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Porosity results for sets of strict contractions

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joint work with Michael Dymond and Simeon Reich

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The main actors

Let (X, ρ_X) and (Y, ρ_Y) be metric spaces. A mapping $f\colon X \to Y$ is called nonexpansive if

$$\rho_Y(f(x), f(y)) \leq \rho_X(x, y) \quad \text{for all} \quad x, y \in X.$$

Moreover f is called a strict contraction if

$$\rho_Y(f(x), f(y)) \leq L\rho_X(x, y) \quad \text{for all} \quad x, y \in X$$

and for some $L < 1$.

Theorem (Brouwer, 1911)

Let $C \subseteq \mathbb{R}^d$ be nonempty, bounded, closed and convex. Then every continuous mapping

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has a fixed point.

Motivation

Theorem (Brouwer, 1911)

Let $C \subseteq \mathbb{R}^d$ be nonempty, bounded, closed and convex. Then every continuous mapping

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has a fixed point.

This is no longer true in infinite dimensions. For example let

$$C := \{g \in \mathcal{C}[0,1] : 0 = g(0) \leq g(t) \leq g(1) = 1 \text{ for } t \in [0,1]\}$$

and

$$T : C \to C, \quad (Tg)(t) := tg(t)$$

Then f is even nonexpansive but has no fixed point.

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Generic properties of nonexpansive mappings

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We define

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equipped with the metric of uniform convergence which makes $\mathcal{M}$ a complete metric space.

Let M be a complete metric space. A set $A \subseteq M$ is of the first Baire category if it is a countable union of nowhere dense sets.

σ-porous sets

A subset $A \subseteq M$ is said to be *porous at* $x \in A$ if there are constants $\alpha > 0$ and $\varepsilon_0 > 0$ with the following property: For all $\varepsilon \in (0, \varepsilon_0)$ there is a point $y \in M$ with $\|y - x\| \leq \varepsilon$ and $B(y, \alpha\,\varepsilon) \cap A = \emptyset$. The set A is called *porous* if it is porous at all of its points.

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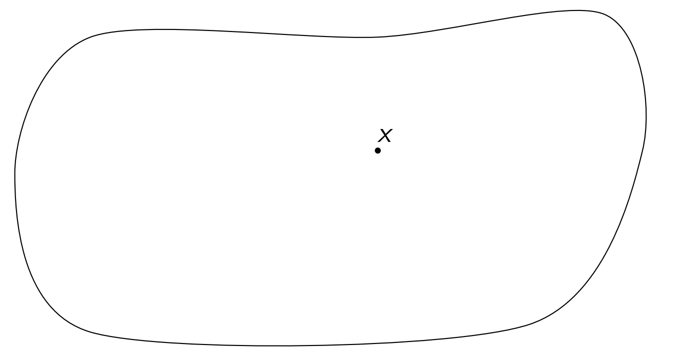

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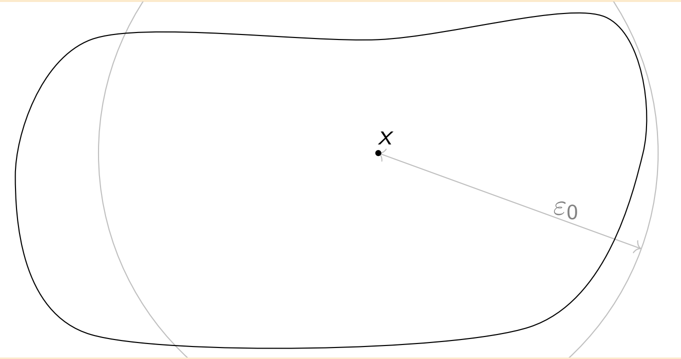

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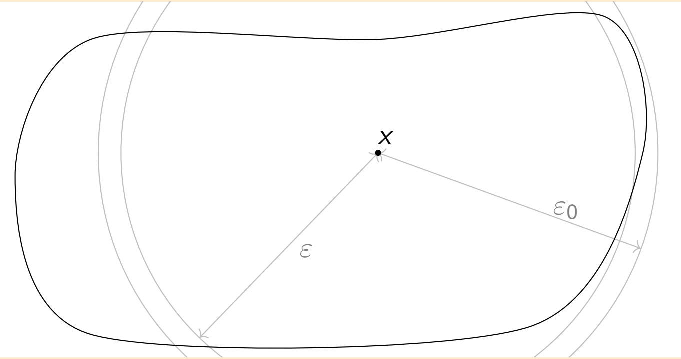

