Multiple summing maps: coordinatewise summability, inclusion theorems and p-Sidon sets

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Nonlinear Functional Analysis 2017

Let $L : c_0 \to \mathbb{K}$ be linear and continuous. Then $\sum_n |L(e_n)| \leq \|L\|$.
Basic question: let $L : c_0 \times c_0 \to \mathbb{K}$ be bilinear and continuous. What can be said on the sequence $(L(e_i, e_j))_{i,j}$?

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Theorem (Littlewood, 1930)

There exists $C > 0$ such that, for all $L : c_0 \times c_0 \to \mathbb{K}$ bilinear and continuous,

$$\left(\sum_{i,j} |L(e_i, e_j)|^{4/3}\right)^{3/4} \leq C\|L\|.$$

Moreover, the exponent $4/3$ is optimal.

Proof - Step 1: an inequality on matrices

Lemma

Let $(a_{i,j})_{1\leq i,j\leq N}$. Then

$$\left(\sum_{i,j}|a_{i,j}|^{4/3}\right)^{3/4}\leq\left(\sum_{i=1}^N\left(\sum_{j=1}^N|a_{i,j}|^2\right)^{1/2}\right)^{1/2}\left(\sum_{j=1}^N\left(\sum_{i=1}^N|a_{i,j}|^2\right)^{1/2}\right)^{1/2}.$$

The proof is done by successive applications of Minkowski and Hölder inequalities.

Proof - Step 2: an application of Khinchine inequality

Lemma

For each $p \in [1,+\infty)$, there exist $A_p, B_p > 0$ such that, for all sequences $(x_i)_{i=1}^N$ of complex numbers,

$$A_p \left(\sum_{i=1}^N |x_i|^2 \right)^{1/2} \leq \left(\int_\Omega \left| \sum_{i=1}^N \varepsilon_i(\omega) x_i \right|^p d\mathbb{P}(\omega) \right)^{1/p} \leq B_p \left(\sum_{i=1}^N |x_i|^2 \right)^{1/2},$$

where $(\varepsilon_i)_{i=1}^N$ is a sequence of independent Bernoulli variables on $(\Omega, \mathcal{A}, \mathbb{P})$.

Proof - Step 2: an application of Khinchine inequality

Let $L : c_0 \times c_0 \to \mathbb{K}$ be bilinear and bounded. Then Later Much later

$$\sum_{i\leq N}\left(\sum_{j\leq N}|L(e_i,e_j)|^2\right)^{1/2} \leq C\sum_{i\leq N}\int_\Omega\left|\sum_{j\leq N}L(e_i,e_j)\varepsilon_j(\omega)\right|d\mathbb{P}(\omega)$$

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$$\le C\int_\Omega\sum_{i\le N}\left|L\left(e_i,\sum_{j\le N}\varepsilon_j(\omega)e_j\right)\right|d\mathbb{P}(\omega)$$

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$$\begin{aligned}
\sum_{i\leq N}\left(\sum_{j\leq N}|L(e_i,e_j)|^2\right)^{1/2} &\leq C\sum_{i\leq N}\int_\Omega\left|\sum_{j\leq N}L(e_i,e_j)\varepsilon_j(\omega)\right|d\mathbb{P}(\omega)\\
&\leq C\int_\Omega\sum_{i\leq N}\left|L\left(e_i,\sum_{j\leq N}\varepsilon_j(\omega)e_j\right)\right|d\mathbb{P}(\omega)\\
&\leq C\int_\Omega\sup_{y\in B_{c_0}}\sum_{i\leq N}|L(e_i,y)|d\mathbb{P}(\omega)
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&\leq C\sup_{y\in B_{c_0}}\sum_{i\leq N}|L(e_i,y)|\\
&\leq C\sup_{y\in B_{c_0}}\sup_{x\in B_{c_0}}|L(x,y)|\\
&\leq C\|L\|.
\end{aligned}$$

Proof - Step 3: optimality

$4/3$ is the best exponent such that, for all $L : c_0 \times c_0 \to \mathbb{K}$ bilinear and continuous,

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Theorem (Kahane,Salem,Zygmund)

For all $m \geq 2$, there exists $C_m > 0$ such that, for all $N \geq 1$, there exists an m-linear form $L : \ell_\infty^N \times \cdots \times \ell_\infty^N \to \mathbb{C}$ which may be written

$$L(z^{(1)}, \dots, z^{(m)}) = \sum_{i_1,\dots,i_m=1}^{N} \pm z_{i_1}^{(1)} \cdots z_{i_m}^{(m)}$$

and which satisfies $\|L\| \leq C_m N^{\frac{m+1}{2}}$.

