Some Problems concerning algebras of holomorphic functions

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Plan of talk

Problems collected over a long period, with many co-authors:

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1. Basic background
2. A Problem involving $\mathcal{H}_b(X)$
3. Problems involving $\mathcal{H}^\infty(B_X)$

1) Basics

X is a complex Banach space with open unit ball B_X. We'll be interested in two algebras of holomorphic functions

- $\mathcal{H}_b(X)$, the (Fréchet) algebra of all entire functions $f : X \to \mathbb{C}$ such that $\|f\|_{nB_X} < \infty, \ \forall n$,
- $\mathcal{H}^\infty(B_X)$, the (Banach) algebra of all holomorphic functions $f : B_X \to \mathbb{C}$ such that $\|f\| = \sup_{x \in B_X} |f(x)| < \infty$. To a lesser extent, we'll look at
- $\mathcal{A}_u(B_X)$, the subalgebra of $\mathcal{H}^\infty(B_X)$ consisting of all uniformly continuous holomorphic functions on B_X.

Let's call any of these three algebras $\mathcal{A}$, for now.

Let $\mathcal{M}(\mathcal{A})$ denote the set of all $\neq 0$ homomorphisms $\varphi : \mathcal{A} \to \mathbb{C}$. In the the case of *Banach algebras*, any such φ is automatically continuous. However, for $\mathcal{A} = \mathcal{H}_b(X)$, the automatic continuity of φ is unknown ("Michael problem"). For this talk, we'll assume automatic continuity of such homomorphisms φ. For Banach algebras $\mathcal{A}$, $\mathcal{M}(\mathcal{A}) \subset \overline{B}_{\mathcal{A}^*}$ and is weak-star compact.
A few more basics later.

2) $\mathcal{H}_b(X)$

By [A & Berner], every $f \in \mathcal{H}_b(X)$ admits an extension (via a canonical map) to $\tilde{f} \in \mathcal{H}_b(X^{**})$. Moreover, $f \to \tilde{f}$ is itself a homomorphism, i.e. it's linear, multiplicative, and continuous. By [Davie & Gamelin], it follows that every $f \in \mathcal{H}^\infty(B_X)$ admits an extension $\tilde{f} \in \mathcal{H}^\infty(B_{X^{**}})$. The same holds for $\mathcal{A}_u(B_X)$. As above, $f \to \tilde{f}$ is a homomorphism.

Examples

1. $X = \mathbb{C}$: $\mathcal{M}(\mathcal{H}(\mathbb{C})) = \{\delta_c \mid c \in \mathbb{C}\}$.
Also $\mathcal{M}(\mathcal{A}_u(B_\mathbb{C})) = \mathcal{M}(\mathcal{A}(\mathbb{D})) = \{\delta_c \mid c \in \mathbb{C},\ |c| \leq 1\}$,
However, $\mathcal{M}(\mathcal{H}^\infty(\mathbb{D}))$ is very complicated and very interesting. The only obvious homomorphisms are $\{\delta_c \mid |c| < 1\}$. But this isn't (cannot be) all, because of the compactness of $\mathcal{M}(\mathcal{H}^\infty(\mathbb{D}))$.

Examples

2. $X = c_0$: It is known that $\mathcal{M}(\mathcal{H}_b(c_0))$ consists of $\{\delta_b \mid b \in c_0\}$ together with all $\{\tilde{\delta}_{b^{**}} \mid b^{**} \in \ell_\infty\}$, where $\tilde{\delta}_{b^{**}}(f) = \tilde{f}(b^{**})$.
Also it is known that $\mathcal{M}(\mathcal{A}_u(B_{c_0}))$ is all $\delta_b,\ b \in c_0,\ |b\| \leq 1$, together with all $\tilde{\delta}_{b^{**}}$, where $b^{**} \in \ell_\infty,\ \|b^{**}\| \leq 1$.
3. $X = \ell_2$. There are many more non-trivial homomorphisms in $\mathcal{M}(\mathcal{H}_b(\ell_2))$ than merely the evaluation homomorphisms $\delta_x, x \in \ell_2$.

Question For a fixed $b^{**} \in X^{**}$, by "performing" $\tilde{\delta}_b^{**} : f \in \mathcal{H}_b(X) \to \tilde{f} \in \mathcal{H}_b(X^{**}) \to \tilde{f}(b^{**})$, we showed that X^{**} can be viewed as a subset of $\mathcal{M}(\mathcal{H}_b(X))$. Can we continue this procedure, going from $f \in \mathcal{H}_b(X) \longrightarrow \tilde{f} \in \mathcal{H}_b(X^{**})$, and then from $\tilde{f} \in \mathcal{H}_b(X^{**}) \longrightarrow \tilde{\tilde{f}} \in \mathcal{H}_b(X^{iv})$? In this way, for each fixed $b^{iv} \in X^{iv}$, can we get *new* homomorphisms $\tilde{\tilde{\delta}}_{b^{iv}} \in \mathcal{M}(\mathcal{H}_b(X))$, $f \rightsquigarrow \tilde{\tilde{f}}(b^{iv})$ Answer: Sometimes yes (when X is "regular"), sometimes no. If X is a C^*-algebra, or if X is reflexive (trivial), then no. Namely, to each $b^{iv} \in X^{iv}$, there corresponds $b^{**} \in X^{**}$ such that $\tilde{\tilde{\delta}}_{b^{iv}} = \tilde{\delta}_{b^{**}}$. However, Example

4. $X = \ell_1$. **Theorem**: There are points $b^{iv} \in \ell_1^{iv}$ such that $\tilde{\tilde{\tilde{\delta}}}_{b^{iv}} \neq \tilde{\delta}_{b^{**}}$ for any $b^{**} \in \ell_1^{**}$.