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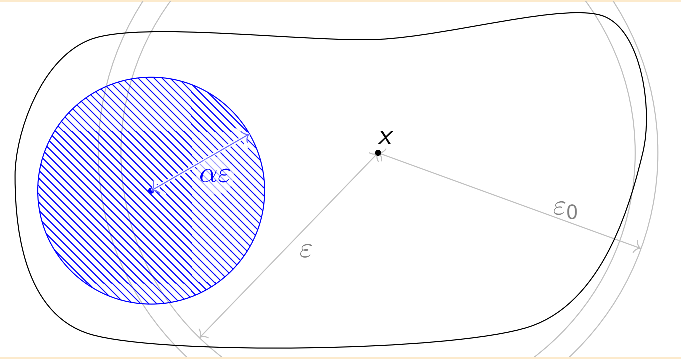

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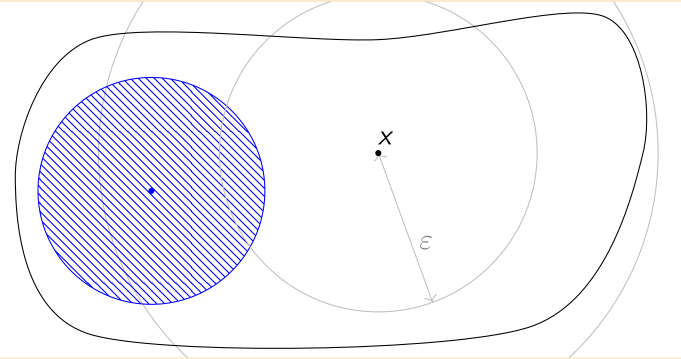

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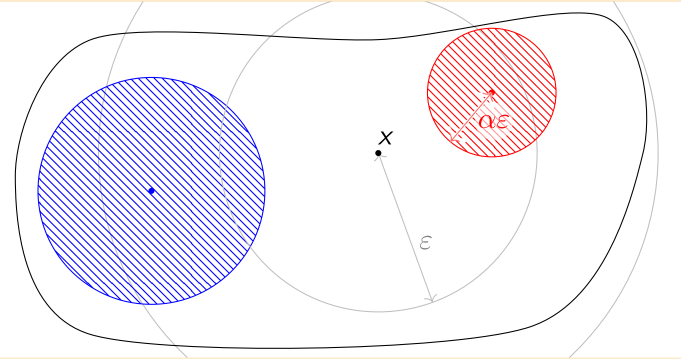

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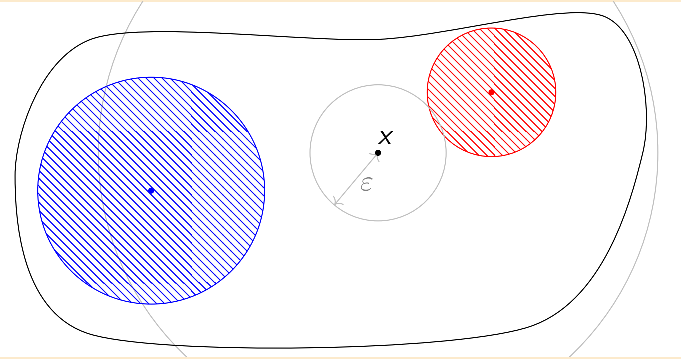

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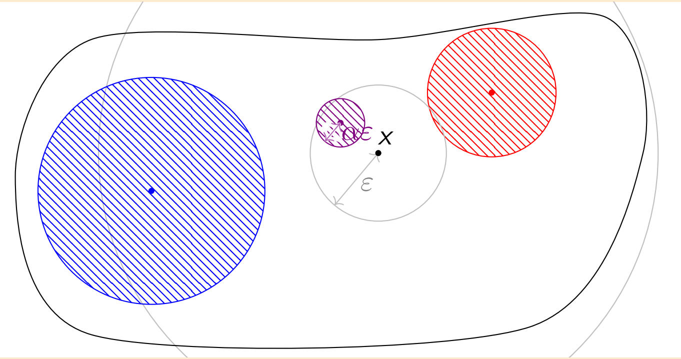