If we apply this for $m = 2$, then for all $N \geq 1$, we get a bilinear form $L_N : \ell_\infty^N \times \ell_\infty^N \to \mathbb{C}$ with $|L(e_i, e_j)| = 1$ and $\|L\| \leq C_2 N^{3/2}$. If p is a convenient exponent, for all $N \geq 1$,

$$\left(\sum_{i=1}^{N} |L(e_i, e_j)|^p\right)^{1/p} \leq C_2 N^{3/2},$$

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$$\left(\sum_{i=1}^{N} |L(e_i, e_j)|^p\right)^{1/p} \leq C_2 N^{3/2},$$

namely

$$N^{2/p} \leq C_2 N^{3/2}.$$

This implies $p \geq 4/3$.

The Bohnenblust-Hille inequality

Theorem (Bohnenblust-Hille, 1931)

There exists $C_m > 0$ such that, for all $L : c_0 \times \cdots \times c_0 \to \mathbb{C}$ m-linear and continuous,

$$\left(\sum_{i_1,\dots,i_m} |L(e_{i_1},\dots,e_{i_m})|^{2m/(m+1)}\right)^{(m+1)/2m} \leq C_m \|L\|.$$

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- It was used by Bohnenblust and Hille to solve a problem on Dirichlet series;
- It was used recently to study problems in complex analysis (the Bohr radius of the polydisc $\mathbb{D}^n$). Here, having an estimate of C_m is important.

Hardy-Littlewood/Praciano Pereira/Dimant-Sevilla Peris

Theorem

Let $1 \leq p_1, p_2, \dots, p_m \leq +\infty$ *such that* $\frac{1}{p_1} + \cdots + \frac{1}{p_m} < 1$. *Define*

$$\frac{1}{\lambda} = 1 - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_m}\right) \text{ and } \frac{1}{\mu} = \frac{1}{m\lambda} + \frac{m-1}{2m}.$$

There exists $C > 0$ *such that, for every* m*-linear* $T : \ell_{p_1} \times \cdots \times \ell_{p_m} \to \mathbb{K}$,

1. *if* $\frac{1}{2} \leq \frac{1}{p_1} + \cdots + \frac{1}{p_m} < 1$, *then* $\sum_{i_1,\dots,i_m} |T(e_{i_1}, \dots, e_{i_m})|^\lambda \leq C\|T\|^\lambda$.
2. *if* $0 \leq \frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{2}$, *then* $\sum_{i_1,\dots,i_m} |T(e_{i_1}, \dots, e_{i_m})|^\mu \leq C\|T\|^\mu$.

Moreover, these exponents are optimal. Main result

An abstract version of Littlewood inequality

Let $T : X \times Y \to \mathbb{C}$ be bilinear and continuous and let $(x_i)_{i=1}^N$, $(y_i)_{i=1}^N$ be two sequences in X (resp. Y). Let $S_1 : \ell_\infty^N \to X$, $e_i \mapsto x_i$, $S_2 : \ell_\infty^N \to Y$, $e_j \mapsto y_j$ and $L = T(S_1, S_2)$. Littlewood's inequality says that

$$\left(\sum_{i,j=1}^N |L(e_i, e_j)|^{4/3}\right)^{3/4} \leq C\|L\| \leq C\|T\| \times \|S_1\| \times \|S_2\|.$$

