged particle moving in the gravitational field g and in an electromagnetic field F:

$$I\colon z \in C'(p_0, p_1) \longmapsto \int_z \sqrt{-g[\dot{z}, \dot{z}]}\mathrm{d}s + \tfrac{q}{m}\int_z \omega,$$

where q is the electric charge of the particle, m its rest mass and ω a 1-form on M such that $\mathrm{d}\omega = F$ (we have set the speed of light $c = 1$).

If z is a C^1 timelike curve and a critical point of I then, after reparameterizing with respect to *proper time* τ, (i.e., $g(z(\tau))[\dot{z}(\tau), \dot{z}(\tau)] = -1$), we get a timelike solution of the *Lorentz force equation*

$$\frac{\mathrm{D}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right) = \frac{q}{m}\hat{F}(z)\left[\frac{\mathrm{d}z}{\mathrm{d}\tau}\right], \tag{1}$$

i.e., a world-line for a charged particle moving in the field F and connecting the points p_0 and p_1. In Eq. (1), $\frac{\mathrm{D}}{\mathrm{d}\tau}$ is the covariant derivative along z induced by the Levi-Civita connection and the map $\hat{F}\colon TM \to TM$ is defined as

$$g(p)[v, \hat{F}(p)[w]] = F(p)[v, w],$$
$$\forall (p, v), (p, w) \in TM.$$

Eq. (1) is the relativistic version, on a curved space-time, of the classical equation of motion of a charged particle in an electric field $\vec{E}$ and a magnetic field $\vec{B}$

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{q}{m}(\vec{E} + \vec{v} \wedge \vec{B}).$$

For $q = 0$ or $F = 0$, Eq. (1) reduces to the geodesic equation.

It can be proved that timelike solutions of (1) locally maximize I (cf. also [4]). On the other hand, it is not clear for us if the functional I is upper semicontinuous. Moreover we are not able to avoid the case that the maximum of I is achieved by a null curve in $C(p_0, p_1)$, actually a future-pointing lightlike geodesic, without conjugate points, which is not in the closure of $C'(p_0, p_1)$ in $C(p_0, p_1)$. For these reasons, we are not able to prove directly a result *a la* Avez and Seifert for Eq. (1). In the next section we will show how such a result can be obtained by using Kaluza-Klein theory.

2. The Kaluza-Klein metric and the main result

Consider the trivial bundle $W = M \times \mathbb{R}$, $\pi\colon W \to M$ endowed with the metric

$$g_{\mathrm{KK}} = \pi^* g + \rho^2(\pi^*_{I\!R}\mathrm{d}y + \pi^*\omega) \otimes (\pi^*_{I\!R}\mathrm{d}y + \pi^*\omega),$$

where ρ is a positive constant, y the coordinate in $I\!R$ and $\pi_{I\!R}$ is the canonical projection on $I\!R$.

The metric g_{KK} on W is Lorentzian. Moreover if V is a timelike vector field on M, then $Y = (V, -\omega[V])$ gives a time orientation to (W, g_{KK}).

It is well known that the projection on M of any timelike future-pointing geodesic with respect to g_{KK} corresponds to a trajectory of a charged particle moving in the field $F = \mathrm{d}\omega$ (see for instance [3]). Namely, assume that $w(s) = (z(s), y(s))$ is a geodesic on (W, g_{KK}), then w satisfies the system:

$$\begin{cases} \frac{\mathrm{D}}{\mathrm{d}s}\dot{z} = \rho^2(\dot{y} + \omega(z)[\dot{z}])\hat{F}(z)[\dot{z}] \\ \frac{\mathrm{d}}{\mathrm{d}s}\big(\rho^2(\dot{y} + \omega(z)[\dot{z}])\big) = 0 \end{cases} \tag{1}$$

From the second equation in (1), we see that the constant of motion $q_w\colon = \rho^2(\dot{y} + \omega(\dot{z}))$ plays the role of the electric charge of the particle moving along the trajectory $z = z(s)$.

Clearly if $w = (z, y)$ is a timelike future-pointing curve on W then

$$\begin{aligned} &0 > g_{\mathrm{KK}}(w)[\dot{w}, \dot{w}] = g(z)[\dot{z}, \dot{z}] + \rho^2(\dot{y} + \omega(z)[\dot{z}])^2 \geq g(z)[\dot{z}, \dot{z}], \\ &0 > g_{\mathrm{KK}}(w)[\dot{w}, Y(w)] = g(z)[\dot{z}, V(z)], \end{aligned}$$

that is also z is timelike and future-pointing. Contracting both hand-sides of the first equation in (1) by $g(z)[\cdot, \dot{z}]$ and using anti-symmetry of F, we see that $g(z)[\dot{z}, \dot{z}]$ is constant along z. We can parameterize z with respect to proper time: the reparameterized curve $z = z(\tau)$ satisfies the equation

$$\frac{\mathrm{D}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right) = \frac{q_w}{m_z}\hat{F}(z)\left[\frac{\mathrm{d}z}{\mathrm{d}\tau}\right], \tag{2}$$

where $m_z = \sqrt{-g[\dot{z}, \dot{z}]}$. Assume that z is causal (with respect to g). If $w = (z, y)$ is a geodesic for g_{KK}, then also $g_{\mathrm{KK}}(w)[\dot{w}, \dot{w}]$ is conserved and

$$g_{\mathrm{KK}}(w)[\dot{w}, \dot{w}] = -m_z^2 + \frac{q_w^2}{\rho^2}.$$

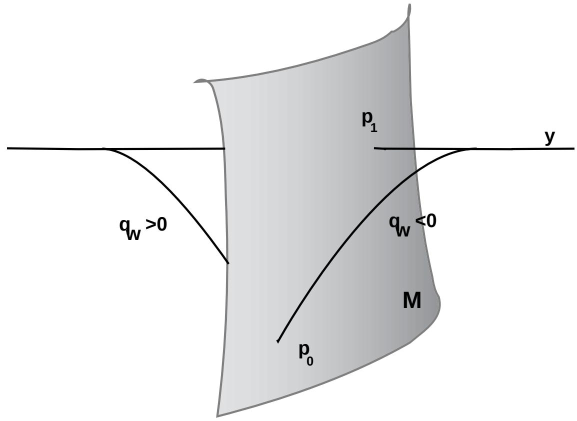


Figure 1: Future-pointing null geodesics with non vanishing charges

Thus the geodesic w on W is null iff

$$m_z^2 = \frac{q_w^2}{\rho^2}.$$

In such a case z is timelike iff $q_w \neq 0$. Therefore we can state that a future-pointing null geodesic w for the Kaluza-Klein metric, starting from a point $w_0 = (p_0, y_0)$, arriving to a point $w_1 = (p_1, y_1)$ and having constant $q_w \neq 0$, provides a future-pointing timelike solution to (1), connecting p_0 and p_1 and having charge-to-mass ratio

$$\frac{q}{m} = \frac{q_w}{m_z} = \pm\rho,$$

with the plus sign if $q_w > 0$ and the minus sign if $q_w < 0$.

In other words, we can fix the charge-to-mass ratio according to the value of the parameter ρ. Our goal then becomes to prove the existence of a future pointing null geodesic w on (W, g_{KK}) having "charge" $q_w \neq 0$. Actually, we will prove that, if $\left|\frac{q}{m}\right|$ is in a suitable neighborhood of $0 \in \mathbb{R}$, there exist at least two future-pointing null geodesics w_1 and w_2 having charges $q_{w_1} > 0$ and $q_{w_2} < 0$ (see Fig. 1)

Theorem 2.1 *Let (M, g) be a time-oriented Lorentzian manifold. Let F be an electromagnetic field on M and ω be a one-form on M such that $F = \mathrm{d}\omega$. Assume that (M, g) is a globally hyperbolic manifold and let p_0 and p_1 be two points on M with p_1 in the chronological future of p_0. Then there exists* *a positive constant $R = R(g, F, p_0, p_1)$, such that (1) has a future-pointing timelike solution connecting p_0 and p_1, for any charge-to-mass ratio such that $\left|\frac{q}{m}\right| < R$.*

Our proof is based on some properties of globally hyperbolic space-times. Namely we can state that:

Proposition 2.2 *If (M, g) is globally hyperbolic, then the manifold $W = M \times \mathbb{R}$ endowed with the metric g_{KK} is globally hyperbolic as well.*

Proof. Show that the projection on M of any inextendible future-pointing causal curve in W is inextendible. Then, if S is a Cauchy surface for M, $\widetilde{S} = S \times I\!R$ is a Cauchy surface for W. Indeed $w(s)$ meets $\widetilde{S}$ as many times as $z(s)$ meets S, and in correspondence of the same value of the parameter. As S is a Cauchy surface for M, $z(s)$ meets S exactly once and $w(s)$ meets $\widetilde{S}$ exactly once. □

We can express R in terms of g, F, p_0 and p_1: let $\mathcal{T}_{p_0,p_1}$ and $\mathcal{N}_{p_0,p_1}$ be the sets, respectively, of all the C^1, future-pointing timelike curves and of all the C^1, future-pointing causal curves connecting p_0 and p_1; then (cf. [1])

$$R = \sup_{z \in \mathcal{T}_{p_0,p_1}} \Big(\frac{\int_z \sqrt{-g[\dot{z}, \dot{z}]} \mathrm{d}s}{\sup_{x \in \mathcal{N}_{p_0,p_1}} \left|\int_x \omega - \int_z \omega\right|} \Big).$$

Notice that R is *gauge invariant*, that is, it is invariant under the replacement $\omega \mapsto \omega + \mathrm{d}f$, where f is any C^2 function on M.

From a physical point of view, the above formula shows that, for sufficiently weak fields F, there exists at least one connecting future-pointing timelike solution to (1) for any charge-to-mass ratio. In fact, under the replacement $\omega \to k\omega$, R scales as $R \to \frac{R}{k}$. Moreover the electron is the free particle with the maximum value of the charge-to-mass ratio, and for sufficiently small k, $\frac{e}{m_e} < R$

We have to prove that $R > 0$. To this end, let γ be a future-pointing timelike geodesic connecting p_0 and p_1 whose length is

$$L = \sup_{z \in \mathcal{N}_{p_0,p_1}} \int_z \sqrt{-g[\dot{z}, \dot{z}]} \mathrm{d}s.$$

For the Avez-Seifert theorem, such a geodesic exists. Choose a gauge such that $\int_\gamma \omega = 0$, and define

$$N = \sup_{z \in \mathcal{N}_{p_0,p_1}} \left| \int_z \omega \right|.$$

It can be proved, by using strong causality and compactness of $C(p_0, p_1)$ that N is finite. Therefore, recalling the definition of R, we get $R \geq \frac{L}{N} > 0$.

Finally we give a sketch of the proof of Theorem 1.

Proof of Theorem 1. Choose $\rho < R$. By definition of R, there exists a timelike curve σ connecting p_0 and p_1 such that

$$\sup_{x \in \mathcal{N}_{p_0,p_1}} \left| \int_x \omega - \int_\sigma \omega \right| < \frac{\int_\sigma \sqrt{-g[\dot\sigma, \dot\sigma]}}{\rho}. \tag{3}$$

Consider its horizontal lift σ^*, with initial point $w_0 = (p_0, y_0)$. As σ^* is timelike, its final point $\widetilde{w}_1 = (p_1, \widetilde{y}_1) = (p_1, y_0 - \int_\sigma \omega)$ belongs to the chronological future of w_0. Let U be the open subset of $\mathbb{R}$ containing all the values y_1 such that $w_1 = (p_1, y_1)$ is in the chronological future of w_0. Moreover let V be the connected component of U containing $\widetilde{y}_1$. Assume that V is given by $]\overline{y}_1, \hat{y}_1[$.

It can be proved that $\overline{y}_1 > -\infty$ and $\hat{y}_1 < +\infty$. Then consider the points $\overline{w} = (p_1, \overline{y}_1)$ and $\hat{w} = (p_1, \hat{y}_1)$. They belong to the boundary of the chronological future of w_0. It is well known that in a globally hyperbolic space-time M, for any $p \in M$, the boundary of the chronological future of p is made up of points which can be connected to p by null geodesics. Thus there exist two null geodesics $\overline{\eta} = (\overline{z}, \overline{y})$ and $\hat{\eta} = (\hat{z}, \hat{y})$ connecting w_0 to $\overline{w}$ and $\hat{w}$, respectively. Now, it is easy to see that

$$\left| \overline{y}_1 - y_0 + \int_{\overline{z}} \omega \right| \geq \frac{\int_{\overline{z}} \sqrt{-g[\dot{\overline{z}}, \dot{\overline{z}}]}}{\rho}$$

and analogously for $\hat{z}$ e $\hat{y}_1$ (otherwise $\overline{w}$ and $\hat{w}$ would belong to the chronological future of w_0). Then, from (3), we get

$$\sup_{x \in \mathcal{N}_{p_0,p_1}} \left| \int_x \omega - \int_\sigma \omega \right| < \left| \overline{y}_1 - y_0 + \int_\sigma \omega \right|,$$

and analogously for $\hat{y}_1$. In particular

$$\left| \int_{\overline{z}} \omega - \int_\sigma \omega \right| < \left| \overline{y}_1 - y_0 + \int_\sigma \omega \right| \tag{4}$$

and analogously for $\hat{z}$ and $\hat{y}_1$. Recalling that $\widetilde{y}_1 = y_0 - \int_\sigma \omega$ and $\hat{y}_1 > \widetilde{y}_1$ and $\overline{y}_1 < \widetilde{y}_1$, from (4) we get

$$q_{\overline{\eta}} = \rho^2 (\overline{y}_1 - y_0 + \int_{\overline{z}} \omega) < 0$$

and analogously for $q_{\hat{\eta}}$

$$q_{\hat{\eta}} = \rho^2(\hat{y}_1 - y_0 + \int_{\hat{z}} \omega) > 0.$$

Therefore we have obtained two timelike future pointing connecting solutions of (1) having charge-to-mass ratios $\frac{q}{m} = -\rho$ and $\frac{q}{m} = +\rho$. Since $\rho < R$ is arbitrary, we get the thesis. □

References

[1] E. Caponio and E. Minguzzi, Solutions to the Lorentz force equation with fixed charge-to-mass ratio in globally hyperbolic spacetimes, *J. Geometry Physics* **49** (2004), 176.

[2] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge Univesity Press, London, (1993).

[3] A. Lichnerowicz, *Thèorie Relativistes de la Gravitation et de l'Électromagnetisme*, Masson Ed., Paris, (1955).

[4] E. Minguzzi, On the existence of maximizing curves for the charged particle action, *Class. Quantum Grav.* **20** (2003), 4169.

PROCEEDINGS OF THE II INTERNATIONAL
MEETING ON LORENTZIAN GEOMETRY
MURCIA, NOVEMBER 12–14, 2003
PUBL. DE LA RSME, VOL. 8 (2004), 76–82

Complete embedded maximal surfaces with isolated singularities in $\mathbb{L}^3$

ISABEL FERNÁNDEZ[1], FRANCISCO J. LÓPEZ[1] AND RABAH SOUAM[2]

[1]*Departamento de Geometría y Topología. Universidad de Granada. e-mail: (first author) isafer@ugr.es (second author) fjlopez@ugr.es*
[2]*Institut de Mathématiques de Jussieu-CNRS UMR 7586. Université Paris 7. e-mail: souam@math.jussieu.fr*

Abstract

We study maximal surfaces in $\mathbb{L}^3$ with isolated (conelike type) singularities, and endow the space of complete maximal surfaces with a fixed number of singularities with a natural structure of real analytical manifold in terms of the position of the singularities and the asymptotic behaviour. The underlying topology is the one of the uniform convergence on compact subsets.

Keywords: spacelike immersions, maximal surfaces, conelike singularities.
2000 Mathematics Subject Classification: 53C50; 58D10, 53C42

1. Introduction

By definition, the Lorentz-Minkowski space $\mathbb{L}^3$ is the space $\mathbb{R}^3$ endowed with the indefinite metric $ds^2 = dx^2 + dy^2 - dz^2$.

Let M be a differentiable surface. An immersion $X : M \to \mathbb{L}^3$ is said to be *spacelike* provided the induced metric on M is a Riemannian metric. Such an immersion is said to be *maximal* provided X is spacelike and its mean curvature vanishes.

Maximal surfaces represent local maxima for the area functional associated to variations of the surface by spacelike surfaces. Furthermore, maximal surfaces can be written locally as the graph of a function u defined on a domain of $\mathbb{R}^2$ and verifing the following elliptic differential equation:

$$(1-u_x^2)u_{yy} + 2u_x u_y u_{xy} + (1-u_y^2)u_{xx} = 0 \quad \text{provided} \quad u_x^2 + u_y^2 < 1$$

As in the case of minimal surfaces in $\mathbb{R}^3$, maximal surfaces admit a Weierstrass type representation in terms of meromorphic data on a Riemann surface, which represents a powerful tool to describe the geometry of the surface (see [3]). Indeed, let M be a differentiable surface and $X : M \to \mathbb{L}^3$ a maximal immersion. Consider the conformal structure on M associated to the Riemannian metric induced by the immersion. Then, there exist a meromorphic function g and a holomorphic 1-form ϕ_3 such that, if we define

$$\phi_1 = \frac{i}{2}\left(\frac{1}{g} - g\right)\phi_3 \qquad \text{and} \qquad \phi_2 = \frac{-1}{2}\left(\frac{1}{g} + g\right)\phi_3,$$

we obtain three holomorphic 1-forms ϕ_1, ϕ_2 and ϕ_3 on M having no common zeroes. Moreover, up to a translation, the immersion X can be recovered as the real part of the integral of these three 1-forms, $X = \mathrm{Re}\int(\Phi_1, \Phi_2, \Phi_3)$.

Either (g, ϕ_3) or (ϕ_1, ϕ_2, ϕ_3) are called the *Weierstrass data* for the immersion X.

In fact, the map g coincides with the hyperbolic stereographic projection of the Gauss map of the immersion to $\overline{\mathbb{C}} - \{|w| = 1\}$. The induced metric on M is given by the expression

$$\mathrm{ds}^2 = |\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 = \frac{1}{4}\left(\frac{1}{|g|} - |g|\right)^2 |\phi_3|^2$$

The first global result in the theory of maximal surfaces was stated by Calabi in [1], who showed that *spacelike planes are the unique complete embedded maximal surfaces.*

However, if we allow the existence of a certain kind of singularities, there exist non-trivial examples, as it is shown in the following pictures.

Figure 2: Classical maximal surfaces with isolated singularities: the Lorentzian catenoid and a Riemann type example.

2. Embedded singularities in a maximal surface: local and global behaviour

Throughout this paper, we study (complete) embedded maximal surfaces with isolated singularities in the following sense:

Definition 2.1 *Let M be a differentiable surface and $X : M \to \mathbb{L}^3$ a topological embedding. We say that X is a maximal embedding with an isolated set of singularities $F_0 \subset M$ if*

- *$X : M - F_0 \to \mathbb{L}^3$ is a maximal embedding,*
- *X cannot be extended to a maximal immersion at any point in F_0.*

We also say that $S = X(M)$ is a maximal embedded surface with singularities at $F = X(F_0)$.

The local behaviour of the surface around a singularity is described in this proposition.