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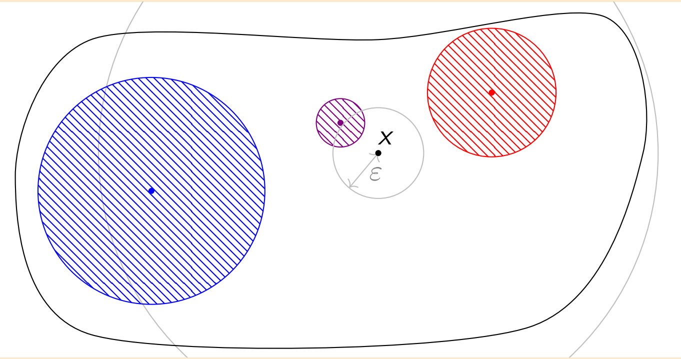

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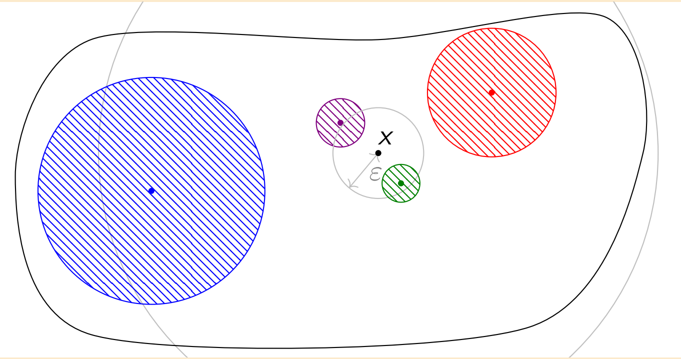

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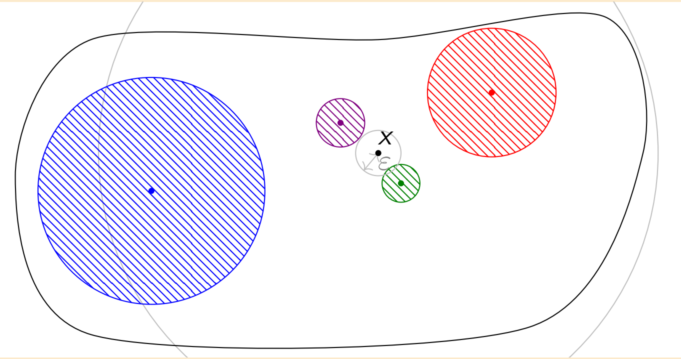

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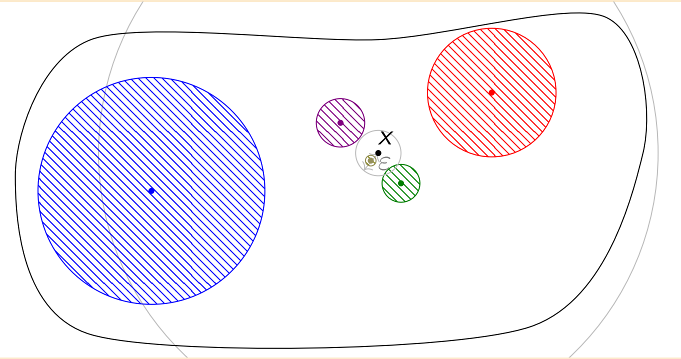

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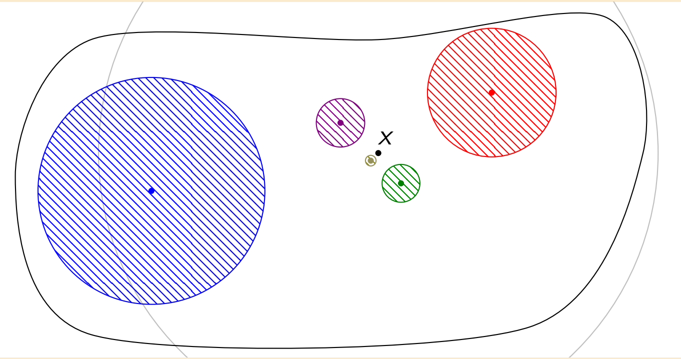

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Note that σ-porous sets are of first category in the sense of the Baire category theorem.

