Preserved extreme points in Lipschitz-free spaces

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This is *not* true if $\mathbb{R}$ is replaced by $\mathbb{C}$.

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- $\mathrm{Lip}_0(X, e) \simeq \mathrm{Lip}_0(X, e')$ under $f \mapsto f - f(e')$

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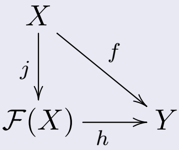
Theorem (Arens, Eells 1956)

$\mathcal{F}(X)^* \simeq \mathrm{Lip}_0(X)$.

Theorem (Weaver 1999)

Let Y be a Banach space. If $f\colon X \to Y$ is Lipschitz and $f(e) = 0$ then there is $h \in \mathcal{B}(\mathcal{F}(X), Y)$ with $\|h\| = L(f)$ and $f = h \circ j$.



de Leeuw transform

$$\Phi : \mathrm{Lip}(X) \to C(\widetilde{X}) \qquad \Phi f(p,q) = \frac{f(p) - f(q)}{d(p,q)}$$

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Elementary molecules

$$u_{pq} = \frac{j(p) - j(q)}{d(p,q)} \in S_{\mathcal{F}(X)}$$

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Elementary molecules

$$u_{pq} = \frac{j(p) - j(q)}{d(p,q)} \in S_{\mathcal{F}(X)}$$

Theorem (Weaver 1995)

If $m \in \mathcal{F}(X)$ is a preserved extreme point of $B_{\mathcal{F}(X)}$ then $m = u_{pq}$ for some $(p,q) \in \widetilde{X}$.

- βX, Stone-Čech compactification of X
  - For $p \in X$, consider $d(p, \cdot)\colon \beta X \to [0, \infty]$

$$d(x, \xi) = \lim_i d(p, x_i), \text{ where } x_i \to \xi$$

  - If $d(p, \xi) < \infty$, the evaluation on ξ is in $\mathrm{Lip}_0(X)^*$

$$\langle j(\xi), f\rangle = f(\xi) = \lim_i f(x_i)$$

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- $\beta\widetilde{X}$, Stone-Čech compactification of $\widetilde{X}$
  - For $f \in \mathrm{Lip}(X)$, consider $\Phi f \in C(\beta\widetilde{X})$

$$\Phi f(\zeta) = \lim_i \Phi f(x_i, y_i), \text{ where } (x_i, y_i) \to \zeta$$

  - $\Phi\colon\ \mathrm{Lip}_0(X) \to C(\beta\widetilde{X})$ is an isometry into
  - $\Phi^*\colon M(\beta\widetilde{X}) \to \mathrm{Lip}_0(X)^*$ is onto

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$$y = \frac{1}{2}(y_1 + y_2),\ y_1, y_2 \in B_Y \quad \Longrightarrow \quad y_1 = y_2 = y$$

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Note: (3) $\Rightarrow$ (2) $\Rightarrow$ (1)

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Proof:

$$\begin{aligned} u_{pq} &= \frac{j(p)-j(q)}{d(p,q)} = \frac{j(p)-j(r)+j(r)-j(q)}{d(p,q)} \\ &= \frac{d(p,r)}{d(p,q)}u_{pr} + \frac{d(r,q)}{d(p,q)}u_{rq}. \qquad \square \end{aligned}$$

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Define $u_{p\xi} = \dfrac{j(p) - j(\xi)}{d(p,\xi)}$, $u_{\xi q} = \dfrac{j(\xi) - j(q)}{d(\xi,q)}$. Both are in $S_{\mathcal{F}(X)^{**}}$.

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$$u_{pq} = \frac{d(p,\xi)}{d(p,q)}u_{p\xi} + \frac{d(\xi,q)}{d(p,q)}u_{\xi q}. \qquad \square$$

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Suppose $\varepsilon(\xi; p, q) > 0$ for all $\xi \in \beta X \setminus \{p, q\}$. Then u_{pq} is a preserved extreme point of $B_{\mathcal{F}(X)}$.

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Then $x_i^{**} = \Phi^* \mu_i$ where $\mu_i \in M(\beta \widetilde{X})$, $\|\mu_i\| = 1$.
$\Phi^* \delta_{(p,q)} = u_{pq} \Rightarrow$ Prove that μ_i is concentrated on (p, q) and (q, p).

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$\Phi^* \delta_{(p,q)} = u_{pq} \Rightarrow$ Prove that μ_i is concentrated on (p, q) and (q, p).