Proposition 2.2 *Let $X : D \to \mathbb{L}^3$ be a maximal embedding from an open disk D with singularity at a point $q \in D$. Then,*

1. *$D - \{q\}$, endowed with the conformal structure associated to the metric induced by X, is biholomorphic to an annulus $A = \{z \in \mathbb{C}\,; r < |z| < 1\}$ and $X : A \to \mathbb{L}^3$ extends analytically to $\gamma = \{|z| = 1\}$ with $X(\gamma) = X(q)$,*
2. *the Weierstrass data of $X : A \to \mathbb{L}^3$, (g, ϕ_3), extend analitically to the mirror of A, $A^* = \{z \in \mathbb{C}\,; 1 < |z| < \frac{1}{r}\}$, satisfying the symmetries $g \circ J = 1/\overline{g}$ and $J^*(\phi_3) = -\overline{\phi_3}$, where $J(z) := 1/\overline{z}$ is the mirror involution,*
3. *ϕ_3 never vanishes on γ, $|g| = 1$ on γ and $g : \gamma \to \mathbb{S}^1$ is injective,*
4. *$X(D)$ is asymptotic to the light cone of $X(q)$. In particular, locally around the singularity the surface is a graph over any spacelike plane.*

Remark 2.3 *The previous proposition is actually a necessary and sufficient condition to construct maximal disks with one singularity. Indeed, take a conformal annulus $A = \{z \in \mathbb{C}\,;\, r < |z| < 1\}$, a meromorphic map g and a holomorphic 1-form ϕ_3 on A satisfying the conditions* 2. *and* 3. *in the proposition and such that ϕ_1, ϕ_2 and ϕ_3 are holomorphic 1-forms having no common zeroes. Then, the map $X = Re \int(\phi_1, \phi_2, \phi_3) : A \cup \{|z| = 1\} \to \mathbb{L}^3$ gives a maximal surface with an embedded singularity at the point $p = X(\{z \in \mathbb{C} \,:\, |z| = 1\})$ (observe that the symmetry of (g, ϕ_3) guarantees that the period problem is solved).*

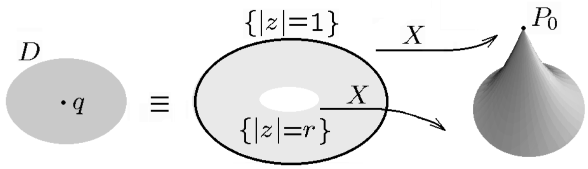


Figure 3: A conelike singularity.

Let $X : M \to \mathbb{L}^3$ be a maximal embedding with (isolated) singularities, then M is a metric space. We say that a maximal surface $X(M)$ is complete if M is a complete metric space.

Proposition 2.4 *Let S be a complete embedded maximal surface in $\mathbb{L}^3$ with isolated singularities. Then S is an entire graph over any spacelike plane. That is to say, the Lorentzian orthogonal projection over any spacelike plane in $\mathbb{L}^3$ is a homeomorphism.*

3. Structure of the space of complete maximal surfaces with a fixed number of singularities

In what follows we will consider *complete embedded maximal surfaces with finitely many (isolated) singularities*, that is to say, *complete maximal graphs with a finite number of singularities.* Our aim is to show that the space of these surfaces having a fixed number of singularities has a natural structure of real analytical manifold and the underlying topology coincides with the one of the uniform convergence of graphs on compact subsets of $\{x_3 = 0\}$.

Theorem 3.1 *Let S be a complete maximal graph with a finite set of singularities $F \subset S$. Then,*

- *$S - F$ is biholomorphic to $\Omega = \mathbb{C} - \bigcup_{j=1}^{n+1} D_j$, where $D_j \subset \mathbb{C}$ are closed pairwise disjoint disks, $j = 1, \ldots, n+1$. The point $\infty \in \overline{\Omega}$ is called the end of the surface,*

- *the Weierstrass data for the maximal immersion $X : \Omega \to \mathbb{L}^3$ extend to the end with $|g(\infty)| \neq 1$,*

- *S is asymptotic to either half catenoid or a plane. Moreover, if we apply a Lorentzian isometry of $\mathbb{L}^3$ so that the tangent plane at the end is horizontal (that is to say, $g(\infty) = 0, \infty$) then the function u such that*

$S = \{(x_1, x_2, u(x_1, x_2)) \ : (x_1, x_2) \in \mathbb{R}^2\}$ *has the following asymptotic expansion*

$$u(x_1, x_2) = c \log R + b + \frac{a_1 x_1 + a_2 x_2}{R^2} + O(R^{-2}), \qquad R = |(x_1, x_2)|$$

for suitable constant $a_1,\ a_2,\ b,\ c \in \mathbb{R}$

The real number c that appears in the above theorem is called the *logarithmic growth* of S at infinity. It plays an important role in the definition of the coordinates of the space of complete maximal graphs having the same number of singularities.

In what follows, we denote by $\mathfrak{G}_n$ the space of all the *complete maximal graphs with* $n+1$ *singularities and with vertical limit normal vector at the end.*

The following theorem is a consequence of the maximum principle for elliptic differential equations and provides a natural way to define coordinates in the space $\mathfrak{G}_n$.

Theorem 3.2 (Uniqueness, [4]) *Two complete maximal graphs with the same set of singularities, the same limit normal vector at the end and the same logarithmic growth at infinity must coincide.*

In order to give a real analytic structure to $\mathfrak{G}_n$ the following notation is required.

Definition 3.3 *A marked maximal graph is a pair* $(S, \mathfrak{m})$ *such that* $S \in \mathfrak{G}_n$ *and* $\mathfrak{m}$ *is an ordering of the set* F *of singular points of the surface* S, $\mathfrak{m} = (q_1, \ldots, q_{n+1})$. *We denote by* $\mathfrak{M}_n = \{(S, \mathfrak{m}) \ : \ S \in \mathfrak{G}_n\}$ *the space of all the marked graphs with* $n+1$ *singularities and having vertical limit normal vector at the end.*

The space $\mathfrak{G}_n$ can be regarded as the quotient of $\mathfrak{M}_n$ under the action of the group of permutations of order $n+1$, $\mathfrak{P}_n$, given by

$$\lambda : \mathfrak{P}_n \times \mathfrak{M}_n \longrightarrow \mathfrak{M}_n$$

$$\lambda(\tau, (S, \mathfrak{m})) = (S, \tau(\mathfrak{m}))$$

In [2] it is shown that $\mathfrak{G}_n$ is non empty for any $n \in \mathbb{N}$ and new examples of maximal surfaces with singularities are given.

Theorem 3.4 (Analytic structure for the space of marked maximal graphs $\mathfrak{M}_n$) $\mathfrak{M}_n$ *is a differentiable manifold of dimension* $3n+4$ *with underlying topology of the uniform convergence on compact subsets of* $\{x_3 = 0\}$.

Figure 4: New examples of maximal surfaces with singularities.

Moreover, the map

$$\psi : \mathfrak{M}_n \longrightarrow \mathbb{R}^{3n+3} \times \mathbb{R}$$
$$\psi(S, \mathfrak{m}) = (\mathfrak{m}, c)$$

where c =logarithmic growth of S at the infinity, is a homeomorphism from $\mathfrak{M}_n$ onto an open subset of $\mathbb{R}^{3n+3} \times \mathbb{R}$ and provides analytic coordinates for the space $\mathfrak{M}_n$.

Sketch of the proof:

Uniqueness Theorem (3.2) gives that ψ is injective. Endow $\mathfrak{M}_n$ with the topology of the uniform convergence on compact subsets. The continuity of ψ follows from the maximum principle and classical Schauder estimates for elliptic differential equations.

To prove the remainder of the theorem we endow $\mathfrak{M}_n$ with a structure of differentiable manifold of dimension $3n+4$ (with the topology of uniform convergence of marked graphs on compact subsets as associated topology). This structure is given in terms of an appropiate bundle of the divisors associated to the Weierstrass data. We prove that ψ is a diffeomorphism when we consider this structure on $\mathfrak{M}_n$ (see [2] for more details). The domain invariance theorem gives that $\psi(\mathfrak{M}_n) \subset \mathbb{R}^{3n+3} \times \mathbb{R}$ is open and finishes the proof.

Theorem 3.5 (Analytic structure for the space of maximal graphs $\mathfrak{G}_n$) *The action $\lambda : \mathfrak{P}_n \times \mathfrak{M}_n \longrightarrow \mathfrak{M}_n$ es discontinuous and therefore the projection map $\pi : \mathfrak{M}_n \longrightarrow \mathfrak{G}_n$ is a covering of $(n+1)!$ sheets which induces in $\mathfrak{G}_n$ a structure of real analytic manifold of dimension $3n+4$.*

Moreover, the underlying topology coincides with the one of the uniform convergence of graphs on compact subsets of $\{x_3 = 0\}$.

Acknowledgments

This work has been partially supported by MCYT-FEDER grant number BFM2001-3489

References

[1] E. CALABI, Examples of the Bernstein problem for some nonlinear equations, *Proc. Symp. Pure Math.*, **vol. 15** (1970), 223-230.

[2] I. FERNÁNDEZ, F. J. LÓPEZ AND RABAH SOUAM, The space of complete embedded maximal surfaces with isolated singularities in the 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$, *preprint, ArXiV e-print archive math.DG/0311330 v1.*

[3] O. KOBAYASHI, Maximal surfaces with conelike singularities, *J. Math. Soc. Japan* **36** (1984), no. 4, 609–617

[4] A. A. KLYACHIN, Description of the set of singular entire solutions of the maximal surface equation, *Math. Sb.* **194** (2003), no. 7, 83–104

Proceedings of the II International
Meeting on Lorentzian Geometry
Murcia, November 12–14, 2003
Publ. de la RSME, Vol. 8 (2004), 83–88

Isometric decomposition of a manifold

Manuel Gutiérrez[1] and Benjamín Olea[2]

[1] *mgl@agt.cie.uma.es*
[2] *benji@agt.cie.uma.es*

Departamento de Álgebra, Geometría y Topología.
Universidad de Málaga.

Abstract

Given a semi-Riemannian manifold, we give some decomposition results using an irrotational and conformal vector field. We have not assumed that the vector field is globally a gradient, in particular we do not use simply connectedness hypothesis.

Keywords: Irrotational vector field, warped product.
2000 Mathematics Subject Classification: 53C50, 53C12 and 53C20.

1. Introduction

As a corollary of the well known decomposition Theorem of De Rham we can conclude that a simply connected semi-Riemannian manifold with a parallel and complete vector field with non null norm is isometric to a direct product $\mathbb{R} \times L$. This result does not hold if the manifold is not simply connected (Example 2.3 and 3.1) because the integral curves of the vector field can intersect the orthogonal leaves at different points. There are decomposition on a non necessarily simply connected manifold using a gradient of a function without critical point [2, 3, 9], because it assures that the integral curves intersect the orthogonal leaves at only one value of its parameter. This results are easily obtained from Proposition 2.2. In this work we obtain a decomposition of a manifold as a warped product using an irrotational and conformal vector field with some additional hypothesis on the orthogonal leaves or on the vector field itself.

2. Irrotational vector fields

Let (M, g) be an n-dimensional semi-Riemannian manifold (we suppose it connected without explicit mention) and E an unit, complete and irrotational vector field. We call $\Phi : \mathbb{R} \times M \to M$ the flow of E, $\varepsilon = g(E, E)$ and L_p the leaf of $E^\perp$ through $p \in M$. It is easy to check that E is a foliate vector field for the orthogonal foliation [6], and therefore $\Phi_t(L_p) = L_{\Phi_t(p)}$ for all $p \in M$ and $t \in \mathbb{R}$. Now, we can construct the local diffeomorphism

$$\begin{aligned} \Psi : \mathbb{R} \times L_p &\longrightarrow M \\ (t, x) &\to \Phi_t(x) \end{aligned}$$

which identifies the canonical foliations of $\mathbb{R} \times L_p$ with the integral curves of E and the foliation $E^\perp$.

Lemma 2.1 *Let M be a semi-Riemannian manifold and E an unit, complete and irrotational vector field. Then the above local diffeomorphism Ψ is onto.*
Proof. Since $Im\Psi$ is open, it is enough to see that it is closed. Take $x \notin Im\Psi = \cup_{t\in\mathbb{R}}\Phi_t(L_p)$, then $x \in \cup_{t\in\mathbb{R}}\Phi_t(L_x) \subset (\cup_{t\in\mathbb{R}}\Phi_t(L_p))^c$, but $\cup_{t\in\mathbb{R}}\Phi_t(L_x)$ is also an open set, so $Im\Psi$ is closed. □

If we take the pull-back metric $h = \Psi^*(g)$ on $\mathbb{R} \times L_p$, then $h = \varepsilon dt^2 + g_t$ where g_t is a metric tensor on L_p for each $t \in \mathbb{R}$, and Ψ becomes a local isometry. The following proposition gives a global decomposition of the manifold as a direct, warped or twisted product depending on the properties of the vector field E. We say that E is orthogonally conformal if there is $\rho \in C^\infty(M)$ such that $g(\nabla_X E, Y) + g(X, \nabla_Y E) = \rho g(X, Y)$ for all vector fields $X, Y \in E^\perp$.

Proposition 2.2 *Let M be a semi-Riemannian manifold and E an unit, complete and irrotational vector field. Take $p \in M$ such that the integral curves with initial values on L_p intersect L_p at only one value of its parameter. Then M is isometric to a product $(\mathbb{R} \times L_p, \varepsilon dt^2 + g_t)$. Moreover,*

1. *If E is parallel then M is isometric to a direct product $(\mathbb{R} \times L_p, \varepsilon dt^2 + g_0)$.*

2. *If E is orthogonally conformal and $grad\, divE$ is proportional to E then M is isometric to a warped product $(\mathbb{R} \times L_p, \varepsilon dt^2 + f^2 g_0)$ where $f(t) = \exp(\int_0^t \frac{divE(\Phi_p(s))}{n-1} ds)$.*

3. *If E is orthogonally conformal then M is isometric to a twisted product $(\mathbb{R}\times L_p, \varepsilon dt^2 + f^2 g_0)$ where $f(t,x)=\exp(\int_0^t \frac{divE(\Phi_x(s))}{n-1}ds)$.*

In any case, $g_0 = g\mid_{L_p}$.

Proof. If the integral curves of E with initial values on L_p intersect L_p at only one value of its parameter then $\Psi:\mathbb{R}\times L_p\to M$ is injective, and therefore a diffeomorphism. Then M is isometric to $(\mathbb{R}\times L_p, h=\varepsilon dt^2+g_t)$. Suppose now that E is orthogonally conformal. Then the foliation $E^\perp$ is umbilic and therefore $h=\varepsilon dt^2+f^2g_0$ where $f:\mathbb{R}\times L_p\to(0,\infty)$ is certain C^∞ function with $f(0,L_p)\equiv 1$ [8]. Since E is irrotational and orthogonally conformal, for all $v\in T_xL_p$, calling $w=\Psi_{*_{(t,x)}}(0,v)\in T_{\Phi_t(x)}M$, we get $\nabla_w E=\frac{divE}{n-1}w$. On the other hand, using the connection formula of a twisted product [8], we obtain $\nabla_v\frac{\partial}{\partial t}=h(\frac{\partial}{\partial t}, grad\ln f)v$. Applying $\Psi_{*_{(t,x)}}$ to both members we get $\frac{\partial}{\partial t}\ln f=\frac{divE}{n-1}$ and then $f(t,x)=\exp(\int_0^t\frac{divE(\Phi_x(s))}{n-1}ds)$.

If moreover $grad\, divE$ is proportional to E then $div\, E$ is constant through the orthogonal leaves and therefore $divE(\Phi_x(s))=divE(\Phi_p(s))$ for all $x\in L_p$ and all $s\in\mathbb{R}$. So, in this case, $f(t)=\exp(\int_0^t\frac{divE(\Phi_p(s))}{n-1}ds)$.

If E is parallel then $div\, E=0$ and we get a direct product. □

If a gradient has never null norm, it is immediate that the integral curves meet the orthogonal leaves at only one value of its parameter. We can assume directly that the vector field is a gradient and state the following: let M be a semi-Riemannian manifold and $f:M\to\mathbb{R}$ a function which gradient has never null norm and $E=\frac{grad\, f}{|grad\, f|}$ complete. If

- $H^f=0$, then M is isometric to a direct product $\mathbb{R}\times L$ [2].
- $H^f=a\cdot g$ then E verifies case two of proposition 2.2 and therefore M is isometric to a warped product $\mathbb{R}\times L$ [3, 9].
- $H^f=a\cdot g+bE^*\otimes E^*$, where $a,b\in C^\infty(M)$, then E verifies case three of the proposition 2.2 and therefore M is isometric to a twisted product $\mathbb{R}\times L$.

This is the easiest way to ensure that the integral curves intersect the leaves at only one point. Nevertheless, although the vector field were parallel we can not ensure the decomposition of the manifold, as is shown in the following example.

Example 2.3 *Take $(\mathbb{R}\times\mathbb{S}^2, -dt^2+g_{can})$ and Γ the group generated by the isometry $\Phi(t,p)=(t+1,-p)$. If $M=\mathbb{R}\times\mathbb{S}^2/\Gamma$ and $p:\mathbb{R}\times\mathbb{S}^2\to M$ is the projection, then $p_*(\frac{\partial}{\partial t})$ is the unique unit and parallel vector field on M. If*

M were isometric to a direct product $\mathbb{R}\times L$ then $\frac{\partial}{\partial t}$ would be identified with $p_*(\frac{\partial}{\partial t})$, but the integral curves of this vector field are all periodic and intersect its orthogonal leaves in two different points. Therefore M can not be a direct product.

3. Irrotational and conformal vector fields

Let U be an irrotational and conformal vector field with never null norm on a semi-Riemannian manifold M. If we call $\lambda = |U|$ and $E = \frac{U}{\lambda}$ then $\nabla_X U = E(\lambda)X$ for all vector field X and we can check that both function λ and $E(\lambda)$ are constant through the orthogonal leaves of U.

If M is a complete Riemannian manifold and U a non trivial irrotational and conformal vector field, then U has at most two zeros. If it has a zero then M is conformal to the Euclidean space. If it has two zeros then M is conformal to an Euclidean sphere [7].

If U has not zeros, then $\tilde{M}$, the universal covering of M, is isometric to a warped product $\mathbb{R}\times L$ as we have seen above. This is true in the indefinite case too.

For semi-Riemannian manifolds an analogous classification does not seem to be known, indeed U could have infinitely many zeros [5].

In the case where U has not zeros, we can not assure the decomposition of M, as is shown in Examples 2.3 or 3.1.

Example 3.1 *Let $\tilde{M} = \mathbb{R}^2$ the Minkowski space and $X = \sqrt{\frac{3}{2}}\frac{\partial}{\partial x} + \sqrt{\frac{1}{2}}\frac{\partial}{\partial y}$. Take Γ the isometry group generated by $\Phi(x,y) = (x, y+1)$, $M = \tilde{M}/\Gamma$ and the canonical projection $p : \tilde{M} \to M$. Since Φ preserves the vector field X, there is a vector field U on M such that $p_*(X) = U$. Since X is parallel, U is parallel, in particular irrotational and conformal. Moreover U is a timelike vector field and M is chronological, but it can not be decomposed as a warped product $\mathbb{R}\times L$ with $\frac{\partial}{\partial t}$ identified with $\frac{U}{|U|}$, since the integral curves of U intersect each orthogonal leaf at infinitely many points. This also provides us a counterexample to Proposition 2 of [1].*

With an additional hypothesis on the orthogonal leaves we can assure the decomposition of the manifold.