Theorem (de Blasi, Myjak, 1989)

Let X be a Banach space and $C \subset X$ be nonempty, bounded, closed and convex. The set of nonexpansive mappings without a fixed point is a σ-porous subset of $\mathcal{M}$.

Generic properties of nonexpansive mappings, ctd.

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What is behind this result? Is it Banach's fixed point theorem? No, at least not in Hilbert spaces ...

Theorem (de Blasi, Myjak, 1989)

Let X be a Hilbert space and $C \subseteq X$ be nonempty, bounded, closed and convex. The set of strict contractions is a σ-porous subset of $\mathcal{M}$.

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Generic properties of nonexpansive mappings, ctd.

Generic existence of fixed points has been shown in many situations. Some examples:

- Self-mappings on nonempty, closed and convex subsets of hyperbolic spaces: Reich and Zaslavski
- Set-valued mappings on nonempty, closed and star-shaped subsets of hyperbolic spaces: de Blasi, Myjak, Reich and Zaslavski, Peng and Luo

How many strict contractions are there?

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In separable spaces, we also have a local version of this result.

Theorem (B., Dymond, 2016)

Let X be a separable Banach space and $C \subseteq X$ be nonempty, bounded, closed and convex. The generic nonexpansive mapping satisfies

$$\mathrm{Lip}(f, x) = \limsup_{r \to 0^+} \left\{ \frac{\|f(x) - f(y)\|}{\|x - y\|} : y \in B(x, r) \setminus \{x\} \right\} = 1$$

at typical points x of its domain C.

What happens beyond Banach spaces?

Metric segments and geodesic metric spaces

Let (X, ρ_X) be a metric space. (X, ρ_X) is called *geodesic* if for every pair $x, y \in X$ there is an isometric embedding

$$c\colon [0, \rho_X(x, y)] \to X$$

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For a metric segment $[x, y]$ and $\lambda \in [0, 1]$ we denote by

$$(1 - \lambda)x \oplus \lambda y$$

the unique point $z \in [x, y]$ satisfying

$$\rho_X(x, z) = \lambda \rho_X(x, y) \quad \text{and} \quad \rho_X(z, y) = (1 - \lambda)\rho_X(x, y).$$

Let (X, ρ_X) be a metric space and $\mathcal{F}$ a family of metric segments in (X, ρ_X).

Convexity conditions in metric spaces

Let (X, ρ_X) be a metric space and $\mathcal{F}$ a family of metric segments in (X, ρ_X).

A set $C \subseteq X$ is called ρ_X-convex if for all $x, y \in C$ there is a metric segment $[x, y] \in \mathcal{F}$ joining x and y and $[x, y] \subseteq C$.

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A set $C \subseteq X$ is called ρ_X-star-shaped if there is a point $c \in C$ such that for all $x \in C$ there is a metric segment $[x, c] \in \mathcal{F}$ joining x and c and $[x, c] \subseteq C$.

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For a ρ_X-star-shaped set $C \subseteq X$, we denote by $\mathrm{star}(C)$ the set of all centres of C, i.e. all points points $c \in C$ such that for $x \in C$ there is a metric segment $[x, c] \in \mathcal{F}$ where $[x, c] \subseteq C$.

Given a metric space (X, ρ_X) and a family $\mathcal{F}$ of metric segments in X, we call the triple $(X, \rho_X, \mathcal{F})$ *hyperbolic* if the following conditions are satisfied:

(i) For each pair $x, y \in X$, there exists a unique metric segment $[x, y] \in \mathcal{F}$ joining x and y.

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(i) For each pair $x, y \in X$, there exists a unique metric segment $[x, y] \in \mathcal{F}$ joining x and y.

(ii) For all $x, y, z, w \in X$ and all $t \in [0, 1]$,

$$\rho_X((1-t)x \oplus ty, (1-t)w \oplus tz) \leq (1-t)\rho_X(x, w) + t\rho_X(y, z).$$

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(iii) The collection $\mathcal{F}$ is closed with respect to subsegements. More precisely, for all $x, y \in X$ and $u, v \in [x, y]$ we have $[u, v] \subseteq [x, y]$.

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for all $x, y, z, w \in X$ is equivalent to

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for all $x, y, z \in X$. The second condition is basically a condition on triangles.