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$$\begin{aligned}
\|S_1\| &= \sup_{(a_i)\in B_{\ell_\infty}} \Big\| \sum_{i\leq N} a_i x_i \Big\| \\
&= \sup_{(a_i)\in B_{\ell_\infty}} \sup_{x^*\in B_{X^*}} \sum_i a_i \langle x^*, x_i\rangle \\
&= \sup_{x^*\in B_{X^*}} \sup_{(a_i)\in B_{\ell_\infty}} \sum_i a_i \langle x^*, x_i\rangle \\
&= \sup_{x^*\in B_{X^*}} \sum_i |\langle x^*, x_i\rangle| .
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Similarly,

$$\|S_2\| = \sup_{y^*\in B_{Y^*}} \sum_i |\langle y^*, y_i\rangle| .$$

An abstract version of Littlewood inequality

Thus Littlewood's inequality says that, for all bilinear forms $T : X \times Y \to \mathbb{C}$, for all finite sequences $(x_i)_{i=1}^N,\ (y_i)_{i=1}^N$, then

$$\left(\sum_{i,j\leq N} \|T(x_i,y_j)\|^{4/3}\right)^{3/4} \leq C\|T\| \times \sup_{x^*\in B_{X^*}} \sum_i |\langle x^*,x_i\rangle| \times \sup_{y^*\in B_{Y^*}} \sum_i |\langle y^*,y_i\rangle|.$$

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This means that T is multiple $(4/3;1)$-summing.

Multiple summing maps

For a sequence $x=(x_i)_{i\in\mathbb{N}}\subset X^{\mathbb{N}}$, its weak ℓ^p-norm is defined by

$$w_p(x) = \sup_{\|x^*\|\leq 1}\left(\sum_{i=1}^{+\infty}|x^*(x_i)|^p\right)^{\frac{1}{p}}.$$

Definition (Bombal,Perez-Garcia,Villanueva/Matos (2003-2004))

Let $r\geq 1$ and $\mathbf{p}=(p_1,\dots,p_m)\in[1,+\infty)^m$. An m-linear map $T:X_1\times\cdots\times X_m\to Y$ is multiple $(r,\mathbf{p})$-summing if there exists a constant $C>0$ such that for all sequences $x(j)\subset X_j^{\mathbb{N}}$, $1\leq j\leq m$,

$$\left(\sum_{\mathbf{i}\in\mathbb{N}^m}\|T(x_{\mathbf{i}})\|^r\right)^{\frac{1}{r}}\leq Cw_{p_1}(x(1))\cdots w_{p_m}(x(m))$$

where $T(x_{\mathbf{i}})$ stands for $T(x_{i_1}(1),\dots,x_{i_m}(m))$.

Multiple summing maps

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$$\left(\sum_{\mathbf{i}\in\mathbb{N}^m} \|T(x_{\mathbf{i}})\|^r\right)^{\frac{1}{r}} \leq C w_{p_1}(x(1)) \cdots w_{p_m}(x(m)).$$

- When $m = 1$, we recover the classical theory of summing linear maps.
- The duality c_0/ℓ_1 says that all linear forms are $(1,1)$-summing.
- Bohnenblust-Hille inequality says that all m-linear forms $T : X_1 \times \cdots \times X_m \to \mathbb{K}$ are multiple $\left(\frac{2m}{m+1}, 1\right)$-summing.

Littlewood again

A crucial ingredient in the proof of Littlewood's inequality was that, for all $x \in B_{c_0}$ and all $y \in B_{c_0}$,

$$\sum_i |L(e_i, y)| \leq \|L\| \text{ and } \sum_j |L(x, e_j)| \leq \|L\|.$$

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In other words, in order to prove that any bilinear form $T : X \times Y \to \mathbb{C}$ is multiple $(4/3, 1)-$summing, a crucial ingredient is that the restrictions $T(x, \cdot)$ and $T(\cdot, y)$ are $(1, 1)$-summing linear maps.

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Question: Let $T : X_1 \times \cdots \times X_m \to Y$ be m-linear such that each restriction map $T(x_1, \ldots, x_{k-1}, \cdot, x_{k+1}, \ldots, x_m)$ is (r_k, p_k)-summing. Can we say something on the multiple summability of T?

Defant,Popa, Schwarting

Definition

Let $T : X_1 \times \cdots \times X_m \to Y$ be m-linear. We say that T is (r,p)-summing in the k-th coordinate if, for all $x_1,\dots,x_{k-1},x_{k+1},\dots,x_m \in X_1 \times \cdots \times X_{k-1} \times X_{k+1} \times \cdots \times X_m$, the linear map $T(x_1,\dots,x_{k-1},\cdot,x_{k+1},\dots,x_m)$ is (r,p)-summing.