Theorem 3.2 *Let M be a chronological Lorentzian manifold and U a timelike, irrotational and conformal vector field with complete unitary. If L is an orthogonal leaf with finite volume then M is isometric to a warped product $(\mathbb{R}\times L, \varepsilon dt^2 + f^2 g_0)$ where $g_0 = g\,|_L$ and $f(t) = \frac{\lambda(\Phi_p(t))}{\lambda(p)}$, being $p \in L$ arbitrary.*

Proof. Let E be the unitary of U, $\lambda = |U|$ and $\Phi : \mathbb{R}\times M \to M$ the flow of E.

Take $p \in L$ and suppose that there is $t_0 > 0$ such that $\Phi_{t_0}(p) \in L$. Since M is chronological and locally a warped product where E is identified to $\frac{\partial}{\partial t}$, there is $\delta > 0$ such that the integral curves of E intersect $B_p(\delta) \subset L$ only one time. Since U is an irrotational and conformal vector field then $\Phi_{t_0} : L \to L_{\Phi_p(t_0)}$ is a conformal diffeomorphism with constant factor $(\frac{\lambda(\Phi_{t_0}(p))}{\lambda(p)})^2$. But λ is constant through the orthogonal leaf L, so $\Phi_{t_0} : L \to L_{\Phi_p(t_0)}$ is an isometry. Then $B_n = \Phi_{nt_0}(B_p(\delta)) \subset L$ and $vol(B_n) = vol(B_p(\delta))$. If $B_m \cap B_n = \emptyset$ for all $n \neq m$ then $vol(L) \geq \sum_{n=0}^{\infty} vol(B_n) = \sum_{n=0}^{\infty} vol(B_p(\delta)) = \infty$. Therefore there are $m < n$ such that $B_m \cap B_n \neq \emptyset$, and then there are $a, b \in B_p(\delta)$ with $\Phi_{(n-m)t_0}(a) = b$. But this is a contradiction since the integral curves of E intersect $B_p(\delta)$ only one time. Now we apply Proposition 2.2. □

With an additional hypothesis on the vector field we can also achieve the decomposition.

Proposition 3.3 *Let M be a semi-Riemannian manifold and U an irrotational and conformal vector field with never null norm and complete unitary. If $div\, U \neq 0$ or $Ric(U) \leq 0$ then M is isometric to a warped product $(\mathbb{R} \times L, \varepsilon dt^2 + f^2 g_0)$ where L is an orthogonal leaf, $g_0 = g\,|_L$ and $f(t) = \frac{\lambda(\Phi_p(t))}{\lambda(p)}$, with $p \in L$ arbitrary.* *Proof.* Since $\nabla U = E(\lambda) \cdot id$ it follows that $divU = n\, E(\lambda)$ and $Ric(U) = -(n-1)U(E(\lambda))$. Take L an orthogonal leaf and γ an integral curve of E with $\gamma(0) \in L$. If there is $t_0 \neq 0$ with $\gamma(t_0) \in L$ then $f(0) = f(t_0)$, where $f(t) = \lambda(\gamma(t))$. But this is a contradiction because $f(t) > 0$ and we are supposing $f'(t) \neq 0$ or $f''(t) \geq 0$ for all $t \in \mathbb{R}$. Using proposition 2.2 we obtain the desired result. □

4. The $\mathbb{S}^1 \times L$ type decomposition

The $\mathbb{S}^1 \times L$ type decomposition is more difficult to obtain than the $\mathbb{R} \times L$ type. This is because in the second case we only have to ensure that the integral curves with initial values on a leaf do not return to it, but in the first case we have to ensure that the integral curves with initial values on a leaf return and intersect the leaf at only one point. Even if we have an irrotational vector field with periodic integral curves with the same period we can not ensure the decomposition of the manifold as $\mathbb{S}^1 \times L$.

Take $(N, g) = (\mathbb{R} \times \mathbb{S}^3(\frac{1}{2}), dt^2 + f^2 g_0)$, where $f(t) = \sqrt{2 + \sin(2t)}$ and g_0 the canonical metric on $\mathbb{S}^3$. The scalar curvature $S = \frac{1}{n(n-1)}\mathbf{C}(Ric)$, of N is

$$S = -2\frac{f''}{nf} + \frac{n-2}{nf^2} S_{\mathbb{S}^3(\frac{1}{2})} - \frac{(n-2)f'^2}{nf^2} = 1.$$

We consider Γ the group generated by the isometry

$$\begin{aligned}\Psi : \mathbb{R} \times \mathbb{S}^3 &\to \mathbb{R} \times \mathbb{S}^3 \\ (t, p) &\to (t + \pi, -p).\end{aligned}$$

Take $M = N/\Gamma$ and $P : N \to M$ the canonical projection. The vector field $V = f\frac{\partial}{\partial t}$ is irrotational and conformal. Since V is preserved by Ψ there exists an irrotational and conformal vector field U on M such that $P_*(V) = U$. Now, M is a complete Riemann manifold furnished with an irrotational and conformal vector field, but it does not split as a warped product $(\mathbb{R} \times L, dt^2 + \lambda^2 h)$ or $(\mathbb{S}^1 \times L, dt^2 + \lambda^2 h)$ where $\frac{\partial}{\partial t}$ is identified with $\frac{U}{|U|}$ since the integral curves of U intersect each orthogonal leaf in two different points. Compare with Theorem 4.3 (ii) of [4].

References

[1] S.G. Harris, A characterization of Robertson-Walker spaces by null sectional curvature, *Gen. Relativity Gravitation* **17** (1985), 493-498.

[2] N. Innami, Splitting theorems of riemannian manifolds, *Compositio Math.* **47** (1982), 237-247.

[3] M. Kanai, On a differential equation characterizing a riemannian structure of a manifold, *Tokyo J. Math.* **6** (1983), 143-151.

[4] W. Kühnel and H.B. Rademacher, Conformal vector fields on pseudo-Riemann spaces, *Differential Geom. Appl.* **7** (1997), 237-250.

[5] W. Kühnel and H.B. Rademacher, Oscillator and pendulum equation on pseudo-Riemannian spaces, *Tohoku Math. J.* **48** (1996) 601-612.

[6] P. Molino, *Riemannian foliations*, Birkhäuser, Boston, (1988).

[7] M. Obata, Conformal transformations of Riemann manifolds, *J. Differential Geometry* **4** (1970) 311-333.

[8] R. Ponge and H. Reckziegel, Twisted products in pseudo-Riemannian Geometry, *Geom. Dedicata* **48** (1993), 15-25.

[9] Y. Tashiro, Complete Riemann manifold and some vector fields, *Trans. Amer. Math. Soc.* **117** (1965), 251-275.

Proceedings of the II International
Meeting on Lorentzian Geometry
Murcia, November 12–14, 2003
Publ. de la RSME, Vol. 8 (2004), 89–94

Volume, energy and spacelike energy of vector fields on Lorentzian manifolds

Ana Hurtado[1]

[1] *Departamento de Geometría y Topología, Universidad de Valencia, 46100 Burjassot, Valencia, Spain*
e-mail: ana.maria.hurtado@uv.es

Abstract

In this paper we will extend the results concerning the spacelike energy of unit timelike vector fields obtained in [2] to the volume and to the energy functionals.

Keywords: Energy; volume; critical point; reference frame; Robertson-walker spacetime; Gödel universe; Hopf vector fields.

2000 Mathematics Subject Classification: 53C50; 53C43; 53C25; 53C80.

1. Introduction

A smooth vector field V on a semi-Riemannian manifold (M, g) can be seen as a map into its tangent bundle endowed with the Sasaki metric g^S, defined by g.

When g is positive definite the energy and the volume can be defined in the space of smooth vector fields in a natural way. The energy of the map V is given, up to constant factors, by $\int_M \|\nabla V\|^2 \mathrm{dv}$ and the volume is defined as the volume of the submanifold $V(M)$ of (TM, g^S).

Many authors have studied the condition for a vector field to be a critical point of these functionals and the existence of minimizers among unit vector fields. Some of these results can be seen in the references of [3] and [4].

If we consider a Lorentzian manifold, the situation is not similar even if we restrict our attention to unit timelike vector fields. The energy is not bounded bellow, so the study of minimizers has no sense and the volume is not always defined. As a consequence, a new functional called spacelike energy, is introduced in [2] on the space of unit timelike vector fields. It is given by the integral of the square norm of the projection of the covariant derivative of the vector field onto its orthonormal complement.

The aim of this paper is to study to what extent the results obtained in [2] are still valid for the volume and for the energy.

The paper is organized as follows. In section 2 we give the definitions, the characterization of critical point and the expression of the second variation of the functionals. In section 3 we exhibit several examples of critical points. Moreover, we analyze the critical character of distinguished observers in spacetimes such as GRW and the classical Gödel universe.

2. Volume, energy and spacelike energy of vector fields

Given a semi-Riemannian manifold (M, g), the *Sasaki metric* g^S on the tangent bundle TM is defined, using g and its Levi-Civita connection ∇, as follows :

$$g^S(\zeta_1, \zeta_2) = g(\pi_* \circ \zeta_1, \pi_* \circ \zeta_2) + g(\kappa \circ \zeta_1, \kappa \circ \zeta_2),$$

where $\pi : TM \to M$ is the projection and κ is the connection map of ∇.

Definition 2.1 *The energy of a vector field V, is given by*

$$E(V) = \frac{n+1}{2} + \frac{1}{2}\int_M \|\nabla V\|^2 \mathrm{dv}.$$

The relevant part of the energy, $B(V) = \int_M b(V)\mathrm{dv}$ where $b(V) = \frac{1}{2}\|\nabla V\|^2$, when considered as a functional on the manifold of unit vector fields, is sometimes called the total bending of the vector field. The first and second variation of B have been widely studied by Wiegmink [6]. The covariant version of these results, as it appears in [4], involves the $1-$form $\omega_V(X) = g(X, \nabla^*\nabla V)$ where $\nabla^*\nabla V$ is the rough Laplacian.

It is easy to see that the similar results also holds for a reference frame (unit timelike vector field) on a Lorentzian manifold. More precisely,

Proposition 2.2 *Given a reference frame Z on a compact Lorentzian manifold (M, g) then*

1. *Z is a critical point of the total bending if and only if $\omega_Z(X) = 0$ for all vector field X orthogonal to Z, where $\omega_Z(X) = g(X, \sum_i \varepsilon_i(\nabla_{E_i}(\nabla Z))(E_i))$.*

2. If Z is a critical point and X is orthogonal to Z then

$$(HessB)_Z(X) = \int_M (-\|X\|^2\omega_Z(Z) + \|\nabla X\|^2)\mathrm{dv}.$$

Definition 2.3 *The volume of a unit vector field V on a Riemannian manifold is defined as the volume of the submanifold $V(M)$ of (T^1M, g^S). Since $(V^*g^S)(X,Y) = g(X,Y) + g(\nabla_X V, \nabla_Y V)$*

$$F(V) = \int_M f(V)\mathrm{dv} = \int_M \sqrt{\det L_V}\mathrm{dv},$$

where $L_V = \mathrm{Id} + (\nabla V)^t \circ \nabla V$.

In contrast with the energy, the volume of a reference frame Z, is not always defined on a Lorentzian manifold, since the $2-$covariant field Z^*g^S can be degenerated. Due to this, we study the volume restricted to unit timelike vector fields for which Z^*g^S is a Lorentzian metric on M. We will denote this set of vector fields by $\Gamma^-(T^{-1}M)$ and it is an open subset of the set of smooth reference frames.

With the same method used in [4] and [5] we have obtained

Proposition 2.4 *Let M be a compact Lorentzian manifold and let Z be a reference frame such that $Z \in \Gamma^-(T^{-1}M)$, then*

a) Z is a critical point of the volume if and only if $\widehat{\omega}_Z(X) = 0$ for all $X \in Z^\perp$, where $\widehat{\omega}_Z = \sum_i (\nabla_{E_i}\widehat{K}_Z)^i$ and $\widehat{K}_Z = f(Z)L_Z^{-1} \circ (\nabla Z)^t$.

b) If Z is a critical point and $X \in Z^\perp$

$$(HessF)_Z(X) = \int_M \Big(- \|X\|^2\widehat{\omega}_Z(Z) - \mathrm{tr}(L_Z^{-1} \circ (\nabla X)^t \circ \nabla Z \circ \widehat{K}_Z \circ \nabla X) + \frac{1}{f(Z)}(\sigma_2(\widehat{K}_Z \circ \nabla X)) + f(Z)\mathrm{tr}(L_Z^{-1} \circ (\nabla X)^t \circ \nabla X)\Big)\mathrm{dv},$$

where $\sigma_2(C) = \mathrm{tr}^2(C) - \mathrm{tr}(C^2)$.

Remark. Let us point out that if we compare these results with those obtained in [5] and [4] for Riemannian metrics, the only difference is the minus sign of the first term of the expression of the Hessian.

The spacelike energy density of Z is defined as

$$\widetilde{b}(Z) = \frac{1}{2}\|A_Z \circ P_Z\|^2 = \frac{1}{2}\sum_{i=1}^n g(\nabla_{E_i}Z, \nabla_{E_i}Z),$$

where $A_Z = -\nabla Z$, $P_Z(X) = X + g(X,Z)Z$ and $\{E_i, Z\}_{i=1}^n$ is an adapted orthonormal local frame.

Definition 2.5 *The spacelike energy is given by*

$$\widetilde{B}(Z) = \int_M \widetilde{b}(Z)\mathrm{dv}.$$

The condition for a reference frame to be spatially harmonic (critical point of the spacelike energy) and the second variation at critical points have been computed in [2]. We summarize here these results.

Proposition 2.6 *Let Z be a reference frame on a compact Lorentzian manifold.*

a) Z is spatially harmonic if and only if the $1-$form $\widetilde{\omega}_Z$ annihilates $Z^\perp$, where $\widetilde{\omega}_Z = -\sum_i(\nabla_{E_i}\widetilde{K}_Z)^i + g(\widetilde{K}_Z(\nabla_Z Z))$ and $\widetilde{K}_Z = (\nabla Z \circ P_Z)^t$.

b) If Z is spatially harmonic and $X \in Z^\perp$, we have

$$\begin{aligned}(Hess\widetilde{B})_Z(X) &= \int_M (\|\nabla X\|^2 + 2g(\nabla_X X, \nabla_Z Z) + \|\nabla_X Z + \nabla_Z X\|^2)\mathrm{dv} \\ &+ \int_M \|X\|^2(\|\nabla_Z Z\|^2 + \widetilde{\omega}_Z(Z))\mathrm{dv}.\end{aligned}$$

As for the energy and the volume, the condition of critical point obtained is tensorial, so we can define critical points even if the manifold is not compact and the functional is not defined.

3. Examples

The easiest examples of spatially harmonic reference frames are those of null spacelike energy. If we write the spacelike energy in terms of the kinematical quantities of the reference frame, then

$$\widetilde{B}(Z) = \frac{1}{2}\int_M (\|\Omega\|^2 + \|\sigma\|^2 + \frac{1}{n}\Theta^2)\mathrm{dv},$$

where Ω, σ and Θ are the rotation, the shear and the expansion respectively.

So, the spacelike energy vanishes when the reference frame is rigid and irrotational. As a consequence, we have the following proposition.

Proposition 3.1 ([2]) *In a static spacetime, the infimum of the spacelike energy is zero and it is attained by the static observer.*

In what concerns energy and volume, computing the Euler-Lagrange equations for this type of vector fields we have shown that

Proposition 3.2 *Let Z be a rigid and irrotational reference frame.*

a) *Z is a critical point of the energy if and only if, $\nabla_Z\nabla_Z Z = \|\nabla_Z Z\|^2 Z$.*

b) *$Z \in \Gamma^-(T^{-1}M)$ is minimal if and only if*

$$Z(\frac{1}{f(Z)})g(X, \nabla_Z Z) + \frac{1}{f(Z)}g(X, \nabla_Z\nabla_Z Z) = 0 \quad \text{for all } X \in Z^\perp.$$

One of the most important cosmological models are the Robertson-Walker spacetimes and the so-called generalized Robertson-Walker spacetimes (see [1] for more details). In [2], using the results obtained for Lorentzian manifolds admitting a closed and conformal timelike vector field, it has been shown that

Proposition 3.3 *The comoving observer ∂_t is spatially harmonic. Furthermore, if M is assumed to be compact and satisfying the null convergence condition, ∂_t is an absolute minimizer of the spacelike energy.*

Following similar arguments, we have shown the following proposition.

Proposition 3.4 *In a GRW spacetime the comoving observer is a minimal immersion.*

Another example concerns the classical Gödel universe that is $\mathbb{R}^4$ endowed with the metric $ds^2 = dx_1^2 + dx_2^2 - \frac{1}{2}e^{2\alpha x_1}dy^2 - 2e^{\alpha x_1}dydt - dt^2$, where α is a positive constant. In this coordinate system we have two distinguished timelike vector fields ∂_t and ∂_y.

Proposition 3.5 ([2]) *In the Gödel universe we have*

1. *The reference frame ∂_t is a critical point of the energy and it is also spatially harmonic.*
2. *The reference frame $Z = \sqrt{2}e^{-\alpha x_1}\partial_y$ is not spatially harmonic.*

In contrast with part 2 of the above proposition, we have shown that

Proposition 3.6 *The reference frame $Z = \sqrt{2}e^{-\alpha x_1}\partial_y$ is a critical point of the energy.*

In what concerns the volume functional, it is easy to see that ∂_t belongs to $\Gamma^-(T^{-1}M)$ and if $Z = \sqrt{2}e^{-\alpha x_1}\partial_y$, then $\det L_Z = (1 - \frac{\alpha^2}{2})(1 + \frac{\alpha^2}{2})$, so $Z \in \Gamma^-(T^{-1}M)$ if and only if $\alpha^2 < 2$.

Proposition 3.7 *In the Gödel Universe*

a) *$\partial_t \in \Gamma^-(T^{-1}M)$ and it is a minimal immersion.*

b) $Z = \sqrt{2}e^{-\alpha x_1}\partial_y \in \Gamma^-(T^{-1}M)$ *if and only if* $\alpha^2 < 2$ *and it is not a critical point of the volume.*

To finish the paper, we would like to remark that another interesting examples are the Hopf vector fields in Lorentzian Berger spheres, that are a particular case of generalized Taub-NUT spacetimes. The study of the spacelike energy in these spaces can be seen in [2], while the study of the energy and the volume has been widely developed in [3].

Acknowledgments

This work has been partially supported by DGI Grant No. BFM2001-3548 and by Ministerio de Educación y Cultura Research Grant. The author wants to thank O. Gil-Medrano for her collaboration in the preparation of this paper.

References

[1] L.J. Alías, A. Romero, M. Sánchez, Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes, *Gen. Rel. and Grav.* **27** (1995), 71-84.

[2] O. Gil-Medrano, A. Hurtado, Spacelike energy of timelike unit vector fields on a Lorentzian manifold, *J. Geom. and Phys.*, **51** (2004), 82-100.

[3] O. Gil-Medrano, A. Hurtado, Volume, energy and generalized energy of unit vector fields on Berger's spheres. Stability of Hopf vector fields, *Preprint*, (2004).

[4] O. Gil-Medrano, E. Llinares-Fuster, Second variation of Volume and Energy of vector fields. Stability of Hopf vector fields, *Math. Ann.*, **320** (2001), 531-545.

[5] O. Gil-Medrano, E. Llinares-Fuster, Minimal unit vector fields , *Tohoku Math. J.*, **54** (2002), 71-84.

[6] G. Wiegmink, Total bending of vector fields on Riemannian manifolds, *Math. Ann.* **303** (1995), 325-344.