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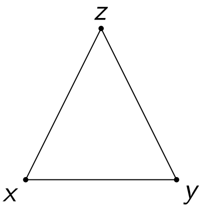

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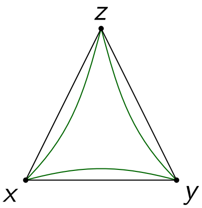

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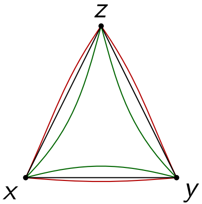
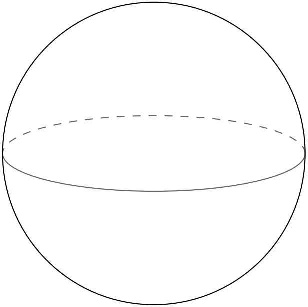


Hyperbolic spaces

Examples of hyperbolic spaces

- Banach spaces
- $\mathrm{CAT}(0)$-spaces (and hence all $\mathrm{CAT}(\kappa)$-spaces for $\kappa \leq 0$)
- The Hilbert ball with the hyperbolic metric

Spherical triangles



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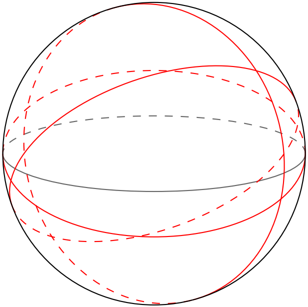

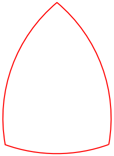

This example shows that the sphere does not even satisfy

$$\rho((1-t)x \oplus tz, (1-t)y \oplus tz) \leq \rho(x, y)$$

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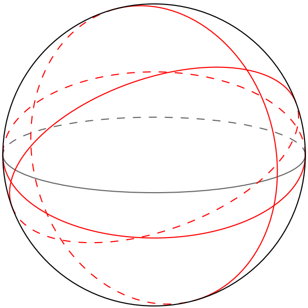

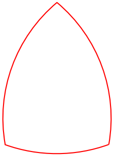

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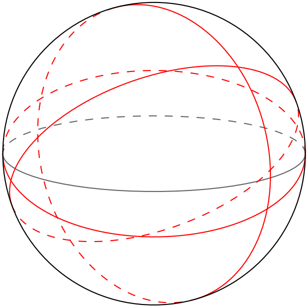

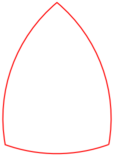

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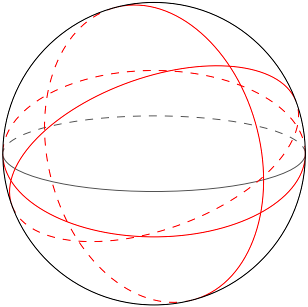

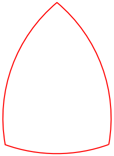

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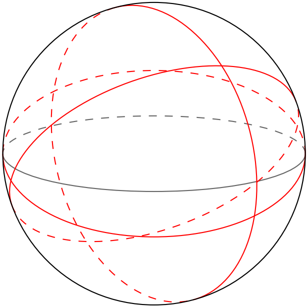

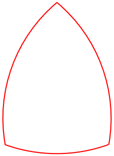

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So if we want to include the sphere we need a weaker condition on the triangles.

Weakly hyperbolic spaces

Given a metric space (X, ρ_X) and a family $\mathcal{F}$ of metric segments in X, we say that the triple $(X, \rho_X, \mathcal{F})$ is *weakly hyperbolic* if the following conditions are satisfied:

(i) There exists a constant $D_X > 0$ s. t. for any $x, y \in X$ with $\rho_X(x, y) < D_X$, there is a unique metric segment $[x, y] \in \mathcal{F}$.

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whenever $z, w \in B(x, \delta_X)$, $[z, y], [w, y] \in \mathcal{F}$ and $t \in [0, \delta_X)$.

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(iii) $\mathcal{F}$ is closed with respect to subsegments, that is, for all metric segments $[x, y] \in \mathcal{F}$ and all points $z, w \in [x, y]$ there is a metric segment $[z, w] \in \mathcal{F}$ with $[z, w] \subseteq [x, y]$.

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(iv) For all $x \in X$ and $r \in (0, D_X/2)$, the ball $B(x, r)$ is a ρ_X-convex subset of X.

Examples of weakly hyperbolic spaces

- Hyperbolic spaces
- $\mathrm{CAT}(\kappa)$-spaces for $\kappa \in \mathbb{R}$.

