$\mathcal{F}$-BASES FROM COUNTABLY GENERATED FILTERS

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Given any filter $\mathcal{F}$ of subsets of $\mathbb{N}$ and any Banach space X we say that a sequence $(e_n)_{n=1}^{\infty}$ is an *$\mathcal{F}$-basis* for X if and only if for each $x \in X$ there is a unique sequence of scalars $(a_n)_{n=1}^{\infty}$ such that

$$x = \mathcal{F}-\lim_{n\to\infty} \sum_{k=1}^{n} a_k e_k$$

in the norm topology of X. In such a case we shall denote $a_n = e_n^*(x)$ and $S_n(x) = \sum_{k=1}^{n} e_k^*(x) e_k$ for $n \in \mathbb{N}$. Of course, $e_k^*\colon X \to \mathbb{R}$ are linear and their continuity, in the case of a *countably generated*[†] filter $\mathcal{F}$, follows from the following result.

Theorem 1. *Let $\mathcal{F}$ be a countably generated filter of subsets of $\mathbb{N}$ and let $(e_n)_{n=1}^{\infty}$ be an $\mathcal{F}$-basis for a Banach space X. Then $e_n^* \in X^*$ for each $n \in \mathbb{N}$.*

Proof. We may assume that $\mathcal{F}$ does not contain any finite sets, since otherwise the definition of $\mathcal{F}$-basis forces X to be finite-dimensional and our assertion is trivial. For every $A \in \mathcal{F}$ we define

$$X_A = \{x \in X : \sup_{\nu\in A} \|S_\nu(x)\| < \infty\}.$$

Obviously, by the uniqueness of the expansion into $\mathcal{F}$-basis, $0 \in X_A$ and X_A is a linear subspace of X. The function $\|\cdot\|_A \colon X_A \to [0,\infty)$ defined by

$$\|x\|_A = \sup_{\nu\in A} \|S_\nu(x)\|$$

is a norm in X_A. To verify the first postulate suppose $\|x\|_A = 0$. For each $\varepsilon > 0$ one may find $B \in \mathcal{F}$ such that $\|S_\nu(x) - x\| < \varepsilon$ for $\nu \in B$, but then for each $\nu \in A \cap B$, which is non-empty as an element of $\mathcal{F}$, we have $S_\nu(x) = 0$, thus $\|x\| < \varepsilon$. By the arbitrariness of ε, we conclude that $x = 0$.

Now, assume $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in $(X_A, \|\cdot\|_A)$. Then for every $\varepsilon > 0$ one may find $m \in \mathbb{N}$ such that for each $n \geq m$ and $\nu \in A$ we have $\|S_\nu(x_m - x_n)\| < \varepsilon/3$. We may choose ν in such a way that $\|S_\nu(x_m) - x_m\| < \varepsilon/3$ and $\|S_\nu(x_n) - x_n\| < \varepsilon/3$. These three inequalities give $\|x_m - x_n\| < \varepsilon$, which shows that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in $(X_A, \|\cdot\|)$. Therefore, there exists x_0 in the $\|\cdot\|$-closure of X_A such that

$$\lim_{n\to\infty} \|x_n - x_0\| = 0. \tag{1}$$

Similarly, for every $\nu \in A$ and $m, n \in \mathbb{N}$ we have

$$\|S_\nu(x_m) - S_\nu(x_n)\| = \|S_\nu(x_m - x_n)\| \leq \|x_m - x_n\|_A,$$

which shows that $(S_\nu(x_n))_{n=1}^{\infty}$ is a Cauchy sequence in $(X, \|\cdot\|)$, and each of its elements lies in $[e_j]_{j\leq\nu}$ (the linear span of $e_1, \ldots, e_\nu$). Hence, there is $y_\nu \in [e_j]_{j\leq\nu}$ such that

$$\lim_{n\to\infty} \|S_\nu(x_n) - y_\nu\| = 0. \tag{2}$$

For every $j \in \mathbb{N}$ denote $\alpha_j = e_j^*(y_\nu)$ for any $\nu \in A$, $j \leq \nu$. This definition does not depend on the choice of such a ν. Indeed, if $k, \ell \in A$ satisfy $j \leq k \leq \ell$, then the continuity of e_j^* on the finite-dimensional subspace $[e_i]_{i\leq\ell}$ gives

$$e_j^*(y_k) = e_j^*(\lim_{n\to\infty} S_k(x_n)) = \lim_{n\to\infty} e_j^*(S_k(x_n)) = \lim_{n\to\infty} e_j^*(S_\ell(x_n)) = e_j^*(y_\ell).$$

[†] A filter $\mathcal{F}$ of subsets of $\mathbb{N}$ is called *countably generated*, if there exist a subfamily $\{A_n\}_{n<\omega}$ of $\mathcal{F}$ such that for every $A \in \mathcal{F}$ we have $A_n \subset A$ for some $n < \omega$.