Proceedings of the II International
Meeting on Lorentzian Geometry
Murcia, November 12–14, 2003
Publ. de la RSME, Vol. 8 (2004), 95–103

START: 4D to 5D generalization of the (Minkowski-)Lorentz Geometry

Jaime Keller

Department of Physics and Theoretical Chemistry,
Facultad de Química and Facultad de Estudios Superiores Cuautitlán,
Universidad Nacional Autónoma de México 04510
México D.F., apartado 70-528, MÉXICO

E-mail: keller@servidor.unam.mx and keller@cms.tuwien.ac.at

Abstract

The quadratic form of the Minkowski space $s^2 = (ct)^2 - (x^2 + y^2 + z^2)$ describes kinematics (motion) in **Space–Time**. To faithfully include matter and interaction (dynamics) we propose a generalization $S^2 = s^2 - w^2$ of this quadratic form, where $w = \kappa_{(0)}\overline{a}$ is an equivalent distance expressing the physical **Action** corresponding to a time interval t during which a distance $l = \sqrt{(x^2 + y^2 + z^2)}$ has been covered. The definitions being such that energy–momentum corresponds to (Planck constant times) the space–time derivatives of w. We show that the resulting space $s^2 \to S^2 = g_{uv}x^u x^v$ can also be pictured as a curved space–time. The corresponding vector (in Clifford algebra a 5-dimensional vector) is $S = \boldsymbol{e}_u x^u$, where the first four components correspond to the (Lorentz-)Minkowski vector $s = e_\mu x^\mu$. The usefulness, and in fact the motivation, of this geometry is illustrated through the analysis of light propagation in a medium and in a gravitational field. In our formulation, including **Relativity Theory** postulates (**START**), the base space for the description is flat. A new approach is presented here pointing out other structures of physics that have also being studied with our formalism.

Keywords: Space-Time-Action, General Relativity, Geometric Optics.
2000 Mathematics Subject Classification: 83D05, 83Exx, 11E88, 15A66(67)

1. Physics and Lorentz Geometry

Physics is understood as the science describing nature as a whole in a useful way. For the Scientific Method (SM) the usefulness requires that two observers will find in their experiments similar phenomena and describe them with similar logical structures. Theoretical Physics provides a Mathematical, SM acceptable, framework for this purpose. In this paper we show that a flat 5-D Lorentz Geometry is the most useful formulation.

- The central purpose of the here formulated START Theory is twofold: first to have a description of nature, the way we perceive it with our senses and experiments, which could be useful in accordance with the Scientific Method, that is which could be a sound basis for physics. Second START is aimed to be a valid general mathematical theory for all the fundamental physical objects and the frame of reference we use for their description. The physical objects are in general aspects of matter. S stands for space, T for time, A for the theoretical description of matter as an energy content for a period of time (action), and finally RT for the mathematical relativity theory constructed from a quadratic space: a Lorentz Geometry.

- By construction Planck's constant is used in the definition of the equivalent distance corresponding to action.

- In our presentation in the present paper optics was taken as an example to guide the reader in the step going from Special Relativity Theory to START which is not only more general but also general enough to be the starting point of a deductive approach to the whole of the basic structures of physics, at least in the chapters mentioned in this paper.

- The 5-D START geometry is presented as a flat Lorentzian manifold time-like oriented in which all light rays correspond to null lines. Massive objects are described as bundles of null trajectories in START. Observers to 4-D time-like trajectories. When interactions are included the trajectories of all physical objects remain to be null trajectories which can also be represented as null geodesics in a curved 4-D subspace embedded in the START manifold. The curvature will obey a set of equations of which Einstein's equations of general relativity are a particular case. The development presented here is new and different from our previous papers in several senses, nevertheless the mathematical structures of previous papers are re-encountered.

- Two examples are used to illustrate how the description of physical phenomena connects the START geometry with the 3-D or 4-D standard language of optics in a refractive medium or in a gravitational field.
- START provides a unified presentation and a common mathematical language. Descriptions with START are simpler. The unity of approach and the easiness of the procedures also provides a badly needed didactical procedure to understand theoretical physics.
- The generality of the mathematical structures, presented here as particular examples, can be seen in the last section.

1.1. The Quadratic Form in Physics

Principia Geometrica Physicae: Quadratic spaces are the fundamental clue to understand the structure of theoretical physics. Classical Physics is embedded within a 5-D flat quadratic space, 3 Space-, one Time- and one Action- like basis manifolds (Lorentz signature 1,4) faithfully providing a Relativistic Theory (START) describing Newtonian, Maxwell, geometrical optics and General Relativity as particular linear and quadratic forms of this (flat) START space.

Our formulation (see [1][2][3][4][5]) claims that there is a useful generalization of the quadratic form which historically started with the Pythagorean formulation:

Quadratic form	Dim/diff. Op	Group
$l^2 = x^2 + y^2 + z^2 \qquad (\triangle t)$	3-D ∇, ∇^2	Galileo
$s^2 = (ct)^2 - (x^2 + y^2 + z^2)$	4-D $D, \Box^2$	Poincaré
$S^2 = (ct)^2 - \left(x^2 + y^2 + z^2\right) - w^2$	5-D $K, \pentagon^2$	Dynamics
$w = \kappa_{(0)}\overline{a}, \qquad \kappa_{(0)} = \frac{d_{(0)}}{h} = \frac{c}{E_{(0)}}$	$(\overline{a})^2 = \sum_\mu a_\mu^2$	START

where l, (x, y, z), c, h, w, a, $E_{(0)}$ are distance, distance components, vacuum speed of light, Planck's constant, distance equivalent to action, action of a system and characteristic energy of the system respectively. In the START space we introduce physics through the **START Relativity Principle**

"**All trajectories are null for all observers**"

"**The vacuum Speed of Light is c for all observers**"

Our geometrical structure allows the development of a new formalism, following a deductive scheme from a set of FUNDAMENTAL PRINCIPLES and POSTULATES, in such a form to obtain a comprehensive theory for Physics.

The set of principles we have introduced are [2]: **START Relativity** (5-D Poincaré group and 5-D Lorentz transformations); **Existence** (physical objects are represented by energy densities); **Least Action** (null, optimal possible, trajectories in START); **Quantized Exchange of Action** (defines systems or subsystems as those among a quanta of action can be exchanged) and, **Choice of Descriptions** (allows all useful physical models to be employed).

2. 5-D Formulation of OPTICS

Geometric optics in a medium is used as a guiding concept to enlarge the representation space of physical phenomena to 5-dimensions. Optics was used in the XX century to create a logical 4-dimensional geometric representation of the frame of reference to describe events in nature, the basic consideration was that of "free space" light rays. We show that considering the concept of light propagation in the medium, where the speed of light is lower, a geometry in 5-dimensions appears as a natural frame of reference. The Lorentz Geometry fundamental quadratic form of this space is $dS^2 = ds^2 - (\kappa_0 da)^2$, here ds^2 is the Poincaré–Einstein-Minkowski 4-D space–time quadratic form. Below we show that General Relativity is also faithfully represented in this quadratic space.

2.1. From Space–Time Geometric Optics to START

The trajectory of a light ray in a medium can be represented in two equivalent forms. Consider $dl = v_e dt = (c/\eta)dt$, for **light propagation in a medium** (refraction index η) and use the quadratic form $dl^2 = dx^2 + dy^2 + dz^2$ for the elementary trajectory in space $dl^2 = (c/\eta)^2 dt^2$, or $(cdt)^2 = \eta^2 dl^2 = (1 + 4\pi\chi)dl^2$, to define

$$dS^2_{light-\boldsymbol{\eta}} = (cdt)^2 - dl^2 - 4\pi\chi dl^2 = (cdt)^2 - dl^2 - \frac{4\pi\chi}{1+4\pi\chi}c^2dt^2 = 0,$$

$$\boxed{dS^2_{light-\boldsymbol{\eta}} = \left(1 - \tfrac{\eta^2-1}{\eta^2}\right) c^2dt^2 - \left(dx^2 + dy^2 + dz^2\right)}$$

The relations above allow a new interpretation of the light propagation in a medium as a **propagation with the vacuum speed of light but in a 5-dimensional quadratic manifold**.

In the 5-D form the fifth term $\left[\left(\eta^2 - 1\right)/\eta^2\right] c^2dt^2$ represents a distance equivalent to the interaction of the medium, with refraction index η, upon the light ray. We now propose the following 5-dimensional construction, including the **action** variable, in the space–time-action quadratic form (see last section)

$$dS^2 = ds^2 - (\kappa_0 da)^2 = (cdt)^2 - (dx^2 + dy^2 + dz^2) - (\kappa_0 da)^2 \qquad (1)$$

where in the case of light the constant $\kappa_0 = c/h\nu$ the inverse of the momentum associated (in vacuum) with the light (one carrier's, one photon) energy $h\nu$. If we associate then the last term above with the last term in the description of the light ray propagation

$$(\kappa_0 da)^2 = \frac{\eta^2 - 1}{\eta^2} c^2 dt^2 = \frac{c^2}{(h\nu)^2} \Delta\left(\mathcal{E}^2\right) dt^2,$$

we can define the equivalent interaction potential as (here $\mathcal{E}$ represents light $\rightarrow$ media interaction energy)

$$\frac{\eta^2 - 1}{\eta^2} = \frac{\Delta\left(\mathcal{E}^2\right)}{(h\nu)^2},$$

therefore the **action** $\mathcal{E}dt$ equivalent to the interaction between light and the medium. In vacuum the action corresponding to light is

$$(da)^2 = \left(\mathcal{E}^2 dt^2 - p_l^2 dl^2\right) = \left((h\nu)^2 - \left(\frac{h}{\lambda} c\right)^2\right) dt^2 = 0$$

whereas in the medium there is a change

$$\begin{aligned}(da')^2 &= \left(\mathcal{E}'^2 dt^2 - p_{l'}'^2 dl'^2\right) = \left(\left(\frac{h\nu - \Delta V}{c}\right)^2 - \left(\frac{h}{\lambda' \eta}\right)^2\right) dt^2 \\ &= \left((h\nu)^2 + \Delta\left(\mathcal{E}^2\right) - \left(\frac{h}{\lambda} c\right)^2\right) dt^2 = \frac{\eta^2 - 1}{\eta^2} (h\nu)^2 dt^2\end{aligned}$$

where the change in the energy term is not compensated by the change in the momentum term. In fact when light of frequency ν travels through a medium of refraction index η where the equivalent speed of light is $c' = c/\eta$ there is a change in the wave length $\lambda \rightarrow \lambda' = \lambda/\eta$ in agreement with our formulation above and simultaneously a change in the length of the trajectory $dl \rightarrow dl' = dl/\eta$ which exactly compensates for the change in momentum. Geometrical optics can be compactly formulated with the additional inclusion of the change in wave length through the 5-dimensional formulation presented here.

In this discussion it is fundamental that these processes take place when the carrier of light is **in interaction with a medium**, the medium being described by an index of refraction η.

3. The photon as a general relativity test particle

General Relativity is considered a comprehensive theory, the best known solutions are developed for the so called matter-free space and a test particle. We show that (1) corresponds to a description of the action distribution which agrees with the conceptual development of General Relativity (GR), this last theory itself being based on the physical postulate that all observers have the right to consider their measurements equally valid..

3.1. Beyond the Schwarzschild solution

There are two fundamental (energy-)carrier structures: the massless (as the photon) and the massive fields with basic relation

$$\mathcal{E}^2 = (\mathcal{E}_0 + \Delta\mathcal{E})^2\,,\ \mathcal{E}^2 - \mathcal{E}_0^2 = (pc)^2, \tag{1}$$

where $\Delta\mathcal{E}$ is any gauge-free energy contribution and $\mathcal{E}_0 = m_0c^2 \Longrightarrow h\nu$ (for a photon).

The concept of test particle (at position$\{r, \theta, \phi\}$) in general relativity in the Schwarzschild solution is compatible with the Newtonian limit for the interaction gravitational energy

$$\Delta\mathcal{E}\,(r) = -m_0\frac{GM}{r}, \tag{2}$$

where M is the total mass of 'the external system' (confined within a radius r_s) which we are exploring with the test particle. START uses the action square difference, writing $\mathcal{E} = \mathcal{E}_0 + \Delta\mathcal{E}$ for large (classical limit) values of $r > r_s$

$$\begin{aligned} \mathcal{E}^2 - \mathcal{E}_0^2 &= \mathcal{E}_0^2 + 2\mathcal{E}_0\Delta\mathcal{E} + (\Delta\mathcal{E})^2 - \mathcal{E}_0^2 = (pc)^2 \\ &= 2\mathcal{E}_0\Delta\mathcal{E} + (\Delta\mathcal{E})^2 \to -2m_0c^2m_0\frac{GM}{r} + \left(m_0\frac{GM}{r}\right)^2, \end{aligned} \tag{3}$$

this corresponds both to the energy (difference square) and radial momentum terms in $(d\mathcal{A})^2 - (d\mathcal{A}')^2$ if $(d\mathcal{A}')^2 = (m_0c^2dt)^2$, and according to the GR basic description principles, substituting (the negative of (3)) in (1) using $\kappa_0 = 1/m_0c$ and space–time spherically symmetric coordinates t, r, θ, ϕ we obtain

$$\begin{aligned} (d\mathbf{S})^2 = &\left(1 - 2\frac{GM}{c^2r} + \left(\frac{GM}{c^2r}\right)^2\right)c^2\,(dt)^2 \\ &- \left(1 - 2\frac{GM}{c^2r} + \left(\frac{GM}{c^2r}\right)^2\right)\left\{(dr)^2 - r^2\left[(d\theta)^2 + \sin^2\theta\,(d\phi)^2\right]\right\}, \end{aligned} \tag{4}$$

with the same physical consequences for the so called "tests of GR" as those of the Schwarzschild [1916] metric in the limit of $r \gg GM/c^2$. The speed of light in this metric remains to be c, this is a boundary condition different from the no-curvature condition of that approach.

3.2. Action and the Quadratic Form in START

$\mathfrak{L} = \left(m_0 c \frac{d\mathbf{S}}{d\tau}\right)^2$, corresponds to the invariant quadratic form of the 4-D relativistic action function in START, the right hand side term corresponding to the proper mass action rate. The variational condition $\delta \int_t \mathfrak{L} d\tau = 0$ replaces the least action principle. The actual (bundles of) physical trajectories are those corresponding to

1. Locally null trajectories, that is $dS^2 = 0$. This condition being universal (in START), for massless or massive carriers.

2. Global least action trajectories, that is $\delta \int_{t_1}^{t_2} dS = 0$. This condition defines the full trajectory. (An example would be a light ray in two refractive media, each one homogeneous, separated by a surface crossed by the ray. In both and each of the two media the condition $dS^2 = 0$ is fulfilled).

3.3. Multivector Representation

The base space $\mathbb{R}^5$ corresponds to the real variables set $\{ct, x, y, z, \kappa_0 \alpha\} \leftrightarrow \{x^u; u = 0, 1, 2, 3, 4\}$ that is: time, 3-D space and action (in units of distance introducing the universal speed of light in vacuum c and the system under observation dependent $\kappa_0 = \lambda_{Compton}^{system\, with\, energy\, mc^2}/h = 1/m_0 c$). For physics time is usually an independent evolution coordinate and action (matter and interaction) is distributed in space, then we consider the functions $x(t), y(t), z(t)$ and $w(t, x, y, z) = \kappa_0 \alpha(t, x, y, z)$. The nested vectors

$$\begin{array}{ll} dS = \sum_\mu dx^u e_u \, ; \, u = 0, 1, 2, 3, 4 & 5 - D \\ ds = \sum_\mu dx^\mu e_\mu \, ; \, \mu = 0, 1, 2, 3 & 4 - D \\ d\mathbf{x} = \sum_i dx^i \mathbf{e}_i \, ; \, i = 1, 2, 3 \, ; \, \mathbf{e}_i = e_0 e_i & 3 - D \end{array}$$

are members of a Clifford algebra generated by the definition of a quadratic form

$$\begin{aligned} dS^2 &\equiv (dS)^2 = \left(\sum\nolimits_\mu dx^u e_u\right)^2 = \sum\nolimits_{\mu\nu} g_{uv}^{START} dx^u dx^v \, , \\ g_{uv}^{START} &= diag\,(1, -1, -1, -1, -1) \, , \, e_u e_v = -e_v e_u \\ e &= e_0 e_1 e_2 e_3 e_4 = -e^\dagger \qquad e_u e = e e_u \end{aligned}$$

The quadratic form which is more relevant for Physics considers that observable objects are extended in space and then an action density α in space-time is required. Then, defining $m(\mathbf{x},t)c^2 = \varepsilon_{total}(\mathbf{x},t)$, and the (Clifford algebra valued) inverse of the space-time volume $e_0e_1e_2e_3/\triangle x\triangle y\triangle z\triangle t$, and the space-time Laplacian operator $\Box = \sum_\mu e^\mu\partial_\mu$ such that along $b = \sum_\mu b^\mu e_\mu$ the directional change operator is $db\cdot\Box = \sum_\mu db^\mu\partial_\mu$ (we apply four times for $b = cte_0, xe_1, ye_2, ze_3$), we can obtain the sum of the **directed** changes of the density of w:

$$\mathbf{a}(\mathbf{x},t)e_4 = \kappa_0\alpha(\mathbf{x},t)e_4 = \kappa_0\frac{m(\mathbf{x},t)c^2\triangle t}{\triangle x\triangle y\triangle z\triangle t}e = \frac{1}{m_0c}\frac{m(\mathbf{x},t)c^2\triangle t}{\triangle x\triangle y\triangle z\triangle t}e = \frac{(m(\mathbf{x},t)/m_0)c\triangle t}{\triangle x\triangle y\triangle z\triangle t}e$$
$$\mathbf{a}(\mathbf{x},t)e_4 = \frac{(m(\mathbf{x},t)/m_0)c\triangle t}{\triangle x\triangle y\triangle z\triangle t}e = \frac{w(\mathbf{x},t)}{\triangle x\triangle y\triangle z\triangle t}e = \mathfrak{w}(\mathbf{x},t)e$$
$$ed\mathfrak{w} = \sum_\mu\left[(\partial_\mu\mathfrak{w}(\mathbf{x},t))\,dx^\mu\right]e_\mu e$$
$$(dS)^2 = (dS)\,(dS)^\dagger = \left(1-(\kappa_0p_0)^2\right)(cdt)^2 +$$
$$-\left(\left(1-(\kappa_0p_1)^2\right)(dx)^2 + \left(1-(\kappa_0p_2)^2\right)(dy)^2 + \left(1-(\kappa_0p_3)^2\right)(dz)^2\right)$$

here $\mathfrak{p}_\mu = \partial_\mu\alpha(\mathbf{x},t)$ is a momentum density. Notice that $\mathfrak{w}(\mathbf{x},t)$ is the distance equivalent to a **reduced action** density, this makes the approach universal for all systems.

Acknowledgments: The technical assistance of Mrs. Irma Vigil de Aragón is gratefully acknowledged. This project was additionally supported by the UNAM program PAPIIT number IN113102-3.

References

[1] J. Keller, The Foundations of Density Functional Theory and Wave Quantum Mechanics, *Rev. Soc. Quim. Mex.* **44**(1) 2000, 22–28

[2] J. Keller, *The Theory of the Electron* ***A Theory of Matter from START***, From the series: Fundamental Theories of Physics, Kluwer Academic Publishers, Dordrecht, Vol. 115 (2001), e-bookhttp://www.wkap.nl/prod/b/0-7923-6819-3

[3] J. Keller, General Relativity from START, in: *Clifford Analysis and Related Topics, K. Gürlebeck X. Ji, W. Sprössig Eds., Adv. in Applied Clifford Algebras* **11**(S2) (2001), 183-204.