Problems: (1) There are *more points* in ℓ_1^{iv} than there are homomorphisms in $\mathcal{M}(\mathcal{H}_b(\ell_1))$. So, which points b^{iv} of the fourth dual yield new homomorphisms and which do not?

(2) Same questions about going to the *sixth dual* of ℓ_1.

Enough!

As before, let $\mathcal{A}$ be one of the following three algebras: $\mathcal{H}_b(X), \mathcal{H}^\infty(B_X), \mathcal{A}_u(B_X)$.

Observation: $X^* \subset \mathcal{A}$. Consequently, for any $x^* \in X^*$ and for any (continuous) homomorphism $\varphi \in \mathcal{M}(\mathcal{A}),\ \varphi(x^*) \in \mathbb{C}$ makes sense.

Define $\Pi : \mathcal{A} \to ???$ by $\Pi(\varphi) = \varphi|_{X^*}$.

So, what is ???. Answer: It has to be the bidual X^{**}.

(Of course, nothing new when $\dim X < \infty$.) For $\mathcal{A} = \mathcal{H}_b(X)$, the range of Π is all of X^{**}, while in the other two cases, $\mathcal{A} = \mathcal{H}^\infty(B_X)$ or $\mathcal{A}_u(B_X)$, the range is $\overline{B}_{X^{**}}$.

Definition

Let z^{**} be in the range of Π. The *fiber* over z^{**} is just $\Pi^{-1}(z^{**})$.

Definition

The *cluster set* of a function $f \in \mathcal{H}^\infty(B_X)$ at the point $z^{**} \in \overline{B}_{X^{**}}$ is the set of all limits of values of f along nets in B_X that converge weak-star to z^{**}.

Let's restrict to $\mathcal{A} = \mathcal{H}^\infty(\mathbb{D})$. Recall that $\delta(\mathbb{D}) \equiv \{\delta_c \mid c \in \mathbb{D}\} \in \mathcal{M}(\mathcal{H}^\infty(\mathbb{D}))$. Also, recall that the natural topology on $\mathcal{M}(\mathcal{H}^\infty(\mathbb{D}))$ is the weak-star topology, induced by all seminorms of the form $\varphi \in \mathcal{M}(\mathcal{H}^\infty(\mathbb{D})) \rightsquigarrow |\varphi(f)|$, where f varies in $\mathcal{H}^\infty(\mathbb{D})$.
Corona Theorem (L. Carleson - 1962) The collection $\delta(\mathbb{D})$ of point evaluations at points of the open unit disc is dense in the space of all homomorphisms $\mathcal{M}(\mathcal{H}^\infty(\mathbb{D}))$ on $\mathcal{H}^\infty(\mathbb{D})$.

Carleson's theorem (288) appeared one year after a somewhat overlooked paper by I. J. Schark (8). In it, among other things I. J. Schark proved

Cluster Value Theorem (I. J. Schark - 1961) Fix $f \in \mathcal{H}^\infty(\mathbb{D})$ and $c \in \overline{\mathbb{D}}$. Then the following sets are equal:
$\{w \in \mathbb{C} \mid \exists (z_n) \subset \mathbb{D}, z_n \to c \text{ and } f(z_n) \to w\}$;
$\{\varphi(f) \mid \varphi \in \mathcal{M}(\mathcal{H}^\infty(\mathbb{D})) \mid \Pi(\varphi) = c\}$.

Remarks 0. Schark's result is trivial if $|c| < 1$.

1. Carleson's theorem $\Rightarrow$ I. J. Schark's theorem, but $\not\Leftarrow$ is false.
2. The analogous result to Carleson's theorem for higher dimensions, e.g. $\mathbb{C}^2$ with the Euclidean or max norms, is unknown. Put briefly, for $\dim X = 1$, there are no known counterexamples; for $\dim X \geq 2$, there are no positive results. On the other hand,
3. There is no known situation in which I. J. Schark's theorem is false.

(3) Problems involving $\mathcal{H}^\infty(B_X)$

First, we're interested in a cluster value theorem, *à la* I. J. Schark. To start, for a given complex Banach space X, observe that $\delta(B_X) \equiv \{\delta_c \mid c \in B_X\} \subset \mathcal{M}(\mathcal{H}^\infty(B_X))$. Also, as before, endow $\mathcal{M}(\mathcal{H}^\infty(B_X))$ with the weak-star topology, considering it as a subspace of $(\mathcal{H}^\infty(B_X)^*$, weak-star).