We shall show that
$$x_0=\mathcal{F}-\sum_{n=1}^\infty \alpha_n e_n,$$
that is, in particular, $S_\nu(x_0)=y_\nu$ for every $\nu\in A$. To this end fix any $\varepsilon>0$ and choose $m\in\mathbb{N}$ such that for each $n\geq m$ we have $\|S_\nu(x_m)-S_\nu(x_n)\|<\varepsilon/3$ (for any $\nu\in A$) and $\|x_m-x_n\|<\varepsilon/3$. Now, let $B\in\mathcal{F}$ be such that for each $\nu\in B$ we have $\|S_\nu(x_m)-x_m\|<\varepsilon/3$. Then $A\cap B\in\mathcal{F}$ and for every $\nu\in A\cap B$ the following estimate holds true:
$$\begin{aligned}\|y_\nu-x_0\|&=\big\|\lim_{n\to\infty}S_\nu(x_n)-\lim_{n\to\infty}x_n\big\|\\&\leq\lim_{n\to\infty}\|S_\nu(x_m)-S_\nu(x_n)\|+\|S_\nu(x_m)-x_m\|+\lim_{n\to\infty}\|x_m-x_n\|\leq\varepsilon,\end{aligned}$$
in view of (1) and (2). This shows that
$$x_0=\mathcal{F}-\lim_{\substack{\nu\to\infty\\ \nu\in A}}y_\nu.$$
Moreover, a similar estimate, for an arbitrary $\nu\in A$ and $m\in\mathbb{N}$ chosen as above, yields
$$\|y_\nu\|\leq\|x_0\|+\frac{1}{3}\varepsilon+\|S_\nu(x_m)\|+\|x_m\|+\frac{1}{3}\varepsilon\leq\frac{2}{3}\varepsilon+\|x_0\|+\|x_m\|_A+\|x_m\|,$$
which implies
$$\sup_{\nu\in A}\|S_\nu(x_0)\|=\sup_{\nu\in A}\|y_\nu\|<\infty,$$
thus $x_0\in X_A$. Now, for any $n\in\mathbb{N}$ we have
$$\begin{aligned}\|x_n-x_0\|_A&=\sup_{\nu\in A}\|S_\nu(x_n)-S_\nu(x_0)\|=\sup_{\nu\in A}\|S_\nu(x_n)-\lim_{m\to\infty}S_\nu(x_m)\|\\&\leq\limsup_{m\to\infty}\sup_{\nu\in A}\|S_\nu(x_n)-S_\nu(x_m)\|,\end{aligned}$$
which shows that $\lim_{n\to\infty}\|x_n-x_0\|_A=0$ and, consequently, $(X_A,\|\cdot\|_A)$ is a Banach space.

Now, we consider the identity map $i_A\colon(X_A,\|\cdot\|_A)\to(X,\|\cdot\|)$ (which is continuous, by $\|\cdot\|_A\geq\|\cdot\|$). By the Open Mapping Theorem, either i_A is surjective, or the image X_A is of the first category in $(X,\|\cdot\|)$. Since $\mathcal{F}$ is countably generated, there is a sequence of $A_n\in\mathcal{F}$ $(n<\omega)$ such that for any $B\in\mathcal{F}$ we have $A_n\subset B$ for some $n<\omega$. For an arbitrary $x\in X$ we may pick a set $B\in\mathcal{F}$ satisfying $\sup_{\nu\in B}\|S_\nu(x)\|<\infty$, hence $x\in X_{A_n}$ provided $A_n\subset B$. Therefore, $X=\bigcup_{n<\omega}X_{A_n}$ and the Baire Category Theorem implies that not all of the subspaces X_{A_n} may be of the first category in $(X,\|\cdot\|)$. Consequently, there is a set $A\in\mathcal{F}$ such that $X_A=X$. Then the inverse operator i_A^{-1} is continuous, i.e. there is a constant $K<\infty$ such that $\|S_\nu(x)\|\leq K\|x\|$ for all $x\in X$ and $\nu\in A$.

Fix any $j\in\mathbb{N}$; we shall show that e_j^* is continuous. Suppose, in search of a contradiction, that there is a sequence $(x_n)_{n=1}^\infty$ of elements of X such that $\|x_n\|=1$ for $n\in\mathbb{N}$ and $e_j^*(x_n)\to\infty$. Pick any index $\nu\in A$, $\nu\geq j$. Obviously, $e_1,\ldots,e_\nu$ are linearly independent and since the finite-dimensional subspace $[e_i]_{i\leq\nu,i\neq j}$ is closed, we infer that
$$\delta:=\inf\big\{\|e_j+y\|:y\in[e_i]_{i\leq\nu,i\neq j}\big\}>0.$$
Since
$$S_\nu(x_n)=e_j^*(x_n)e_j+\sum_{i=1,i\neq j}^\nu e_i^*(x_n)e_i,$$
we have
$$\|S_\nu(x_n)\|\geq\delta\cdot|e_j^*(x_n)|\underset{n\to\infty}{\longrightarrow}\infty,$$
which contradicts the fact that S_ν is continuous. □