[4] J. Keller, General Relativity as a Symmetry of a Unified Space-Time-Action Geometrical Space, *Proceedings of Institute of Mathematics of NAS of Ukraine,* **43**(2) also J. Keller, A Derivation of Quantum Mechanics from START, *idem* **50**, Editors A.G. Nikitin, V.M. Boyko and R.O. Popovych (and I.A.Yehorchenko), Kyiv, Institute of Mathematics, (2002). 557-568 and (2004).811-820 http://www.imath.kiev.ua/activities.php

[5] J. Keller, Unification of electrodynamics and Gravity form START, *Annales de la Fond. Louis de Broglie*, **27**(3) (2002) 359. http://www.ensmp.fr/aflb/AFLB-272/aflb272p359.htm

Proceedings of the II International
Meeting on Lorentzian Geometry
Murcia, November 12–14, 2003
Publ. de la RSME, Vol. 8 (2004), 104–112

Nonlinear connections and exact solutions in Einstein and extra dimension gravity

Sergiu I. Vacaru

Departamento de Fisica, Instituto Superior Tecnico, Av. Rovisco Pais 1, Lisbon 1049-001, Portugal, and Departamento de Fisica Aplicada, Facultad de Ciencias Experimentales, Universidad de Huelva, 21071 Huelva, Spain

Abstract

We outline a new geometric method of constructing exact solutions of gravitational field equations parametrized by generic off–diagonal metrics, anholonomic frames and possessing, in general, nontrivial torsion and nonmetricity. The formalism of nonlinear connections is elaborated for (pseudo) Riemannian and Einstein–Cartan–Weyl spaces.

1. Introduction: Nonlinear Connections

We consider a $(n+m)$–dimensional manifold V^{n+m}, provided with general metric and linear connection structure and of necessary smooth class. It is supposed that in any point $u \in V^{n+m}$ there is a local splitting into $n-$ and m–dimensional subspaces, $V_u^{n+m} = V_u^n \oplus V_u^{mn}$. The local/abstract coordinates are denoted $u = (x,y)$, or $u^\alpha = \left(x^i, y^a\right)$, where $i,j,k,... = 1,2,...,n$ and $a,b,c,... = n+1, n+2, ..., n+m$. The metric is parametrized in the form

$$\mathbf{g} = g_{ij}(u)\mathbf{e}^i \otimes \mathbf{e}^j + h_{ab}(u)\mathbf{e}^a \otimes \mathbf{e}^b \qquad (1)$$

where

$$\vartheta^\mu = [\vartheta^i = dx^i, \vartheta^a = dy^a + N_i^a(u)dx^i] \qquad (2)$$

is the dual frame to

$$\mathbf{e}_\nu = [e_i = \frac{\partial}{\partial x^i} - N_i^a(u)\frac{\partial}{\partial y^a}, e_a = \frac{\partial}{\partial y^a}]. \quad (3)$$

Let us denote by $\pi^\top : TV^{n+m} \to TV^n$ the differential of a map $\pi : V^{n+m} \to V^n$ defined by fiber preserving morphisms of the tangent bundles TV^{n+m} and TV^n. The kernel of $\pi^\top$is just the vertical subspace vV^{n+m} with a related inclusion mapping $i : vV^{n+m} \to TV^{n+m}$.

Definition 1.1 *A nonlinear connection (N–connection)* $\mathbf{N}$ *on space* V^{n+m} *is defined by the splitting on the left of an exact sequence*

$$0 \to vV^{n+m} \to TV^{n+m} \to TV^{n+m}/vV^{n+m} \to 0,$$

i. e. by a morphism of submanifolds $\mathbf{N} :\ TV^{n+m} \to vV^{n+m}$ *such that* $\mathbf{N} \circ \mathbf{i}$ *is the unity in* vV^{n+m}.

Equivalently, a N–connection is defined by a Whitney sum of horizontal (h) subspace (hV^{n+m}) and vertical (v) subspaces,

$$TV^{n+m} = hV^{n+m} \oplus vV^{n+m}. \quad (4)$$

A space provided with N–connection structure will be denoted V_N^{n+m}. We shall use boldfaced indices for the geometric objects adapted to N–connections. The well known class of linear connections consists a particular subclass with the coefficients being linear on y^a, i. e. $N_i^a(u) = \Gamma_{bj}^a(x)y^b$.

To any sets $N_i^a(u)$, we can associate certain anholonomic frames (2) and (3), with associated N–connection structure, satisfying the anholonomy relations

$$[\vartheta_\alpha, \vartheta_\beta] = \vartheta_\alpha\vartheta_\beta - \vartheta_\beta\vartheta_\alpha = W_{\alpha\beta}^\gamma\vartheta_\gamma$$

with (antisymmetric) nontrivial anholonomy coefficients $W_{ia}^b = \partial_a N_i^b$ and $W_{ji}^a = \Omega_{ij}^a$, where $\Omega_{ij}^a = e_{[i}N_{j]}^a$ are the coefficients of the N–connection curvature.

Essentially, the method to be presented in this work is based on the notion of nonlinear connection (N–connection) and considers a Whitney-like splitting of the tangent bundle to a manifold into a horizontal (see discussion and bibliography in Refs. [1, 2, 3]). Here we emphasize that the geometrical aspects of the N–connection formalism has been studied since the first papers of E. Cartan [4] and A. Kawaguchi [5, 6] (who used it in component form for in Finsler geometry), then it should be mentioned the so called Ehressman connection [7]) and the work of W. Barthel [8] where the global definition of

N–connection was given. The monograph [9] consider the N–connection formalism elaborated and applied to geometry of generalized Finsler–Lagrange and Cartan–Hamilton spaces. There is a set of contributions by Spanish authors, see, for instance, [10, 11, 12].

We considered N–connections for Clifford and spinor structures [13, 14], on superbundles and (super) string theory [15] as well in noncommutative geometry and gravity [16]. The idea to apply the N–connections formalism as a new geometric method of constructins exact solutions in gravity theories was suggested in Refs. [17, 18] and developed and applied in a number of works, see for instance, Ref. [19, 20, 21]). This contribution outlines the author's and co–authors' results.

2. N–distinguished Torsions and Curvatures

The geometric constructions can be adapted to the N–connection structure:

Definition 2.1 *A distinguished connection (d–connection)* $\mathbf{D} = \{\mathbf{\Gamma}^{\alpha}_{\beta\gamma}\}$ *on* V_N^{n+m} *is a linear connection conserving under parallelism the Whitney sum (4).*

Any d–connection $\mathbf{D}$ is represented by irreducible h- v–components $\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \left(L^i_{jk}, \widetilde{L}^a_{bk}, C^i_{jc}, \widetilde{C}^a_{bc}\right)$ stated with respect to N–elongated frames (2) and (3). This defines a N–adapted splitting into h– and v–covariant derivatives, $\mathbf{D} = D^{[h]} + D^{[v]}$, where $D^{[h]} = (L, \widetilde{L})$ and $D^{[v]} = (C, \widetilde{C})$. A d–tensor (distinguished tensor, for instance, a d–metric like (1)) formalism and d–covariant differential and integral calculus can be elaborated [1] for spaces provided with general N–connection, d–connection and d–metric structure and nontrivial nonmetricity

$$\mathcal{Q}_{\alpha\beta} \doteq -\mathbf{D}\mathbf{g}_{\alpha\beta}.$$

The simplest way is to use N–adapted differential forms like $\mathbf{\Gamma}^{\alpha}_{\beta} = \mathbf{\Gamma}^{\alpha}_{\beta\gamma}\vartheta^{\gamma}$ with the coefficients defined with respect to (2) and (3).

Theorem 2.2 *The torsion* $\mathcal{T}^{\alpha} \doteqdot \mathbf{D}\vartheta^{\alpha} = d\vartheta^{\alpha} + \Gamma^{\alpha}_{\beta} \wedge \vartheta^{\beta}$ *of a d–connection has the irreducible h- v– components (d–torsions),*

$$\begin{aligned} T^i{}_{jk} &= L^i{}_{[jk]},\ T^i{}_{ja} = -T^i{}_{aj} = C^i{}_{ja},\ T^a{}_{ji} = \Omega^a{}_{ji}, \\ T^a{}_{bi} &= T^a{}_{ib} = \frac{\partial N^a_i}{\partial y^b} - L^a{}_{bi},\ T^a{}_{bc} = C^a{}_{[bc]}. \end{aligned} \quad (1)$$

Proof. *By a straightforward calculation we can verify the formulas.*

The Levi–Civita linear connection $\nabla = \{{}^{\nabla}\mathbf{\Gamma}^{\alpha}_{\beta\gamma}\}$, with vanishing torsion and nonmetricity, is not adapted to the global splitting (4). One holds:

Proposition 2.3 *There is a preferred, canonical d–connection structure, $\widehat{\mathbf{\Gamma}}$, on V^{n+m}_N, constructed only from the metric coefficients $[g_{ij}, h_{ab}, N^a_i]$ and satisfying the conditions $\widehat{\mathbf{Q}}_{\alpha\beta} = 0$ and $\widehat{T}^i{}_{jk} = 0$ and $\widehat{T}^a{}_{bc} = 0$.*

Proof. *By straightforward calculations with respect to the N–adapted bases (2) and (3), we can verify that the connection*

$$\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma} = {}^{\nabla}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} + \widehat{\mathbf{P}}^{\alpha}_{\beta\gamma} \tag{2}$$

with the deformation d–tensor

$$\widehat{\mathbf{P}}^{\alpha}_{\beta\gamma} = (P^i_{jk} = 0, P^a_{bk} = \frac{\partial N^a_k}{\partial y^b}, P^i_{jc} = -\frac{1}{2} g^{ik}\Omega^a{}_{kj} h_{ca}, P^a_{bc} = 0)$$

satisfies the conditions of this Proposition. It should be noted that, in general, the components $\widehat{T}^i{}_{ja}$, $\widehat{T}^a{}_{ji}$ and $\widehat{T}^a{}_{bi}$ are not zero. This is an anholonomic frame (or, equivalently, off–diagonal metric) frame effect.

The torsion of the connection (2) is denoted $\widehat{\mathbf{T}}^{\alpha}_{\beta\gamma}$. In a similar form we can prove:

Theorem 2.4 *The curvature $\mathcal{R}^{\alpha}{}_{\beta} \doteqdot \mathbf{D}\mathbf{\Gamma}^{\alpha}_{\beta} = d\mathbf{\Gamma}^{\alpha}_{\beta} - \mathbf{\Gamma}^{\gamma}_{\beta} \wedge \mathbf{\Gamma}^{\alpha}_{\gamma}$ of a d–connection $\mathbf{\Gamma}^{\alpha}_{\gamma}$ has the irreducible h- v- components (d–curvatures),*

$$\begin{aligned}
R^i{}_{hjk} &= e_k L^i{}_{hj} - e_j L^i{}_{hk} + L^m{}_{hj} L^i{}_{mk} - L^m{}_{hk} L^i{}_{mj} - C^i{}_{ha}\Omega^a{}_{kj}, \\
R^a{}_{bjk} &= e_k L^a{}_{bj} - e_j L^a{}_{bk} + L^c{}_{bj} L^a{}_{ck} - L^c{}_{bk} L^a{}_{cj} - C^a{}_{bc}\Omega^c{}_{kj}, \\
R^i{}_{jka} &= e_a L^i{}_{jk} - D_k C^i{}_{ja} + C^i{}_{jb} T^b{}_{ka}, \\
R^c{}_{bka} &= e_a L^c{}_{bk} - D_k C^c{}_{ba} + C^c{}_{bd} T^c{}_{ka}, \\
R^i{}_{jbc} &= e_c C^i{}_{jb} - e_b C^i{}_{jc} + C^h{}_{jb} C^i{}_{hc} - C^h{}_{jc} C^i{}_{hb}, \\
R^a{}_{bcd} &= e_d C^a{}_{bc} - e_c C^a{}_{bd} + C^e{}_{bc} C^a{}_{ed} - C^e{}_{bd} C^a{}_{ec}.
\end{aligned} \tag{3}$$

Contracting the components of (3) we prove:

Corollary 2.5 *a) The Ricci d–tensor $\mathbf{R}_{\alpha\beta} \doteqdot \mathbf{R}^{\tau}{}_{\alpha\beta\tau}$ has the irreducible h-v–components*

$$R_{ij} \doteqdot R^k{}_{ijk}, \quad R_{ia} \doteqdot -R^k{}_{ika}, \; R_{ai} \doteqdot R^b{}_{aib}, \; R_{ab} \doteqdot R^c{}_{abc}. \tag{4}$$

b) The scalar curvature of a d–connection is

$$\overleftarrow{\mathbf{R}} \doteqdot \mathbf{g}^{\alpha\beta}\mathbf{R}_{\alpha\beta} = g^{ij}R_{ij} + h^{ab}R_{ab}.$$

c) The Einstein d–densor is computed $\mathbf{G}_{\alpha\beta} = \mathbf{R}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\overleftarrow{\mathbf{R}}$.

In modern gravity theories one considers more general linear connections generated by deformations of type $\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \widehat{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma} + \mathbf{P}^{\alpha}_{\beta\gamma}$.We can split all geometric objects into canonical and post-canonical pieces which results in N–adapted geometric constructions. For instance,

$$\mathcal{R}^{\alpha}_{\ \beta} = \widehat{\mathcal{R}}^{\alpha}_{\ \beta} + \mathbf{D}\mathcal{P}^{\alpha}_{\ \beta} + \mathcal{P}^{\alpha}_{\ \gamma} \wedge \mathcal{P}^{\gamma}_{\ \beta} \tag{5}$$

for $\mathcal{P}^{\alpha}_{\beta} = \mathbf{P}^{\alpha}_{\beta\gamma}\vartheta^{\gamma}$.

3. Anholonomic Frames and Nonmetricity in String Gravity

For simplicity, we investigate here a class of spacetimes when the nonmetricity and torsion have nontrivial components of type

$$\mathcal{T} \doteqdot \mathbf{e}_{\alpha}\rfloor\mathcal{T}^{\alpha} = \kappa_0\phi,\ \mathcal{Q} \doteqdot \frac{1}{4}\mathbf{g}^{\alpha\beta}\mathcal{Q}_{\alpha\beta} = \kappa_1\phi,\ \mathbf{\Lambda} \doteqdot \vartheta^{\alpha}\mathbf{e}^{\beta}\rfloor(\mathcal{Q}_{\alpha\beta} - \mathcal{Q}\mathbf{g}_{\alpha\beta}) = \kappa_2\phi \tag{1}$$

where $\kappa_0, \kappa_1, \kappa_2 = const$ and $\phi = \phi_{\alpha}\vartheta^{\alpha}$. The abstract indices in (1) are "upped" and "lowed" by using $\eta_{\alpha'\beta'}$ and its inverse defined from the vielbein decompositions of d–metric, $\mathbf{g}_{\alpha\beta} = \mathbf{e}^{\alpha'}_{\alpha}\mathbf{e}^{\beta'}_{\beta}\eta_{\alpha'\beta'}$.

Let us consider the strengths $\mathbf{H}_{\nu\mu} \doteqdot \widehat{\mathbf{D}}_{\nu}\phi_{\mu} - \widehat{\mathbf{D}}_{\mu}\phi_{\nu} + W^{\gamma}_{\mu\nu}\phi_{\gamma}$ (intensity of ϕ_{γ}) and $\widehat{\mathbf{H}}_{\nu\lambda\rho} \doteqdot \mathbf{e}_{\nu}\mathbf{B}_{\lambda\rho} + \mathbf{e}_{\rho}\mathbf{B}_{\nu\lambda} + \mathbf{e}_{\lambda}\mathbf{B}_{\rho\nu}$ (antysimmetric torsion of the $\mathbf{B}_{\rho\nu} = -\mathbf{B}_{\nu\rho}$ from the bosonic model of string theory with dilaton field Φ) and introduce

$$\begin{aligned} \mathbf{H}_{\nu\lambda\rho} &\doteqdot \widehat{\mathbf{Z}}_{\nu\lambda\rho} + \widehat{\mathbf{H}}_{\nu\lambda\rho}, \\ \widehat{\mathbf{Z}}_{\nu\lambda} &\doteqdot \widehat{\mathbf{Z}}_{\nu\lambda\rho}\vartheta^{\rho} = \mathbf{e}_{\lambda}\rfloor\widehat{\mathbf{T}}_{\nu} - \mathbf{e}_{\nu}\rfloor\widehat{\mathbf{T}}_{\lambda} + \frac{1}{2}(\mathbf{e}_{\nu}\rfloor\mathbf{e}_{\lambda}\rfloor\widehat{\mathbf{T}}_{\lambda})\vartheta^{\gamma}. \end{aligned}$$

We denote the energy–momentums of fields:

$$\mathbf{\Sigma}^{[\phi]}_{\alpha\beta} \doteqdot \mathbf{H}_{\alpha}^{\ \mu}\mathbf{H}_{\beta\mu} - \frac{1}{4}\mathbf{g}_{\alpha\beta}\mathbf{H}^{\nu\mu}\mathbf{H}_{\nu\mu} + \mu^2(\phi_{\alpha}\phi_{\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\phi_{\nu}\phi^{\nu}),$$

$\mu^2 = const$, $\mathbf{\Sigma}^{[mat]}_{\alpha\beta}$ is the source from any possible matter fields and $\mathbf{\Sigma}^{[T]}_{\alpha\beta}(\widehat{\mathbf{T}}_{\nu}, \Phi)$ contains contributions of torsion and dilatonic fields.

Theorem 3.1 *The dynamics of sigma model of bosonic string gravity with generic off–diagonal metrics, effective matter and torsion and nonmetricity (1) is defined by the system of field equations*

$$\begin{aligned} \widehat{\mathbf{R}}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\overleftarrow{\widehat{\mathbf{R}}} &= k(\mathbf{\Sigma}^{[\phi]}_{\alpha\beta} + \mathbf{\Sigma}^{[mat]}_{\alpha\beta} + \mathbf{\Sigma}^{[T]}_{\alpha\beta}), \\ \widehat{\mathbf{D}}_{\nu}\mathbf{H}^{\nu\mu} &= \mu^2\phi^{\mu},\ \widehat{\mathbf{D}}^{\nu}(\mathbf{H}_{\nu\lambda\rho}) = 0, \end{aligned} \tag{2}$$

where $k = const$, and the Euler–Lagrange equations for the matter fields are considered on background V_N^{n+m}).

Proof. *See details in Ref. [2].*

In terms of differential forms the eqs. (2) are written

$$\eta_{\alpha\beta\gamma} \wedge \widehat{\mathcal{R}}^{\beta\gamma} = \widehat{\Upsilon}_\alpha, \qquad (3)$$

where, for the volume 4–form $\eta \doteqdot *1$ with the Hodje operator "$*$", $\eta_\alpha \doteqdot \mathbf{e}_\alpha \rfloor \eta$, $\eta_{\alpha\beta} \doteqdot \mathbf{e}_\beta \rfloor \eta_\alpha$, $\eta_{\alpha\beta\gamma} \doteqdot \mathbf{e}_\gamma \rfloor \eta_{\alpha\beta}, ...,$ $\widehat{\mathcal{R}}^{\beta\gamma}$ is the curvature 2–form and Υ_α denote all possible sources defined by using the canonical d–connection. The deformation of connection (2) defines a deformation of the curvature tensor of type (5) but with respect to the curvature of the Levi–Civita connection, ${}^{\nabla}\mathcal{R}^{\beta\gamma}$. The gravitational field equations (3) transforms into

$$\eta_{\alpha\beta\gamma} \wedge {}^{\nabla}\mathcal{R}^{\beta\gamma} + \eta_{\alpha\beta\gamma} \wedge {}^{\nabla}\mathcal{Z}^{\beta\gamma} = \widehat{\Upsilon}_\alpha, \qquad (4)$$

where ${}^{\nabla}\mathcal{Z}^{\beta}{}_{\gamma} = \nabla \mathcal{P}^{\beta}{}_{\gamma} + \mathcal{P}^{\beta}{}_{\alpha} \wedge \mathcal{P}^{\alpha}{}_{\gamma}$.