Consider the set

$$D_{(p,q)} = \big\{\zeta \in \beta\widetilde{X} : \text{if } f \in \mathrm{Lip}_0(X) \text{ attains its norm at } (p, q), \text{ then } f \text{ also attains its norm at } \zeta\big\}.$$

(f *attains its norm* at ζ if $|\Phi f(\zeta)| = L(f)$)

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There is $f \in \mathrm{Lip}_0(X)$ such that $L(f) = 1$, $\Phi f(p,q) = 1$, $|\Phi f(\zeta)| < 1$.
There are $c < 1$ and $U(\zeta)$ such that $|\Phi f(\zeta')| \leq c$ for $\zeta' \in U(\zeta)$.

$$\langle \mu_i, \Phi f \rangle = \int_{\beta\widetilde{X}} (\Phi f)\, d\mu_i = \int_{U(\zeta)} + \int_{\beta\widetilde{X}\setminus U(\zeta)} \leq 1 - (1-c)\left|\mu_i\right|(U(\zeta))$$

$$\Rightarrow\ 1 = \langle f, u_{pq} \rangle = \left\langle \frac{1}{2}(\mu_1 + \mu_2), \Phi f \right\rangle \leq 1 - \frac{1-c}{2}(|\mu_1| + |\mu_2|)(U(\zeta))$$

$$\Longrightarrow |\mu_1|\,(U(\zeta)) = |\mu_2|\,(U(\zeta)) = 0$$

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By regularity

$$\left|\mu_i\right|(\beta\widetilde{X} \setminus D_{(p,q)}) = \sup_{\substack{K \subset \beta\widetilde{X} \setminus D_{(p,q)} \\ K \text{ compact}}} \left|\mu_i\right|(K)$$

For any K, find a cover $K \subset \bigcup_{j=1}^{n} U(\zeta_j)$.

$$\left|\mu_i\right|(K) \leq \sum_{j=1}^{n} \left|\mu_i\right|(U(\zeta_j)) = 0$$

$\Rightarrow \left|\mu_i\right|(\beta\widetilde{X} \setminus D_{(p,q)}) = 0$. $\square$

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For $D_{(p,q)} = \{(p, q), (q, p)\}$ we get:

Theorem (AG 2017)

If $(p, q) \in \widetilde{X}$ and for every $\zeta \in \beta\widetilde{X}$, $\zeta \neq (p, q), (q, p)$ there is $f \in \mathrm{Lip}_0(X)$ such that $|\Phi f(p, q)| = L(f)$ and $|\Phi(\zeta)| < L(f)$, then u_{pq} is a preserved extreme point of $B_{\mathcal{F}(X)}$.

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Theorem (de Leeuw 1961)

If $(p,q) \in \widetilde{X}$ and there is $f \in \mathrm{Lip}_0(X)$ that peaks at (p,q), then u_{pq} is a preserved extreme point of $B_{\mathcal{F}(X)}$.

(f *peaks at* (p,q) if $|\Phi f| = L(f)$ in $(p,q),(q,p)$ and $< L(f)$ elsewhere)

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Theorem (de Leeuw 1961)

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Theorem (García-Lirola, Procházka, Rueda Zoca 2017)

Let X be a pointed metric space and $(p,q) \in \widetilde{X}$. TFAE:
(i) u_{pq} is a strongly exposed point of $B_{\mathcal{F}(X)}$
(ii) There is $f \in \mathrm{Lip}_0(X)$ that peaks at (p,q)

Step 2?

If $\varepsilon(\xi;p,q)>0$ for all $\xi\in\beta X\setminus\{p,q\}$, then

$$D_{(p,q)}=\{(p,q),(q,p)\}$$

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Counterexample:

If $p_n \to p$ and $\frac{\varepsilon(p_n;p,q)}{d(p_n,p)} \to 0$ then $D_{(p,q)}$ contains a point that lies over p.

($\zeta \in \beta\widetilde{X}$ *lies over* p if $(x_i, y_i) \to \zeta$ where $x_i, y_i \to p$)

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$$D_{(p,q)} = \{(p,q),(q,p)\} \cup \{\text{points that lie over } p \text{ or } q\}$$

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Proof: Let $(x_i, y_i) \to \zeta \in D_{(p,q)}$, $x_i \to \xi$, $y_i \to \eta$, where $\xi, \eta \in \beta X$.
We must show that $\zeta \in D_{(p,q)} \Rightarrow \xi, \eta \in \{p, q\}$.
Suppose $\xi, \eta \neq p, q$ (the other case is similar).

$$\varepsilon(\xi; p, q), \varepsilon(\eta; p, q) > 0 \Rightarrow \inf \frac{\varepsilon(x_i; p, q)}{d(x_i, q)}, \inf \frac{\varepsilon(y_i; p, q)}{d(y_i, q)} \geq c > 0$$

Define $g \in \mathrm{Lip}(\{p, q, x_i, y_i\})$ as

$$g(x) = \begin{cases} d(x, q) & \text{if } x = p \\ (1 - c) \cdot d(x, q) & \text{if } x = q \text{ or } x_i \text{ or } y_i \end{cases}$$

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- $\Phi g(p,q) = 1$
- $|\Phi g(x,y)| \leq 1 - c$ if $x, y \neq p$
- $|\Phi g(x,p)| \leq 1$ if $x \neq p, q$ (definition of c)

$\Rightarrow L(g) = 1$

Extend g to X and let $f = g - g(e)$. Then $f \in \mathrm{Lip}_0(X)$, $L(f) = 1$, $\Phi f(p,q) = 1$, and $|\Phi f(\zeta)| = \lim_i |\Phi f(x_i, y_i)| \leq 1 - c$.
$\Rightarrow \zeta \notin D_{(p,q)}$. □

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In our case, μ_1 and μ_2 are concentrated on (p,q) and (q,p).

