A NEW CHARACTERIZATION OF WEAK COMPACTNESS

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Outline

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- Fundamental Question

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- Motivations
- A Few Known Answers
- Contributions

Fundamental Question

(FQ)

Let C be a bounded, closed convex subset of a Banach space. Assume that whenever $K \subset C$ is closed and convex, then every uniformly Lipschitz affine mapping $f\colon K \to K$ has a fixed point. Is C necessarily weakly compact ?

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where $f^p = f \circ f \circ \cdots \circ f$ (p-times), and **Affine** when

$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda) f(y), \quad \forall \lambda \in [0,1],\ \forall x, y \in K.$$

Remarks

Def. Let $\mathcal{F}$ be a class of maps. Then C is said to have the $\mathcal{G}$-FPP wrt $\mathcal{F}$ if whenever $K \subset C$ is closed and convex then every map $f \in \mathcal{F}$ with $f(K) \subset K$ has a fixed point.

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- A Banach space X is said to be an L-embedded if X is an L-summand in X^{**}.

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  - (ii) If X is either c_0 of J_p, then C is weakly compact iff C has the $\mathcal{G}$-FPP for **Uniformly Lipschitz Affine** Maps.
  - (iii) If X is L-embedded Banach space, then C is weakly compact iff C has $\mathcal{G}$-FPP for **1-Lipschitz Affine** Maps.
- (Strategy) Assume that C is not weakly compact and try to find a basic sequence $(x_n) \subset C$ with very nice structural properties.

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- *So, the fixed point problem* (FQ) *also has a Banach space structural nature.*

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- Characterization of Weak Compactness
  - Fact: *Weak Compactness do not characterize* $\mathcal{G}$*-FPP for* 1*-Lipschitz Maps*

Thm (Accepted under minor revisions in AIF, 2017)

Weak Compactness Characterizes $\mathcal{G}$-FPP for the class $\mathcal{F}$ of **Bi-Lipschitz Affine** *Maps.*

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$$K = \overline{\mathrm{co}}\{(x_n)\} = \left\{ \sum_{n=1}^{\infty} t_n x_n : \; t_n \geq 0 \; \& \; \sum_{n=1}^{\infty} t_n = 1 \right\}$$

Next, choose suitable sequences $(\lambda^{(n)})_k$ so that

- $\alpha_n := 1 - \sum_{k=n}^{\infty} \lambda_k^{(n)}$ is non-decreasing in $(0, 1)$
- $1/2 \leq \alpha_n \to 1$
- $$\sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \lambda_k^{(n)} < \frac{\varepsilon}{4\mathcal{K}} \frac{\inf_n \|x_n\|}{\sup_n \|x_n\|},$$

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with,

$$z_n = \alpha_{n+1} x_n + \sum_{k=n+1}^{\infty} \lambda_k^{(n+1)} x_k$$

Lemma

(z_n) *is equivalent to* (x_n)

Proof. Partially thanks to a result of Hájek and Johanis (JDE, 2010).

Corollary. f is bi-Lipschitz, affine and fixed point free.

A positive answer under unconditionality

Thm

If X contains an unconditional basis, then a bounded, closed convex set $C \subset X$ is weakly compact iff it has the $\mathcal{G}$-FPP for the class of uniformly Lipschitz affine maps.

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Sketch... If C is not weakly compact and contains no ℓ_1-basic sequence, then we can find a wide-(s) sequence $(y_n) \subset C$ without weakly convergent subsequences. Since X has an unconditional basis, the space $[(y_n)]$ has property (u) of Pełczyński. Thus, some convex block of (y_n) must be equivalet to the summing basis of c_0. This shows that C contains a c_0-summing basic sequence. The proof is essentially done.

Thanks!!!