Harder Problem: Is the Cluster Value Theorem still true? Namely, for a fixed $f \in \mathcal{H}^\infty(B_X)$ and a fixed point $z^{**} \in \overline{B}_{X^{**}}$, are the following two sets equal?

$\{w \in \mathbb{C} \mid \exists \text{ net } (z_\alpha)_\alpha \in B_X,\ z_\alpha \to z^{**}\ weak - * \ \&\ f(z_\alpha) \to w\}$;

$\{\varphi(f) \mid \varphi \in \mathcal{M}(\mathcal{H}^\infty(B_X)),\ \Pi(\varphi) = z^{**}\}$.

Remark Unlike the case $\dim X < \infty$, the fiber over *any, even an interior* point of $B_{X^{**}}$ is rich. In particular, $\beta\mathbb{N} \subset \Pi^{-1}(0)$. Even in this case, the easier (?) problem is open in general:
Easier Problem: For a fixed $f \in \mathcal{H}^\infty(B_X)$, are the following two sets equal?

$$\{w \in \mathbb{C} \mid \exists \text{ net } (z_\alpha)_\alpha \in B_X,\ z_\alpha \to 0 \text{ weakly } \&\ f(z_\alpha) \to w\};$$
$$\{\varphi(f) \mid \varphi \in \mathcal{M}(\mathcal{H}^\infty(B_X)),\ \Pi(\varphi) = 0\}.$$

Yes, even to the "harder" question, if $X = c_0$. Namely,
Theorem. Fix $f \in \mathcal{H}^\infty(B_{c_0})$ and $z^{**} \in \overline{B}_{\ell_\infty}$. Then the two sets

$$\{w \in \mathbb{C} \mid \exists \text{ net } (z_\alpha)_\alpha \in B_X,\ z_\alpha \to z^{**} \ weak - * \ \& \ f(z_\alpha) \to w\}$$

and

$$\{\varphi(f) \mid \varphi \in \mathcal{M}(\mathcal{H}^\infty(B_X)),\ \Pi(\varphi) = z^{**}\}$$

are equal.

The same result is unknown, even for the apparently easier case of $X = \ell_2$.

One basic idea for proof of harder problem, $X = c_0$. Notation: For $g \in \mathcal{H}^\infty(B_{c_0})$ and $n \in \mathbb{N}$, define $g_n \in \mathcal{H}^\infty(B_{c_0})$ by $g_n(x_1, ..., x_n, x_{n+1}, ...) \equiv g(0, ..., 0, x_{n+1}, ...)$.

Lemma

Fix $\varphi \in \mathcal{M}(\mathcal{H}(^\infty(B_{c_0})))$ *so that* $\Pi(\varphi) = 0$. *For any* $g \in \mathcal{H}^\infty(B_{c_0})$ *and any* $n \in \mathbb{N}$, $\varphi(g) = \varphi(g_n)$.

Remark The lemma is false if c_0 is replaced by ℓ_2 (and so we're stuck).

Problems about fibers in $\mathcal{M}(\mathcal{H}^\infty(B_X))$. Recall: For a complex Banach space X,
$\Pi : \mathcal{M}(\mathcal{H}^\infty(B_X)) \to X^{**}, \ \Pi(\varphi) \equiv \varphi|_{X^*}$.

Fix X and two points z^{**} and w^{**} in $\overline{B}_{X^{**}}$.
Problem What is the relation between the two fibers $\Pi^{-1}(z^{**})$ and $\Pi^{-1}(w^{**})$?

Suppose $X = \ell_2$. If $\|z\| = \|w\| = 1$, then $\Pi^{-1}(z) \backsimeq \Pi^{-1}(w)$. The same result holds if $\|z\|$ and $\|w\|$ are both < 1. What if $1 = \|z\| > \|w\|$?

Suppose $X = c_0$. Then $\|z\|, \|w\| < 1 \Rightarrow \Pi^{-1}(z) \backsimeq \Pi^{-1}(w)$. But for $\|z\| = \|w\| = 1$, the situation is murky.

For the special cases $\mathcal{H}^\infty(D)$ and $\mathcal{H}^\infty(D^2)$, what is known is that $\Pi^{-1}(1) \backsimeq \Pi^{-1}(a, b)$, if one of $|a|, |b| = 1$ and the other is < 1.
Also, $\Pi^{-1}(1)$ and $\Pi^{-1}(1, 1)$ are not homeomorphic.
(But the argument really uses dimension 1.)

Remark Even if $\dim X < \infty$ (so $B_X = B_{X^{**}}$) and even if $\|z\|, \|w\| < 1$, the problem, of whether $\pi^{-1}(z)$ and $\pi^{-1}(w)$ are (somehow) the "same'," is apparently unknown in general. The problem is that, in general, it isn't known if $\Pi^{-1}(z) = \{\delta_z\}$ if $\dim X < \infty$ and $\|z\| < 1$.