Corollary 3.2 *A subclass of solutions of the gravitational field equations for the canonical d–connection defines also solutions of the Einstein equations for the Levi–Civita connection if and only if $\eta_{\alpha\beta\gamma} \wedge {}^{\nabla}\mathcal{Z}^{\beta\gamma} = 0$ and $\widehat{\Upsilon}_\alpha = {}^{\nabla}\Upsilon_\alpha$, (i. e. the effective source is the same for both type of connections).*

Proof. It follows from the Theorem 3.1.

This property is very important for constructing exact solutions in Einstein and string gravity, parametrized by generic off–diagonal metrics and ahnolonomic frames with associated N–connection structure (see Refs. in [1, 2] and [3]) and equations (4)).

Let us consider a five dimensional ansatz for the metric (1) and frame (2) when $u^\alpha = (x^i, y^4 = v, y^5); i = 1, 2, 3$ and the coefficients

$$\begin{aligned} g_{ij} &= diag[g_1 = \pm 1, g_2(x^2, x^3), g_3(x^2, x^3)], h_{ab} = diag[h_4(x^k, v), h_5(x^k, v)], \\ N_i^4 &= w_i(x^k, v), N_i^5 = n_i(x^k, v) \end{aligned} \qquad (5)$$

are some functions of necessary smooth class. The partial derivative are briefly denoted $a^\bullet = \partial a/\partial x^2, a^{'} = \partial a/\partial x^3, a^* = \partial a/\partial v$.

4. Main results:

Theorem 4.1 *The nontrivial components of the Ricci d–tensors (4) for the canonical d–connection (2) are*

$$
\begin{aligned}
R_2^2 &= R_3^3 = -\frac{1}{2g_2g_3}[g_3^{\bullet\bullet} - \frac{g_2^{\bullet}g_3^{\bullet}}{2g_2} - \frac{(g_3^{\bullet})^2}{2g_3} + g_2^{''} - \frac{g_2^{'}g_3^{'}}{2g_3} - \frac{(g_2^{'})^2}{2g_2}], \\
R_4^4 &= R_5^5 = -\frac{1}{2h_4h_5}[h_5^{**} - h_5^{*}(\ln\left|\sqrt{|h_4h_5|}\right|)^{*}], \qquad (1)\\
R_{4i} &= -w_i\frac{\beta}{2h_5} - \frac{\alpha_i}{2h_5},\ R_{5i} = -\frac{h_5}{2h_4}[n_i^{**} + \gamma n_i^{*}],
\end{aligned}
$$

$\alpha_i = \partial_i h_5^{*} - h_5^{*}\partial_i \ln\left|\sqrt{|h_4h_5|}\right|$, $\beta = h_5^{**} - h_5^{*}[\ln\left|\sqrt{|h_4h_5|}\right|]^{*}$, $\gamma = 3h_5^{*}/2h_5 - h_4^{*}/h_4$ $h_4^{*} \neq 0$ *and* $h_5^{*} \neq 0$.

Proof. It is provided in Ref. [2].

Corollary 4.2 *The Einstein equations (3) for the ansatz (5) are compatible for vanishing sources and if and only if the nontricial components of the source, with respect to the frames (3) and (2), are any functions of type*

$$\widehat{\Upsilon}_2^2 = \widehat{\Upsilon}_3^3 = \Upsilon_2(x^2, x^3, v),\ \widehat{\Upsilon}_4^4 = \widehat{\Upsilon}_5^5 = \Upsilon_4(x^2, x^3)\ \textit{and}\ \widehat{\Upsilon}_1^1 = \Upsilon_2 + \Upsilon_4.$$

Proof. *The proof, see details in [2], follows from the Theorem 4.1 with the nontrivial components of the Einstein d-tensor,* $\widehat{\mathbf{G}}^{\alpha}{}_{\beta} = \widehat{\mathbf{R}}^{\alpha}{}_{\beta} - \frac{1}{2}\delta^{\alpha}{}_{\beta}\overleftarrow{\hat{\mathbf{R}}}$, *computed to satisfy the conditions*

$$G_1^1 = -(R_2^2 + R_4^4), G_2^2 = G_3^3 = -R_4^4(x^2, x^3, v), G_4^4 = G_5^5 = -R_2^2(x^2, x^3).$$

Having the values (1), we can proove [2] the

Theorem 4.3 *The system of gravitational field equations (2) (equivalently, of (3)) for the ansatz (5) can be solved in general form if there are given certain values of functions* $g_2(x^2, x^3)$ *(or, inversely,* $g_3(x^2, x^3)$*),* $h_4(x^i, v)$ *(or, inversely,* $h_5(x^i, v)$*) and of sources* $\Upsilon_2(x^2, x^3, v)$ *and* $\Upsilon_4(x^2, x^3)$.

Finally, we note that we have elaborated a new geometric method of constructing exact solutions in extra dimension gravity and general relativity theories. The classes of solutions define very general integral varieties of the vacuum and nonvacuum Einstein equations, in general, with torsion and nonmetricity and corrections from string theory, and/or noncommutative/quantum variables. For instance, in five dimensions, the metrics are generic off–diagonal and depend on four coordinates. So, we have proved in explicit form how it is possible to solve the Einstein equations on nonholonomic manifolds (see mathematical problems analized in Refs. [22, 23]), in our case, provided with nonlinear connection structure.

Acknowledgement: The author is grateful to the Organizers for supporting his participation at the Conference “Geometría de Lorentz, Murcia 2003, 12-14 de Noviembre”, partial support from “Consejería de Educación de la Junta de Andalucía” and the “Grupo de Estructura de la Materia” of the University of Huelva for his kind hospitality. He also thanks the referee for valuable remarks and D. Leites, M. de León and F. Etayo for important discussions on the geometry of nonholonomic manifolds and nonlinear connection formalism.

References

[1] S. VACARU, Generalized Finsler Geometry in Einstein, String and Metric–Affine Gravity, hep-th/0310132.

[2] S. VACARU, E. Gaburov and D. Gontsa, A Method of Constructing Off–Diagonal Solutions in Metric–Affine and String Gravity, hep–th/0310133.

[3] S. VACARU AND E. GABUROV, Noncomnmutative Symmetries and Stability of Black Ellipsoids in Metric–Affine and String Gravity, hep–th/0310134.

[4] E. CARTAN, Les Espaces de Finsler (Herman, Paris, 1935)

[5] A. KAWAGUCHI, Beziehung zwischen einer metrischen linearen Ubertangung and einer nichtmetrischen in einem allgemeinen metrischen Raume. Akad. Wetensch. Amsterdam Proc. 40 (1937) 596–601.

[6] A. KAWAGUCHI, On the Theory of Non–linear Connections, I. Tensor, N. S. 2 (1952) 123–142; II. Tensor, N. S. 6 (1956) 165–199.

[7] C. EHRESMANN, Les connexions infinitesimales dans un espace fibre differentiable. Coll. Topologia, Bruxelles (1955) 29–55.

[8] W. BARTHEL, Nichtlineare Zusammenhange und deren Holonomie gruppen. J. Reine Angew. Math. 212 (1963) 120–149.

[9] R. MIRON, D. HRIMIUC, H. SHIMADA AND V. S. SABAU, The Geometry of Hamilton and Lagrange Spaces (Kluwer, 2000).

[10] M. DE LEON AND C. VILLAVERDE, C. R. Acad. Sci. Paris, serie I, 293 (1981) 51–54.

[11] F. ETAYO, The geometry of good squares of vector bundles, Riv. Mat. Univ. Parma 17 (1991) 131–147.

[12] A. FERRÁNDEZ, Nearly-Kaehler curvature operators. Rev. Roumaine Math. Pures Appl. 29 (1984), no. 3, 225–229

[13] S. VACARU, Spinor Structures and Nonlinear Connections in Vector Bundles, Generalized Lagrange and Finsler Spaces, J. Math. Phys. 37 (1996) 508-523.

[14] S. VACARU AND N. VICOL, Nonlinear Connections and Spinor Geometry, Int. J. Math. and Math. Sciences. (IJMMS), 23 (2004) 1189-1237.

[15] S. VACARU, Interactions, Strings and Isotopies in Higher Order Anisotropic Superspaces (Hadronic Press, Palm Harbor, FL, USA, 1998), 450 pages, math-ph/0112065.

[16] S. VACARU, Gauge and Einstein Gravity from Non-Abelian Gauge Models on Noncommutative Spaces, Phys. Lett. B 498 (2001) 74-82.

[17] S. VACARU, Anholonomic Soliton-Dilaton and Black Hole Solutions in General Relativity, JHEP, 04 (2001) 009; gr-qc/0005025.

[18] S. VACARU, Locally Anisotropic Kinetic Processes and Thermodynamics in Curved Spaces, Ann. Phys. (N.Y.), 290, N 2 (2001) 83-123.

[19] S. VACARU AND O. TINTAREANU-MIRCEA, Anholonomic Frames, Generalized Killing Equations, and Anisotropic Taub NUT Spinning Spaces, Nucl. Phys. B 626 (2002) 239-264.

[20] S. VACARU AND D. SINGLETON, Warped, Anisotropic Wormhole / Soliton Configurations in Vacuum 5D Gravity, Class. Quant. Gravity, 19 (2002), 2793-2811.

[21] S. VACARU AND H. DEHNEN, Locally Anisotropic Structures and Nonlinear Connections in Einstein and Gauge Gravity, Gen. Rel. Grav. 35 (2003) 209-250.

[22] D. LEITES, E.POLETAEVA, V.SERGANOVA, On Einstein equations on manifolds and supermanifolds, J.Nonlin.Math.Phys. 9 (2002) 394-425.

[23] D. LEITES, The Riemann tensor for nonholonomic manifolds, math.RT/0202213.

Proceedings of the II International
Meeting on Lorentzian Geometry
Murcia, November 12–14, 2003
Publ. de la RSME, Vol. 8 (2004), 113–118

Einstein-Rosen waves and microcausality

J. Fernando Barbero G.[1], Guillermo A. Mena Marugán[2]
and Eduardo J. S. Villaseñor[3]

[1] *Instituto de Matemáticas y Física Fundamental, CSIC, Serrano 113bis, 28006 Madrid, Spain*
[2] *Instituto de Estructura de la Materia, CSIC, Serrano 121, 28006 Madrid, Spain*
[3] *Departamento de Matemáticas, Universidad Carlos III de Madrid, Avd. de la Universidad 30, 28911 Leganés, Spain*

Abstract

Einstein-Rosen waves provide a non trivial, but tractable, symmetry reduction of general relativity with local degrees of freedom. It is described by an axially symmetric scalar field evolving in an auxiliary Minkowskian background. We discuss the use of commutators of this scalar field to study the quantum causal structure of space-time, taking into account the fact that the physical Hamiltonian of the system is a *bounded* function of a free field Hamiltonian. We will show that these commutators have the features corresponding to interacting theories and study, in a quantitative way, the smearing of light cones at the Planck scale and the semiclassical limit of the model.

Keywords: Einstein-Rosen waves, microcausality, asymptotic analysis.
2000 Mathematics Subject Classification: 83C45, 83C47, 81T99

1. Introduction

The main obstacles behind the unfruitful attempts to quantize general relativity are usually attributed to two properties of the classical theory. First, the absence of background structures makes it a formidable task to construct the quantum algebra of observables. Second, general relativity is a field theory so its quantum counterpart must belong to the class of quantum field theories, with all the subtle functional problems inherent to it.

In order to deal with these issues, several families of solutions to the Einstein equations have been discussed in the literature. In particular, Bianchi models have been used to explore the problems related to the algebra of observables [1]. However, the phase space of Bianchi models is finite dimensional, so the field-theoretical character of general relativity is absent in this approach. To avoid this, we are forced to consider genuine field theories which do not require a background space-time metric. In this sense, the Einstein-Rosen waves [2, 3, 4] provide a valuable toy model to explore quantum gravity effects because the degrees of freedom are described by an axially symmetric scalar field. In this paper, we will show how to compute, in a quantitative way, the smearing, at the Planck scale, of light cones of the Einstein-Rosen space-times. Throughout the paper, we set $c = \hbar = 1$ and denote $G = \hbar G_3$, with G_3 the gravitational constant per unit length in the axis direction.

2. Classical Einstein-Rosen vaves

Einstein-Rosen waves, also known as linearly polarized cylindrical gravitational waves, are space-times whose metric $\mathbf{g}$ satisfies the vacuum Einstein equations and admit a two-parametric, Abelian, and orthogonally transitive group of isometries [3]. These space-times have two linearly independent, spacelike Killing fields, one of them axial, $\xi_{(\theta)}$, and the other translational, $\xi_{(Z)}$. The Killing fields are assumed to be mutually orthogonal, $\mathbf{g}(\xi_{(Z)}, \xi_{(\theta)}) = 0$, commuting, $[\xi_{(\theta)}, \xi_{(Z)}] = 0$, and generate a space-time foliation orthogonal to the isometry orbits. Under these hypotheses it is possible to choose a system of symmetry-adapted coordinates $(t, R, \theta, Z) \in \mathbb{R} \times \mathbb{R}^+ \times [0, 2\pi) \times \mathbb{R}$ such that the line element takes the form

$$ds^2 = e^{-\psi}\left[-N^2 dt^2 + e^{\gamma}(dR + N^R dt)^2 + r^2 d\theta^2\right] + e^{\psi} dZ^2 .$$

The smooth fields ψ, γ, r, N (lapse), and N^R (radial shift) depend only on the (t, R) coordinates and satisfy suitable boundary conditions at the symmetry axis $R = 0$ and at spatial infinity $R \to \infty$ [2, 3].

After the gauge fixing [5] given by $r(t, R) = R$, $\pi_\gamma(t, R) = 0$ (π_γ is the canonical momentum of the γ field), and introducing the new time coordinate $T = e^{-\gamma_\infty/2} t$, where $\gamma_\infty = \lim_{R\to\infty} \gamma$, the line element takes the form

$$ds^2_{\text{gf}} = e^{\gamma-\psi}(-dT^2 + dR^2) + e^{-\psi} R^2 d\theta^2 + e^{\psi} dZ^2 .$$

Then, it is straightforward to show that the Einstein equations are equivalent to the equation of a free, massless, cylindrically symmetric scalar field evolving in a Minkowski space-time

$$\frac{\partial^2\psi}{\partial T^2} - \frac{\partial^2\psi}{\partial R^2} - \frac{1}{R}\frac{\partial\psi}{\partial R} = 0\,. \tag{1}$$

Thus, using the T-time and imposing regularity at the origin $R = 0$ [2], the classical solutions for the field ψ can be expanded in the form (J_0 is the zero order Bessel function)

$$\psi(R,T) = \sqrt{4G}\int_0^\infty J_0(Rk)[A(k)e^{-ikT} + A^\dagger(k)e^{ikT}]\,dk\,,$$

with an associated free Hamiltonian

$$\gamma_\infty = H_0 = \int_0^\infty kA^\dagger(k)A(k)\,dk\,.$$

Nevertheless, to reach a unit asymptotic timelike Killing vector field in the actual four-dimensional spacetime, with respect to which one can truly introduce a physical notion of energy per unit length [6], one must make use of the original system of coordinates, namely (t, R, θ, Z). In these coordinates the solutions take the form

$$\psi_E(R,t) = \sqrt{4G}\int_0^\infty J_0(Rk)[A(k)e^{-ikte^{-\gamma_\infty/2}} + A^\dagger(k)e^{ikte^{-\gamma_\infty/2}}]\,dk$$

and the true physical Hamiltonian is given by

$$H = E(H_0) = \frac{1}{4G}\left(1 - e^{-4GH_0}\right)\,.$$

Notice that the physical Hamiltonian is bounded both from above and below. This fact leaves a strong footprint on microcausality.

3. Quantum Einstein-Rosen waves: microcausality

Owing to the linear character of the field equation (1), the quantization of the field ψ can be carried out in a standard way (technical details can be found in [2, 4]). If we use the Schrödinger picture, operators do not evolve, and we can introduce a Fock space in which the quantum counterpart of $\psi(R,0) = \psi_E(R,0)$ is an operator-valued distribution $\hat\psi(R,0) = \hat\psi_E(R,0)$. Its action is determined by those of $\hat A(k)$ and $\hat A^\dagger(k)$, the usual annihilation and creation operators, whose only non-vanishing commutators are

$$[\hat A(k_1), \hat A^\dagger(k_2)] = \delta(k_1 - k_2).$$

In terms of these operators

$$\hat{\psi}(R,0) = \hat{\psi}_E(R,0) = \sqrt{4G}\int_0^\infty J_0(Rk)\left[\hat{A}(k)+\hat{A}^\dagger(k)\right]\,dk.$$

In constrast, in the Heisenberg picture, operators change in time. In this case we can consider the evolution of the quantum field $\hat{\psi}$ from its value at $T = t = 0$ given by the two Hamiltonians that appear in the problem

$$\hat{H}_0 = \int_0^\infty k\hat{A}^\dagger(k)\hat{A}(k)\,dk\,, \quad \hat{H} = E(\hat{H}_0) = \frac{1}{4G}\left(1 - e^{-4G\hat{H}_0}\right).$$

The physical Hamiltonian $\hat{H}$ is related to the physical time t, and the auxiliary massless scalar field Hamiltonian $\hat{H}_0$ is associated with the time T of the auxiliary three-dimensional Minkowski spacetime. The time evolution is provided by the unitary operators $\hat{U}_0(T) = \exp(-iT\hat{H}_0)$ and $\hat{U}(t) = \exp(-it\hat{H})$ according to

$$\hat{\psi}(R,T) = \hat{U}_0^\dagger(T)\hat{\psi}(R,0)\hat{U}_0(T) = \sqrt{4G}\int_0^\infty J_0(Rk)[\hat{A}(k)e^{-ikT} + \hat{A}^\dagger(k)e^{ikT}]dk,$$

$$\hat{\psi}_E(R,t) = \hat{U}^\dagger(t)\hat{\psi}_E(R,0)\hat{U}(t) = \sqrt{4G}\int_0^\infty J_0(Rk)[\hat{A}_E(k,t) + \hat{A}_E^\dagger(k,t)]dk,$$

where

$$\hat{A}_E(k,t) := \exp\left[-itE(k)e^{-4G\hat{H}_0}\right]\hat{A}(k), \quad \hat{A}_E^\dagger(k,t) := \hat{A}^\dagger(k)\exp\left[itE(k)e^{-4G\hat{H}_0}\right].$$

We will show now how the field commutators of the scalar field ψ, that encodes the physical information in linearly polarized cylindrical waves, can be used as an alternative to the metric operator to extract physical information about quantum spacetime. In particular we will see how the light cones get smeared by quantum corrections in a precise and quantitative way.