Proof: Let $\mu_i = \lambda_i + \lambda_i'$ where λ_i is concentrated on $\{(p,q),(q,p)\}$ and λ_i' on $D_{(p,q)} \setminus \{(p,q),(q,p)\}$.
Choose $f \in \mathrm{Lip}_0(X)$ with $L(f) = 1$, $\Phi f(p,q) = 1$ and f constant in neighborhoods of p and q.
Then $\Phi f(\zeta) = 0$ if $\zeta \in \beta\widetilde{X}$ lies over p or q. So $\int_{\beta\widetilde{X}} (\Phi f)\, d\lambda_i' = 0$.

$$1 = \langle f, u_{pq}\rangle = \left\langle \frac{1}{2}(\mu_1+\mu_2), \Phi f \right\rangle = \frac{1}{2}\left(\int_{\beta\widetilde{X}} (\Phi f)\, d\lambda_1 + \int_{\beta\widetilde{X}} (\Phi f)\, d\lambda_2 \right)$$

$$\leq \frac{1}{2}\left\|\Phi f\right\|_\infty (\|\lambda_1\| + \|\lambda_2\|) \leq \frac{1}{2}\left\|\Phi f\right\|_\infty (\|\mu_1\| + \|\mu_2\|) \leq 1$$

$\Rightarrow \|\lambda_1'\| = \|\lambda_2'\| = 0$. $\square$

Theorem (AG 2017)

Let X be a pointed metric space and $p \neq q \in X$. TFAE:
(i) u_{pq} is a preserved extreme point of $B_{\mathcal{F}(X)}$
(ii) $\varepsilon(\xi; p, q) > 0$ for all $\xi \in \beta X \setminus \{p, q\}$

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Theorem (Mayer-Wolf 1981)

Let X, Y be concave metric spaces. TFAE:
(i) $\mathrm{Lip}_0(X) \simeq \mathrm{Lip}_0(Y)$
(ii) $\mathcal{F}(X) \simeq \mathcal{F}(Y)$
(iii) There is a dilation from X onto Y

($f : X \to Y$ is a *dilation* if $\dfrac{d(f(x), f(y))}{d(x, y)}$ is constant)

Conjecture (Weaver 1999)

If X is a compact metric space such that $d(p,q) < d(p,r) + d(q,r)$ for all distinct $p,q,r \in X$, then X is concave.

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A compact metric space X is concave iff $d(p,q) < d(p,r) + d(q,r)$ for all distinct $p,q,r \in X$.

Corollary

Compact Hölder spaces are concave.

Theorem

Let X be a pointed metric space and $m \in \mathcal{F}(X)$. TFAE:
(i) m is a preserved extreme point of $B_{\mathcal{F}(X)}$
(ii) $m = u_{pq}$ where $p \neq q \in X$ have the property:

(P*) For any $\varepsilon > 0$, there is $\delta > 0$ such that, for all $r \neq p, q$

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(E1*) All preserved extreme points of $B_{\mathcal{F}(X)}$ are of the form u_{pq}.

(E2*) u_{pq} is preserved extreme iff p, q have property (P*).

Theorem (García-Lirola, Procházka, Rueda Zoca 2017)

Let X be a pointed metric space and $m \in \mathcal{F}(X)$. TFAE:
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(P') There is $C > 0$ such that, for all $r \neq p, q$
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Conjecture

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- (E2) is true if X is compact
- (E1) is true if X is compact and $\mathrm{lip}_0(X)$ separates points uniformly (in this case, $\mathcal{F}(X) = \mathrm{lip}_0(X)^*$)

$$\mathrm{lip}_0(X) = \{ f \in \mathrm{Lip}_0(X) : \forall \varepsilon > 0 \, \exists \delta > 0 \text{ such that } d(p,q) < \delta \Rightarrow |\Phi f(p,q)| < \varepsilon \}.$$

$\mathrm{lip}_0(X)$ *separates points uniformly* if $\exists C \geq 1$ such that $\forall p, q \in X$ $\exists f \in \mathrm{lip}_0(X)$ with $|\Phi f(p,q)| = 1$ and $L(f) \leq C$.

Examples:

- Cantor middle-thirds set
- Compact Hölder spaces
- (Dalet 2015) Compact countable spaces
- (Dalet 2015) Compact ultrametric spaces

Corollary

If X is

- the Cantor middle-thirds set,
- a compact Hölder space,
- a countable compact space, or
- an ultrametric compact space,

then

$$\operatorname{Ext} B_{\mathcal{F}(X)} = \Big\{ u_{pq} : p, q \in X, p \neq q,$$

$$d(p,q) < d(p,r) + d(q,r) \text{ for all } r \in X \setminus \{p,q\} \Big\}$$