Remark 1. A filter $\mathcal{F}$ of subsets of $\mathbb{N}$ is called *almost principal*, if there is a set $A\in\mathcal{F}$ such that for every $B\in\mathcal{F}$ the set $A\setminus B$ is finite. The convergence of some sequence $(x_n)_{n=1}^\infty$ with respect to such a filter $\mathcal{F}$ is equivalent to the ordinary convergence of the subsequence $(x_{n_k})_{k=1}^\infty$, where $A=\{n_1,n_2,\ldots\}$. Of course, almost principal filters are countably generated.

However, the filter $\mathcal{F}=\mathcal{F}_{st}$ of sets whose complement is of natural density zero is not countably generated. This follows, e.g., from the fact that $\mathcal{F}_{st}$ is a *P-filter*, that is for every countable sequence $(A_n)_{n<\omega}$ in $\mathcal{F}_{st}$

there is a set $A \in \mathcal{F}_{st}$ such that $A \subseteq^* A_n$ for each $n < \omega$, i.e. A is *almost included*[†] in every A_n. This implies that $\mathcal{F}_{st}$ cannot be generated by $(A_n)_{n<\omega}$, since we may find an infinite set $C \subset A$ of density zero and we have $A \setminus C \in \mathcal{F}_{st}$, whereas none of A_n is contained in $A \setminus C$.

Remark 2. As it was pointed out by Professor D.H. Fremlin, the above method works also in models where the Baire Category Theorem holds true for "not small" uncountable number of meager sets.

For instance, if we assume the Martin axiom then the following Booth's lemma is valid: Given a family $\mathcal{R} \subseteq \mathcal{P}(\omega)$ of cardinality less than continuum such that for all $A_1, \ldots, A_n \in \mathcal{R}$ the intersection $A_1 \cap \ldots \cap A_n$ is an infinite set, there exists an infinite set B satisfying $B \subseteq^* A$ for each $A \in \mathcal{R}$. Consequently, if a filter $\mathcal{F}$ has character (the minimal cardinality of a family generating $\mathcal{F}$) less than continuum, then for some infinite set B we have $B \subseteq^* A$ for each $A \in \mathcal{F}$, hence $X_B = X$ and we may apply the Open Mapping Theorem for the identity operator $i_B\colon (X_B, \|\cdot\|_B) \to (X, \|\cdot\|)$ getting the following result.

Theorem 2. *Assume the Martin axiom. If $\mathcal{F}$ is a filter on $\mathbb{N}$ with character less than continuum and $(e_n)_{n=1}^{\infty}$ is an $\mathcal{F}$-basis for a Banach space X, then $e_n^* \in X^*$ for each $n \in \mathbb{N}$.*

Remark 3. A slight modification of Example 1 from the paper [J. Connor, M. Ganichev, V. Kadets, *A characterization of Banach spaces with separable duals via weak statistical convergence*, J. Math. Anal. Appl. 244 (2000), 251–261] shows that in general our strategy, based on proving $X_A = X$ for some $A \in \mathcal{F}$, does not work. Namely, let $\mathcal{F}_{st}$ be the filter of statistical convergence and let $(e_n)_{n=1}^{\infty}$ be the standard basis in ℓ_2 with coordinate functionals $(e_n^*)_{n=1}^{\infty}$. Put also $x_n = \sum_{i=1}^{n} e_i$. Then, as it is shown in the paper mentioned above, $(x_n)_{n=1}^{\infty}$ is an $\mathcal{F}_{st}$-basis in ℓ_2 with the coordinate functionals given by $x_n^* = e_n^* - e_{n+1}^*$. They are, of course, continuous but for any increasing sequence of natural numbers $n_1 < n_2 < \ldots$ we may define an element $x = \sum_{k=1}^{\infty} a_k e_k$ of ℓ_2 such that

$$\sup\{\|S_{n_k}(x)\| : k \in \mathbb{N}\} = \infty. \tag{3}$$

To this end choose an increasing subsequence $\{m_i\}_{i=1}^{\infty} \subset \{n_i\}_{i=1}^{\infty}$ with $m_i > i^4$ and put

$$a_k = \begin{cases} 1/\sqrt[4]{k} & \text{if there is an } i \in \mathbb{N} \text{ such that } k = m_i + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, repeating the argument from the paper cited above, we obtain (3) which shows that in this case $(\ell_2)_A \subsetneq \ell_2$ for every infinite set $A \subset \mathbb{N}$.

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[†]We say that A is *almost included* in B, and we write $A \subseteq^* B$, if the set $A \setminus B$ is finite.