Let us start by computing the field commutator for the free Hamiltonian:

$$[\hat{\psi}(R_1,T_1),\hat{\psi}(R_2,T_2)] = 8iG\int_0^\infty J_0(R_1k)J_0(R_2k)\sin[(T_2-T_1)k]\,dk\,. \qquad (1)$$

This commutator is a c-number (proportional to the identity operator in the Fock space) and shows the typical light cone structure found in standard perturbative quantum field theories. It is easy to prove that $[\hat{\psi}(R_1,T_1),\hat{\psi}(R_2,T_2)]$ vanishes outside the light cone, i.e. when $0 < |T_2 - T_1| < |R_2 - R_1|$ (Region I). Inside the light cone $[\hat{\psi}(R_1,T_1),\hat{\psi}(R_2,T_2)]$ can be written in terms of the complete elliptic integrals

$$\mathbf{E}(k) := \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}\,d\theta\,, \quad \mathbf{K}(k) := \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\,.$$

Explicitly, in the region $|R_2 - R_1| < |T_2 - T_1| < R_2 + R_1$ (Region II) we get

$$[\hat{\psi}(R_1,T_1),\hat{\psi}(R_2,T_2)] = \frac{i}{\pi}\frac{8G}{\sqrt{R_1R_2}}\mathbf{K}\left(\sqrt{\frac{(T_2-T_1)^2-(R_2-R_1)^2}{4R_1R_2}}\right).$$

Finally, its value in the region $R_1 + R_2 < |T_2 - T_1|$ (Region III) is

$$[\hat{\psi}(R_1,T_1),\hat{\psi}(R_2,T_2)]=\frac{16Gi}{\sqrt{\pi^2[(T_2-T_1)^2-(R_2-R_1)^2]}}\mathbf{K}\left(\sqrt{\frac{4R_1R_2}{(T_2-T_1)^2-(R_2-R_1)^2}}\right).$$

On the other hand, the commutator for the physical Hamiltonian, as it usually happens for interacting theories, is no longer a c-number, so one has to consider its matrix elements. We will focus here on the vacuum expectation value (v.e.v.)

$$\begin{aligned}&\langle 0|[\hat{\psi}_E(R_1,t_1),\hat{\psi}_E(R_2,t_2)]|0\rangle\\ &\qquad = 8iG\int_0^\infty J_0(R_1k)J_0(R_2k)\sin\left[\frac{t_2-t_1}{4G}\left(1-e^{-4Gk}\right)\right]dk\,. \qquad (2)\end{aligned}$$

In order to relate this v.e.v. to the free commutator (1), we will consider the semiclassical limit corresponding to radial distances (and time intervals) much larger than the Planck scale. To do this, it is convenient to introduce the dimensionless parameters τ, ρ, and λ

$$R_2 = \rho R_1\,,\quad t_2 - t_1 = \tau R_1\,,\quad \lambda = \frac{R_1}{4G}\,.$$

Then, it is possible to show [7] that the asymptotic behavior of (2) when $\lambda \to \infty$ (semiclassical limit) in the different regions of the (ρ,τ) plane is given by:

• If $\rho = 0$

$$\frac{2i}{\lambda}\Im\left[\frac{i}{\sqrt{\tau^2-1}}+\frac{e^{i\lambda[\tau-\log\tau-1]}}{\sqrt{\log\tau}}\right]+O\left(\frac{1}{\lambda^2}\right),\quad \text{if}\quad \tau>1\,,$$

$$\frac{i}{\lambda^2}\frac{\tau(1+2\tau^2)}{(1-\tau^2)^{5/2}}+O\left(\frac{1}{\lambda^3}\right),\quad \text{if}\quad 0\le\tau<1\,.$$

• If $\rho \neq 0$

○ Region I

$$\begin{aligned}&\frac{-2i\tau}{\pi\lambda^2\sqrt{(1+\rho)^2-\tau^2}}\left\{\frac{\tau^2}{[\rho^4+(\tau^2-1)^2-2(1+\tau^2)\rho^2]}\mathbf{K}\left(\sqrt{\frac{4\rho}{(1+\rho)^2-\tau^2}}\right)\right.\\ &\left.-\frac{[1+\rho^4+2\tau^2-3\tau^4+2\rho^2(\tau^2-1)]}{[(1+\rho)^2-\tau^2](1-\rho+\tau)^2(\rho-1+\tau)^2}\mathbf{E}\left(\sqrt{\frac{4\rho}{(1+\rho)^2-\tau^2}}\right)\right\}+O\left(\frac{1}{\lambda^3}\right).\end{aligned}$$

∘ Region II

$$\frac{2i}{\pi\lambda\sqrt{\rho}}\mathbf{K}\left(\sqrt{\frac{\tau^2-(\rho-1)^2}{4\rho}}\right)+\frac{2i}{\lambda}\Im\left[\frac{e^{-\frac{\pi i}{4}}e^{i\lambda[\tau+|\rho-1|(1+\log\frac{\tau}{|1-\rho|})]}}{\sqrt{2\pi\lambda\rho|1-\rho|\log\frac{\tau}{|1-\rho|}}}\right]+O\left(\frac{1}{\lambda^{5/2}}\right).$$

∘ Region III

$$\frac{4i}{\pi\lambda\sqrt{\tau^2-(1-\rho)^2}}\mathbf{K}\left(\sqrt{\frac{4\rho}{\tau^2-(1-\rho)^2}}\right)$$
$$+\frac{2i}{\lambda}\Im\left[\frac{e^{-\frac{\pi i}{4}}e^{i\lambda[\tau-|\rho-1|(1+\log\frac{\tau}{|1-\rho|})]}}{\sqrt{2\pi\lambda\rho|1-\rho|\log\frac{\tau}{|1-\rho|}}}+\frac{e^{\frac{\pi i}{4}}e^{i\lambda[\tau+(1+\rho)(\log\frac{1+\rho}{\tau}-1)]}}{\sqrt{2\pi\lambda\rho(1+\rho)\log\frac{\tau}{1+\rho}}}\right]+O\left(\frac{1}{\lambda^{5/2}}\right).$$

The terms proportional to $1/\lambda$ in the above asymptotic expansions correspond to the commutator obtained from the free Hamiltonian $\hat{H}_0$, both in the axis $\rho = 0$ and outside the axis. The remaining terms are corrections to this free commutator that fall off to zero faster than $1/\lambda$ (except when $\rho = 0$ and $\tau > 1$), and have an additional, non-perturbative dependence in λ (different from an inverse power dependence). Inasmuch as the free commutator defines a characteristic light cone structure, these terms are responsible for the smearing of the light cones of the model.

Acknowledgments

This work was supported by the Spanish MCYT under the research projects BFM2001-0213 and BFM2002-04031-C02-02.

References

[1] S. Carlip, *Knots and Quantum Gravity*, edited by J. Baez, Clarendon Press, Oxford (1994); A. Ashtekar, R. S. Tate, and C. Uggla, *Int. J. Mod. Phys.* **D2** (1993), 15; N. Manojlovic and G. A. Mena Marugán, *Phys. Rev.* **D48** (1993), 3704.

[2] A. Ashtekar and M. Pierri, *J. Math. Phys.* **37** (1996), 6250.

[3] J. D. Romano and C. G. Torre, *Phys. Rev.* **D53** (1996), 5634.

[4] J. F. Barbero G., G. A. Mena Marugán, and E. J. S. Villaseñor, *Phys. Rev.* **D67** (2003), 124006.

[5] M. E. Angulo and G. A. Mena Marugán, *Int. J. Mod. Phys.* **D9** (2000), 669.

[6] A. Ashtekar and M. Varadarajan, *Phys. Rev.* **D50** (1994), 4944.

[7] J. F. Barbero G., G. A. Mena Marugán, and E. J. S. Villaseñor, submited to Journal of Mathematical Physics.

Posters

Proceedings of the II International
Meeting on Lorentzian Geometry
Murcia, November 12–14, 2003
Publ. de la RSME, Vol. 8 (2004), 121–126

Pointwise Osserman four-dimensional manifolds with local structure of twisted product

Miguel Brozos-Vázquez, Eduardo García-Río
and Ramón Vázquez-Lorenzo

Department of Geometry and Topology, Faculty of Mathematics, Univ. of Santiago de Compostela, 15782 Santiago de Compostela (Spain)

emails: mbrozos@usc.es, xtedugr@usc.es, ravazlor@usc.es

Abstract

Four-dimensional pointwise Osserman manifolds with a local structure of twisted product are analyzed, showing that they must be necessarily of constant sectional curvature.

Keywords: Jacobi operator, Osserman manifold, twisted product
2000 Mathematics Subject Classification: 53C50

1. Introduction

The classification of pointwise Osserman (semi)-Riemannian manifolds in dimension four is still an open problem. In this work we deal with such manifolds assuming they have a local structure of twisted product. Under this assumption we show that the pointwise Osserman condition is equivalent to the constancy of the sectional curvature. Since in any pointwise Osserman manifold there is a local choice of orientation such that the manifold is Einstein self-dual (or Einstein anti-self-dual), the result (cf. Theorem 4.2) is obtained as a consequence of two facts. On the one hand, if the dimension of

[0]Work partially supported by project BFM2003-02949 Spain.

the fibre of the twisted product is greater than one then the Einstein condition lets us reduce the twisted product structure to a warped product one [4]. On the other hand, self-duality and anti-self-duality are equivalent for four-dimensional semi-Riemannian manifolds which are locally a twisted product with fibre of dimension different from two (cf. Theorem 3.1), and also for those which are locally a warped product, even if the fibre is two-dimensional (cf. Theorem 4.1).

2. Preliminaries

First we fix some notation and criteria to be used in what follows. Let (M, g) be an n-dimensional semi-Riemannian manifold with Levi-Civita connection ∇. The Riemannian curvature tensor R is the $(1, 3)$-tensor field on M defined by $R(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$, for all vector fields $X, Y, Z \in \mathfrak{X}(M)$. The Ricci curvature is the contraction of the curvature tensor given by $Ric(X, Y) = Tr\{U \rightsquigarrow R(X, U)Y\}$, for all $X, Y \in \mathfrak{X}(M)$. Finally, the scalar curvature is obtained by contracting the Ricci tensor, $Sc = Tr(Ric)$.

2.1. Warped and twisted product metrics

Let (B, g_B) and (F, g_F) be semi-Riemannian manifolds and $P = B \times F$ the product manifold. The product metric tensor on P is given by $\pi^*(g_B)+\sigma^*(g_F)$, where π and σ denote the projections of $B \times F$ onto B and F, respectively. Considering a smooth real-valued function $f > 0$ on B, the *warped product* $P = B \times_f F$ is the product manifold $B \times F$ endowed with the metric tensor

$$g = \pi^*(g_B) + (f \circ \pi)^2\, \sigma^*(g_F).$$

It is interesting to emphasize that warped products appear quite naturally as open dense submanifolds of certain complete manifolds (see [2] for more details).

Now, if the function $f > 0$ is defined on the product manifold $B \times F$, the resulting metric $g = \pi^*(g_B)+f^2\sigma^*(g_F)$ is called the *twisted product* $P = B\times_f F$ [9]. In any of the cases above, the factors (B, g_B) and $(F g_F)$ are referred as the *base* and the *fibre* of the product structure. The differences among the product structures above can be expressed in terms of the geometry of the leaves and the fibres of the product as in [9]. For our purpose here, the following criteria for a twisted product to be warped is of interest

Lemma 2.1 [4] *Let $P = B \times_f F$ be a twisted product of semi-Riemannian manifolds (B, g_B) and (F, g_F), with $\dim F > 1$, such that P is Einstein. Then it is possible to write P as a warped product of (B, g_B) and $(F, \widetilde{g_F})$, where $\widetilde{g_F}$ is a metric tensor conformally equivalent to g_F.*

2.2. The Weyl tensor and curvature decomposition

The curvature tensor R of any semi-Riemannian manifold can be decomposed as $R = U \oplus Z \oplus W$, [7], where

$$\begin{aligned} U &= \tfrac{S_c}{n(n-1)} R^0 \\ Z &= \tfrac{1}{n-2}\left(Ric - \tfrac{S_c}{n} g\right) \bullet g \\ W &= R - U - Z = R - \tfrac{1}{n-2}\left(Ric - \tfrac{S_c}{2(n-1)}\, g\right) \bullet g. \end{aligned}$$

Here W denotes the *Weyl tensor* of the semi-Riemannian manifold and $\bullet$ expresses the Kulkarni-Nomizu product of symmetric $(0,2)$-tensors, defined by:

$$\begin{aligned} (A \bullet B)(X,Y,Z,W) &= A(X,Z)B(Y,W) + A(Y,W)B(X,Z) \\ &\quad -A(X,W)B(Y,Z) - A(Y,Z)B(X,W). \end{aligned}$$

The manifold is said to be *Einstein* if $Z = 0$ and *locally conformally flat* if $W = 0$. In what follows we will use the following

Lemma 2.2 [7] *Let (M,g) be a semi-Riemannian manifold. (M,g) has constant sectional curvature if and only if $Z = W = 0$, i.e., (M,g) is Einstein and locally conformally flat.*

2.3. Pointwise Osserman and self-duality/anti-self-duality

Let (M,g) be a semi-Riemannian manifold and denote by $S^{\pm}(M)$ the bundle of unit tangent vectors. The *Jacobi operator* associated to $X \in S^{\pm}(M)$ is defined by

$$\mathcal{J}(X) : Y \in \langle X\rangle^{\perp} \mapsto R(X,Y)X \in \langle X\rangle^{\perp}.$$

(M,g) is said to be *Osserman* if the eigenvalues of the Jacobi operator $\mathcal{J}$ are constant on $S^{\pm}(M)$ and it is said to be *pointwise Osserman* if the eigenvalues of $\mathcal{J}$ are constant on $S^{\pm}(T_pM)$ for all $p \in M$, but possibly changing from point to point (cf. [5]).

In what follows (M,g) is assumed to be 4-dimensional and let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal local frame. Denoting by Λ the space of 2-forms, i.e., $\Lambda = \langle\{e^i \wedge e^j : i,j \in \{1,2,3,4\},\ i<j\}\rangle$, the Hodge operator $\star : \Lambda \longrightarrow \Lambda$ is defined by:

$$e^i \wedge e^j \wedge \star(e^k \wedge e^l) = (\delta^i_k \delta^j_l - \delta^i_l \delta^j_k)\, \varepsilon_i \varepsilon_j\, e^1 \wedge e^2 \wedge e^3 \wedge e^4,$$

where $\varepsilon_i = \langle e_i, e_i\rangle$. Now, note that the Hodge operator satisfies $\star^2 = id$ if the metric signature is $(++++)$ or $(++--)$ and it is a complex structure on Λ for Lorentzian signature. Let Λ^+ and Λ^- be the eigenspaces corresponding to the eigenvalues 1 and -1, respectively, that is,

$$\Lambda = \Lambda^+ \oplus \Lambda^-, \quad \Lambda^+ = \{\alpha \in \Lambda : \star\alpha = \alpha\}, \quad \Lambda^- = \{\alpha \in \Lambda : \star\alpha = -\alpha\},$$

and denote by $W^\pm$ the restriction of W to the spaces $\Lambda^\pm$. Viewing the curvature tensor R as $R : \Lambda \longrightarrow \Lambda$, we say that R is *self-dual* (resp., *anti-self-dual*) if $W^- = 0$ (resp., $W^+ = 0$).

Theorem 2.3 [5] *Let (M, g) be a 4-dimensional semi-Riemannian manifold. Then the following statements are equivalent:*

– *(M, g) is pointwise Osserman.*

– *Locally there is a choice of orientation for M such that it is Einstein self-dual (or Einstein anti-self-dual).*

It is worth to emphasize at this point that a pointwise Osserman manifold is necessarily of constant curvature in Lorentzian signature [5]. Furthermore note that self-duality and anti-self-duality are also equivalent for Lorentzian manifolds.

3. Four-dimensional twisted product manifolds

Let $P = B \times_f F$ be a 4-dimensional twisted product and $\{e_1, e_2, e_3, e_4\}$ be an orthonormal local frame. Now local basis of the spaces of self-dual and anti-self-dual 2-forms can be constructed as

$$\Lambda^\pm = \langle \{E_1^\pm, E_2^\pm, E_3^\pm\} \rangle$$

where

$$E_1^\pm = \frac{e^1 \wedge e^2 \pm \varepsilon_3\varepsilon_4 e^3 \wedge e^4}{\sqrt{2}},\ E_2^\pm = \frac{e^1 \wedge e^3 \mp \varepsilon_2\varepsilon_4 e^2 \wedge e^4}{\sqrt{2}},\ E_3^\pm = \frac{e^1 \wedge e^4 \pm \varepsilon_2\varepsilon_3 e^2 \wedge e^3}{\sqrt{2}}$$

and $\varepsilon_k = g(e_k, e_k)$. Further, one can write the matrices corresponding to the self-dual and anti-self-dual components of the Weyl tensor in the basis above to obtain:

$(*)$ If $P = B \times_f F$ is a twisted product with 3-dimensional fibre, then

$$W^+ = \begin{pmatrix} \varepsilon_1\varepsilon_2 W_{11}^+ & \varepsilon_1\varepsilon_2 W_{12}^+ & \varepsilon_1\varepsilon_2 W_{13}^+ \\ \varepsilon_1\varepsilon_3 W_{12}^+ & \varepsilon_1\varepsilon_3 W_{22}^+ & \varepsilon_1\varepsilon_3 W_{23}^+ \\ \varepsilon_1\varepsilon_4 W_{13}^+ & \varepsilon_1\varepsilon_4 W_{23}^+ & \varepsilon_1\varepsilon_4 W_{33}^+ \end{pmatrix} = \begin{pmatrix} \varepsilon_1\varepsilon_2 W_{11}^- & \varepsilon_1\varepsilon_2 W_{12}^- & \varepsilon_1\varepsilon_2 W_{13}^- \\ \varepsilon_1\varepsilon_3 W_{12}^- & \varepsilon_1\varepsilon_3 W_{22}^- & \varepsilon_1\varepsilon_3 W_{23}^- \\ \varepsilon_1\varepsilon_4 W_{13}^- & \varepsilon_1\varepsilon_4 W_{23}^- & \varepsilon_1\varepsilon_4 W_{33}^- \end{pmatrix} = W^-$$

$(**)$ If $P = B \times_f F$ is a twisted product with 1-dimensional fibre, then

$$W^+ = \begin{pmatrix} \varepsilon_1\varepsilon_2 W_{11}^+ & \varepsilon_1\varepsilon_2 W_{12}^+ & \varepsilon_1\varepsilon_2 W_{13}^+ \\ \varepsilon_1\varepsilon_3 W_{12}^+ & \varepsilon_1\varepsilon_3 W_{22}^+ & \varepsilon_1\varepsilon_3 W_{23}^+ \\ \varepsilon_1\varepsilon_4 W_{13}^+ & \varepsilon_1\varepsilon_4 W_{23}^+ & \varepsilon_1\varepsilon_4 W_{33}^+ \end{pmatrix} = \begin{pmatrix} \varepsilon_1\varepsilon_2 W_{11}^- & \varepsilon_1\varepsilon_2 W_{12}^- & -\varepsilon_1\varepsilon_2 W_{13}^- \\ \varepsilon_1\varepsilon_3 W_{12}^- & \varepsilon_1\varepsilon_3 W_{22}^- & -\varepsilon_1\varepsilon_3 W_{23}^- \\ -\varepsilon_1\varepsilon_4 W_{13}^- & -\varepsilon_1\varepsilon_4 W_{23}^- & \varepsilon_1\varepsilon_4 W_{33}^- \end{pmatrix}$$

As a consequence of the above relations $(*)$ and $(**)$ we get the following:

Theorem 3.1 *A four-dimensional semi-Riemannian manifold which is locally a twisted product with fibre of dimension one or three is self-dual if and only if it is anti-self-dual, i.e.,*

$$W^- = 0 \Leftrightarrow W^+ = 0.$$

Note in the theorem above that $W^+ = W^-$ if the fibre is of dimension 3, but not necessarily in the case of 1-dimensional fibre.