Spaces of nonexpansive mappings

Let (X, ρ_X) and (Y, ρ_Y) be two complete metric spaces. In addition let $C_X \subseteq X$ and $C_Y \subseteq Y$ be closed subsets. We choose $\theta \in C_X$ and set

$$\mathcal{M}(C_X, C_Y) := \{f \colon C_X \to C_Y \colon \operatorname{Lip}(f) \leq 1\}$$

and

$$d_\theta(f, g) := \sup_{x \in C_X} \frac{\rho_Y(f(x), f(x))}{1 + \rho_X(x, \theta)} \quad \text{for} \quad f, g \in \mathcal{M}(C_X, C_Y).$$

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Then,

- $(\mathcal{M}(C_X, C_Y), d_\theta)$ is a complete metric space.
- For $\theta_1 \neq \theta$, the metrics d_θ and d_{θ_1} are equivalent.

Theorem (B., Dymond, Reich, 2016)

Let X and Y be complete weakly hyperbolic spaces and $C_X \subseteq X$ and $C_Y \subseteq Y$ nonempty, non-singleton, closed and ρ_X- and ρ_Y-star-shaped subsets, respectively. In addition assume that $C_Y \subseteq B(\mathrm{star}(C_Y), D_Y)$. Then the set

$$\{f\colon C_X \to C_Y\colon \mathrm{Lip}(f) < 1\}$$

is a σ-porous subset of $\mathcal{M}(C_X, C_Y)$.

σ-porosity of the set of strict contractions—local version

Theorem (B., Dymond, Reich, 2016)

Suppose that in addition C_X is separable and ρ_X-convex. Then there exists a σ-porous set $\widetilde{\mathcal{N}} \subseteq \mathcal{M}(C_X, C_Y)$ such that for every mapping $f \in \mathcal{M}(C_X, C_Y) \setminus \widetilde{\mathcal{N}}$, the set

$$R(f) = \{x \in C_X \colon \operatorname{Lip}(f, x) = 1\}$$

is a residual subset of C_X.

Special case I: Nonexpansive self-mappings

Corollary

Let (X, ρ_X) be a complete hyperbolic space and $C_X \subset X$ a nonempty, non-singleton, closed and ρ_X-star-shaped subset. The set of strict contractions is a σ-porous subset of the space $\mathcal{M}(C_X, C_X)$ of nonexpansive self-mappings of C_X.

Corollary

Suppose C_X is a separable and ρ_X-convex subset of a weakly hyperbolic space X. There exists a σ-porous set $\widetilde{\mathcal{N}} \subseteq \mathcal{M}(C_X, C_X)$ such that for every $f \in \mathcal{M}(C_X, C_X) \setminus \widetilde{\mathcal{N}}$, the set

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Special case II: Set-valued mappings

Given a set C, we denote by $\mathcal{B}(C) := \{A \subset C \colon A \text{ bounded, closed}\}$ the hyperspace of bounded and closed subsets of C. We equip $\mathcal{B}(C)$ with the Hausdorff distance.

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Corollary

Let X be a complete hyperbolic space and $C \subseteq X$ be a non-empty, non-singleton, closed, ρ-star-shaped subset. Then the set of strict contractions is a σ-porous subsets of the space

$$\mathcal{M}(C, \mathcal{B}(C)) := \{f\colon C \to \mathcal{B}(C)\colon \operatorname{Lip}(f) \leq 1\}$$

of all nonexpansive $\mathcal{B}(C)$-valued mappings equipped with the metric d_θ.

Some remarks

- In addition to the hyperspace of bounded and closed sets, the previous result is also true for
  - the space of singletons;
  - the space of compact subsets;
  - the space of bounded, closed and ρ_X-convex sets and
  - the space of compact and ρ_X-convex sets.

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- In addition to the hyperspace of bounded and closed sets, the previous result is also true for
  - the space of singletons;
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  - the space of bounded, closed and ρ_X-convex sets and
  - the space of compact and ρ_X-convex sets.
- The presented porosity results are also true if we replace the space $\mathcal{M}(C_X, C_Y)$ by the space of bounded nonexpansive mappings with the metric

$$d_\infty(f,g) := \sup_{x \in C_X} \rho_Y(f(x), g(x)).$$

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