Remark 3.2 If $P = B \times_f F$ is a twisted product with two-dimensional fibre, then self-duality and anti-self-duality are not equivalent properties. Indeed, one has the following relations between W^+ and W^-:

$$W^+ = \begin{pmatrix} \varepsilon_1\varepsilon_2 W_{11}^+ & \varepsilon_1\varepsilon_2 W_{12}^+ & \varepsilon_1\varepsilon_2 W_{13}^+ \\ \varepsilon_1\varepsilon_3 W_{12}^+ & \varepsilon_1\varepsilon_3 W_{22}^+ & 0 \\ \varepsilon_1\varepsilon_4 W_{13}^+ & 0 & \varepsilon_1\varepsilon_4 W_{33}^+ \end{pmatrix}; \quad W^- = \begin{pmatrix} \varepsilon_1\varepsilon_2 W_{11}^- & \varepsilon_1\varepsilon_2 W_{12}^- & \varepsilon_1\varepsilon_2 W_{13}^- \\ \varepsilon_1\varepsilon_3 W_{12}^- & \varepsilon_1\varepsilon_3 W_{22}^- & 0 \\ \varepsilon_1\varepsilon_4 W_{13}^- & 0 & \varepsilon_1\varepsilon_4 W_{33}^- \end{pmatrix},$$

where $W_{11}^+ = W_{11}^-$, $W_{22}^+ = W_{22}^-$ and $W_{33}^+ = W_{33}^-$, but W_{12}^+ and W_{13}^+ are not necessarily equal to W_{12}^- and W_{13}^-, respectively. This can be easily checked in $P = \mathbb{R}^2 \times_f \mathbb{R}^2$, where one takes $f(x_1, x_2, x_3, x_4) = e^{x_1 x_3 - x_2 x_4}$ as twisting function.

4. Four-dimensional pointwise Osserman twisted products

In this section we prove the main results of this paper.

Theorem 4.1 *Let (P, g) be a 4-dimensional semi-Riemannian manifold which is locally a warped product. Then (P, g) is pointwise Osserman if and only if it has constant sectional curvature.*

Proof. First note that for a 4-dimensional warped product with 2-dimensional fibre the self-dual and the anti-self-dual conditions are equivalent, since

$$W^+ = \begin{pmatrix} \varepsilon_1\varepsilon_2 W_{11}^+ & 0 & 0 \\ 0 & \varepsilon_1\varepsilon_3 W_{22}^+ & 0 \\ 0 & 0 & \varepsilon_1\varepsilon_4 W_{33}^+ \end{pmatrix} = \begin{pmatrix} \varepsilon_1\varepsilon_2 W_{11}^- & 0 & 0 \\ 0 & \varepsilon_1\varepsilon_3 W_{22}^- & 0 \\ 0 & 0 & \varepsilon_1\varepsilon_4 W_{33}^- \end{pmatrix} = W^-.$$

Next suppose the warped product is pointwise Osserman. From Theorem 2.3 we can choose an orientation such that the manifold is Einstein and self-dual. Therefore, the above equivalence together with Theorem 3.1 imply that $W = 0$. Thus the warped product is Einstein and locally conformally flat, and then it must have constant sectional curvature. The converse is clearly true. □

Theorem 4.2 *Let (P, g) be a 4-dimensional semi-Riemannian manifold which is locally a twisted product. Then (P, g) is pointwise Osserman if and only if it has constant sectional curvature.*

Proof. Suppose the twisted product is pointwise Osserman. First note that if the fibre is 1 or 3-dimensional, then the proof is analogous to the proof of the previous result just using Theorem 3.1. Now, if the fibre is 2-dimensional, since any pointwise Osserman manifold is Einstein, then Lemma 2.1 lets us reduce the twisted product to a warped product, and thus the constancy of the sectional curvature follows from Theorem 4.1. □

References

[1] N. BLAŽIĆ, N. BOKAN AND Z. RAKIĆ, Osserman pseudo-Riemannian manifolds of signature (2,2), *J. Austral Math. Soc.* **71** (2001), 367-395.

[2] A.L. BESSE, *Einstein manifolds*, Springer-Verlag, Berlin, (1987).

[3] Q.S. CHI, A curvature characterization of certain locally rank-one symmetric spaces, *J. Diff. Geom.* **28** (1988), 187-202.

[4] M. FERNÁNDEZ-LÓPEZ, E. GARCÍA-RÍO, D.N. KUPELI AND B. ÜNAL, A curvature condition for a twisted product to be a warped product, *Manuscripta Math.* **106** (2001), 213-217.

[5] E. GARCÍA-RÍO, D.N. KUPELI AND R. VÁZQUEZ-LORENZO, *Osserman manifolds in semi-Riemannian geometry*, Lecture Notes in Math., Springer-Verlag, Berlin, 1777 (2002).

[6] H. KAMADA AND Y. MACHIDA, Self-duality of metrics of type (2,2) on four-dimensional manifolds, *Tôhoku Math. J.* **49** (1997), 259-275.

[7] W. KÜHNEL, *Differential Geometry. Curves-Surfaces-Manifolds*, Student Mathematical Library, Amer. Math. Soc., 16 (2002).

[8] B. O'NEILL, *Semi-Riemannian Geometry, with aplications to relativity*, Academic Press, New York, 103 (1983).

[9] R. PONGE AND H. RECKZIEGEL, Twisted products in pseudo-riemannian geometry, *Geom. Dedicata* **48** (1993), 15-25.

Proceedings of the II International
Meeting on Lorentzian Geometry
Murcia, November 12–14, 2003
Publ. de la RSME, Vol. 8 (2004), 127–132

The Reflection Principle for flat surfaces in $\mathbb{S}^3_1$

José M. Espinar

Departamento de Geometría y Topología, Universidad de Granada.

Abstract

In this paper we derive a reflection principle for spacelike flat surfaces in the three-dimensional De-Sitter space.

Keywords: Reflection principle, flat surfaces.

1. Introduction

In the past two decades there has been a special interest in the study of surfaces admitting a conformal representation, mainly due to the fact that the global results from Complex Analysis can be applied to their study.

The most representative example is the theory of minimal surfaces in $\mathbb{R}^3$, but there are many others, like maximal surfaces in $\mathbb{L}^3$, surfaces with $H = 1$ in $\mathbb{H}^3$ and $\mathbb{S}^3_1$, flat surfaces in $\mathbb{H}^3$ and $\mathbb{S}^3_1$, etc.

In this paper we extend the classical Schwarz reflection principle for minimal surfaces of $\mathbb{R}^3$ to the case of spacelike flat surfaces in $\mathbb{S}^3_1$. Essentially, we prove that a spacelike flat surface in $\mathbb{S}^3_1$ meeting orthogonally a totally geodesic de Sitter plane $\mathcal{P}$ can be analytically reflected across $\mathcal{P}$ so that the extended surface is flat and symmetric with respect to $\mathcal{P}$. For this, we use the complex representation of flat surfaces in $\mathbb{S}^3_1$ derived in [2] (and [1]). The Schwarz reflection principle has been extended to other classes of surfaces, like Bryant surfaces and flat surfaces in $\mathbb{H}^3$ and maximal surfaces in $\mathbb{L}^n$ (see [3, 4, 6]).

2. Reflection Principle

Let us denote by $\mathbb{L}^4$ the 4-dimensional Lorentz-Minkowski space given as the vector space $\mathbb{R}^4$ with the Lorentzian metric $\langle , \rangle$ induced by the quadratic form $-x_0^2+x_1^2+x_2^2+x_3^2$, and consider the de Sitter 3-space, realized as the Lorentzian submanifold

$$\mathbb{S}^3_1 = \left\{(x_0, x_1, x_2, x_3) \in \mathbb{L}^4 \ : \ -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\right\}.$$

Let $\wedge$ be the cross product of $\mathbb{S}^3_1$ associated to the induced metric,

$$u \wedge v = p \times u \times v \qquad \forall u, v \in T_p\mathbb{S}^3_1, \tag{1}$$

where $p\times u\times v$ is the unique vector in $\mathbb{L}^4$ such that $\langle p\times u\times v, w\rangle = \det(p, u, v, w)$ for all $w \in \mathbb{L}^4$. We will also consider the positive null cone, given by

$$\mathbb{N}^3_+ = \left\{(x_0, x_1, x_2, x_3) \in \mathbb{L}^4 \ : \ -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0,\ x_0 > 0\right\}.$$

The quotient $\mathbb{N}^3_+/\mathbb{R}^+$, which can be considered as the upper connected component of the ideal boundary of $\mathbb{S}^3_1$ at infinity, inherits a natural conformal structure. Observe that, by means of the map

$$(x_0, x_1, x_2, x_3) \longmapsto \frac{x_1 + ix_2}{x_0 + x_3},$$

we can identify $\mathbb{N}^3_+/\mathbb{R}^+$ with a sphere $\mathbb{S}^2_{+\infty} \equiv \mathbb{C} \cup \{\infty\}$.

In addition, $\mathbb{L}^4$ will be considered as the space of 2×2 Hermitian matrices, Herm(2), in the following way

$$(x_0, x_1, x_2, x_3) \longmapsto \begin{pmatrix} x_0 - x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 + x_3 \end{pmatrix},$$

where $\langle m, m\rangle = -\det(m)$ for all $m \in \mathrm{Herm}(2)$. Thus, $\mathbb{S}^3_1$ corresponds to the set of matrices with determinant -1. Moreover, the action of $\mathbb{SL}(2,\mathbb{C})$ on Herm(2)

$$g \cdot m = gmg^*, \qquad g \in \mathrm{Herm}(2) \ \text{ and } \ g^* = {}^t\overline{g},$$

preserves the inner product, orientations and, therefore, $\mathbb{S}^3_1$ remains unchanged.

In this model the positive null cone can be regarded as the set of positive semi-definite Hermitian matrices with vanishing determinant and its elements can be written as $w\,{}^t\overline{w}$, where ${}^tw = (w_1, w_2)$ is a non zero vector in $\mathbb{C}^2$ uniquely determined, up to multiplication, by an unimodular complex number. Moreover, the map $w\,{}^t\overline{w} \to [(w_1, w_2)] \in \mathbb{CP}^1$ induces one from $\mathbb{N}^3/\mathbb{R}^+$ which identifies $\mathbb{S}^2_{+\infty}$ with $\mathbb{CP}^1$. Thereby, the natural action of $\mathbb{SL}(2,\mathbb{C})$ on $\mathbb{S}^2_{+\infty}$ is the action of $\mathbb{SL}(2,\mathbb{C})$ on $\mathbb{CP}^1$ by Möbius transformations.

Next, let us consider $\psi : S \longrightarrow \mathbb{S}^3_1$ a flat spacelike immersion from a simply-connected surface S in the de Sitter space, and let η be its Gauss map. We will denote by $G^+ := [\psi+\eta]$ and $G^- := [\psi-\eta]$ the hyperbolic Gauss maps of ψ.

Observe that the geodesic curve in $\mathbb{S}^3_1$ passing through a point $\psi(p)$ with speed $\eta(p)$, is given by

$$\gamma_{\psi(p),\eta(p)}(t) = \cosh(t)\psi(p) + \sinh(t)\eta(p)\,,\ t \in \mathbb{R}$$

and therefore has two ends, one in each component of the boundary of $\mathbb{S}^3_1$ at infinity. Then, up to an isometry of $\mathbb{S}^3_1$, we have that $(\psi+\eta)(S) \subset \mathbb{N}^3_+$ and $(\psi-\eta)(S) \subset \mathbb{N}^3_-$, and we can take

$$G = (G^+, G^-) : S \longrightarrow \mathbb{S}^2_{+\infty} \times \mathbb{S}^2_{-\infty}.$$

Now, see [1] or [2], flat surfaces in $\mathbb{S}^3_1$ have a Weierstrass representation in terms of a meromorphic map, h, and an holomorphic 1-form, $\omega = f\,dz$, called *Weierstrass data* of the immersion. Specifically,

$$\psi = g\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}g^* \quad \text{and} \quad \eta = gg^*, \tag{2}$$

where $g : S \longrightarrow \mathbb{SL}(2,\mathbb{C})$ is a meromorphic curve such that

$$g^{-1}dg = \begin{pmatrix} 0 & \omega \\ dh & 0 \end{pmatrix}. \tag{3}$$

Let $(h(z), f(z)dz)$ be the Weierstrass data associated to ψ, being z an arbitrary parameter and f a holomorphic function. Then, if we put

$$g = \begin{pmatrix} C & H \\ D & J \end{pmatrix},$$

it follows from (3) that

$$\frac{d}{dz}\begin{pmatrix} C & H \\ D & J \end{pmatrix} = \begin{pmatrix} C & H \\ D & J \end{pmatrix}\begin{pmatrix} 0 & f \\ h_z & 0 \end{pmatrix}$$

and consequently

$$H = \frac{1}{h_z}C_z, \quad J = \frac{1}{h_z}D_z.$$

Therefore, C and D are linearly independent solutions of the differential equation

$$X_{zz} - \frac{h_{zz}}{h_z} X_z - f h_z X = 0. \tag{4}$$

Conversely, if C, D are linearly independent solutions of (4), then $\frac{1}{h_z}(CD_z - DC_z)$ is constant. Thus, we can choose C, D such that

$$g = \begin{pmatrix} C & \frac{1}{h_z} C_z \\ D & \frac{1}{h_z} D_z \end{pmatrix} \in \mathbb{SL}(2, \mathbb{C}). \tag{5}$$

Hence, the hyperbolic Gauss maps become

$$G^+ := [\psi + \eta] = \frac{C}{D}, \quad G^- := [\psi - \eta] = \frac{C_z}{D_z}$$

from where

$$G^+ - G^- = \frac{h_z}{DD_z}, \quad G^+_z = -\frac{h_z}{D^2}$$

So,

$$\xi(z) := D(z)^{-1} = c_0 \exp \int_{z_0}^{z} \frac{dG^+}{G^+ - G^-}$$

being $z_0 \in S$ a fixed point and $c_0 = D(z_0)^{-1}$.

In this way we obtain an expression for g in terms of its hyperbolic Gauss maps:

$$g = \begin{pmatrix} G^+/\xi & \xi G^-/(G^+ - G^-) \\ 1/\xi & \xi/(G^+ - G^-) \end{pmatrix} \in \mathbb{SL}(2, \mathbb{C}). \tag{6}$$

Let $\mathbb{S}^2_1$ denote the set $\{(x_0, x_1, x_2, x_3) \in \mathbb{S}^3_1 : \quad x_2 = 0\}$. Let $\mathcal{O}$ denote a domain in the complex plane which is symmetric with respect to the real axis, and set

$$\begin{aligned} \mathcal{O}^+ &:= \mathcal{O} \cap \{z \in \mathbb{C} : \quad \mathrm{Im} z > 0\}, \\ \mathcal{O}^- &:= \mathcal{O} \cap \{z \in \mathbb{C} : \quad \mathrm{Im} z < 0\}, \\ \mathcal{I} &:= \mathcal{O} \cap \{z \in \mathbb{C} : \quad \mathrm{Im} z = 0\}. \end{aligned}$$

Assume also that $\mathcal{I}$ is an open set of $\mathbb{R}$.

Theorem 2.1 *Let $\psi \in C^1(\mathcal{O}^+ \cup \mathcal{I}, \mathbb{S}^3_1)$ be a flat spacelike surface such that $\psi|_{\mathcal{I}} \subset \mathbb{S}^2_1$. Assume that the unit normal vector field, η, to ψ verifies $\eta|_{\mathcal{I}} \subset \mathbb{S}^2_1$. Then ψ can be extended across $\mathcal{I}$ on all of $\mathcal{O}$ by reflection about $\mathbb{S}^2_1$, and the extension is defined by*

$$(\psi_0(\bar{z}), \psi_1(\bar{z}), \psi_2(\bar{z}), \psi_3(\bar{z})) = (\psi_0(z), \psi_1(z), -\psi_2(z), \psi_3(z)), \quad z \in \mathcal{O}^-$$

Proof: Using the hypothesis

$$\mathrm{Im}\, G^{+}\big|_{\mathcal{I}} = 0, \quad \mathrm{Im}\, G^{-}\big|_{\mathcal{I}} = 0$$

and by applying Schwarz Reflection Principle for holomorphic functions to G^+ and G^- we get

1. G^+ and G^- can be extended across $\mathcal{I}$ on all of $\mathcal{O}$, and the extension is defined by
$$G^{+}(z) = \overline{G^{+}(\bar{z})}, \quad G^{-}(z) = \overline{G^{-}(\bar{z})},$$
and from here
$$\left|G^{+}(z)\right| = \left|G^{+}(\bar{z})\right|, \quad \left|G^{-}(z)\right| = \left|G^{-}(\bar{z})\right|.$$

2. ξ can be extended across $\mathcal{I}$ on all of $\mathcal{O}$, and the extension is defined by
$$\overline{\xi(z)} = \frac{\overline{c_0}}{c_0}\xi(\bar{z}),$$
and
$$\left|\xi(z)\right| = \left|\xi(\bar{z})\right|.$$

From (2)

$$\psi = \frac{1}{|\xi|^2 |G^+ - G^-|^2}\begin{pmatrix} |G^+|^2 |G^+ - G^-|^2 - |\xi|^4 |G^-|^2 & |G^+ - G^-|^2 G^+ - |\xi|^4 G^- \\ |G^+ - G^-|^2 \bar{G^+} - |\xi|^4 \bar{G^-} & |G^+ - G^-|^2 - |\xi|^4 . \end{pmatrix}$$

Then we have

$$\begin{aligned} \psi_0 &= \frac{|G^+ - G^-|^2 (|G^+|^2 + 1) - |\xi|^4 (|G^-|^2 + 1)}{2|\xi|^2 |G^+ - G^-|^2} \\ \psi_1 &= \frac{|G^+ - G^-|^2 \,\mathrm{Re} G^+ - |\xi|^4 \,\mathrm{Re} G^-}{|\xi|^2 |G^+ - G^-|^2} \\ \psi_2 &= \frac{|G^+ - G^-|^2 \,\mathrm{Im} G^+ - |\xi|^4 \,\mathrm{Im} G^-}{|\xi|^2 |G^+ - G^-|^2} \\ \psi_3 &= \frac{|G^+ - G^-|^2 (|G^+|^2 - 1) - |\xi|^4 (|G^-|^2 - 1)}{2|\xi|^2 |G^+ - G^-|^2}. \end{aligned} \tag{7}$$

The formula we wanted to proof follows from a straightforward computation in (7) using 1. and 2.

References

[1] J. M. ESPINAR, Linear Weingarten Surfaces of Bianchi type in $\mathbb{S}^3_1$, *Preprint*, (2003).

[2] J.A. GÁLVEZ, A. MARTÍNEZ AND F. MILÁN, Flat surfaces in $\mathbb{L}^4$, *Ann. Global Anal. Geom.*, **20** (2001), 243–251.

[3] J.A. GÁLVEZ AND P. MIRA, An initial value problem for flat surfaces in $\mathbb{H}^3$ and applications, *Preprint*.

[4] P. MIRA, Ph. D. Thesis, *University of Murcia*, (2003).

[5] R. OSSERMAN, A survey of minimal surfaces, Van Nostrand, New York, 1969.

[6] P. ROITMAN, Flat surfaces in $\mathbb{H}^3$ as normal surfaces to a congruence of geodesics, *Preprint*.