Theorem (Defant, Popa, Schwarting (2010))

Let $T : X_1 \times \cdots \times X_m \to Y$ be m-linear with Y a cotype q space. Let $r \in [1,q]$ and assume that T is $(r,1)$-summing in each coordinate. Then T is multiple $(s,1)$-summing, with

$$\frac{1}{s} = \frac{m-1}{mq} + \frac{1}{mr}.$$

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Bohnenblust-Hille inequality follows immediately (with $r = 1$ and $q = 2$)!

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We recall that Y has cotype $q \geq 2$ if there exists $C > 0$ such that, for all finite sequences $y_1, \ldots, y_N$ of elements of Y,

$$\left(\sum_{i=1}^{N} \|y_i\|^q\right)^{1/q} \leq C \int_\Omega \left\|\sum_{i=1}^{N} \varepsilon_i(\omega) y_i\right\| d\mathbb{P}(\omega).$$

proof

Main results

Hardy-Littlewood

Theorem

Let $T : X_1 \times \cdots \times X_m \to Y$ be $m-$linear with Y a cotype q space. Assume that T is (r,p)-summing in each coordinate and let $t \geq p$.

- *If $\frac{1}{r} + \frac{1}{p^*} - \frac{m}{t^*} > \frac{1}{q}$, then T is multiple (s,t)-summing with*

$$\frac{1}{s} = \frac{m-1}{mq} + \frac{1}{mr} + \frac{1}{mp^*} - \frac{1}{t^*}.$$

- *If $0 < \frac{1}{r} + \frac{1}{p^*} - \frac{m}{t^*} \leq \frac{1}{q}$, then T is multiple (s,t)-summing with*

$$\frac{1}{s} = \frac{1}{r} + \frac{1}{p^*} - \frac{m}{t^*}.$$

When $1 \leq p = t \leq 2$ and $q = 2$, the above values of s are optimal.

Main results

Theorem

Let $T : X_1 \times \cdots \times X_m \to Y$ with Y a cotype q space and $\mathbf{p} \in [1, +\infty)^m$. Assume that T is (r_k, p_k)-summing in the k-th coordinate and that there exists $\theta < 0$ such that $\frac{1}{r_k} - \frac{1}{p_k} = \theta$ for all k. Set

$$\frac{1}{\gamma} = 1 + \theta - \sum_{k=1}^{m} \frac{1}{p_k^*}.$$

1. *If $\gamma \in (0, q)$, then T is multiple $(s, \mathbf{p})$-summing with*

$$\frac{1}{s} = \frac{m-1}{mq} + \frac{1}{\gamma m}.$$

2. *If $\gamma \geq q$, then T is multiple $(\gamma, \mathbf{p})$-summing.*

Main results

We have a very general version of this kind (allowing different values for p_k and r_k, allowing summability on bigger sets of coordinates). In particular, we also get

Corollary (Popa, Sinnamon (2013))

Let $T : X_1 \times \cdots \times X_m \to Y$ be m-linear with Y a cotype q space. Let $r_1, \dots, r_m \in [1, q]$ and assume that T is $(r_k, 1)$-summing in the k-th coordinate. Then T is multiple $(s, 1)$-summing, with

$$s = \frac{qR}{1+R} \textit{ and } R = \sum_{k=1}^{m} \frac{r_k}{q - r_k}.$$

Sidon sets Third tool

Hardy-Littlewood/Praciano Pereira/Dimant-Sevilla Peris

Theorem (Abstract form)

Let $T : X_1 \times \cdots \times X_m \to \mathbb{C}$ *be m-linear and let* $\mathbf{p} = (p_1, \ldots, p_m) \in [1, +\infty)^m$. *We set*

$$\frac{1}{\gamma} = 1 - \sum_{k=1}^{m} \frac{1}{p_k^*}.$$

1. *If* $\gamma \in (0, 2)$ *then* T *is multiple* $(s, \mathbf{p})$*-summing with*

$$\frac{1}{s} = \frac{m-1}{2m} + \frac{1}{m\gamma}.$$

2. *If* $\gamma \geq 2$, *then* T *is multiple* $(\gamma, \mathbf{p})$*-summing.*

Hardy-Littlewood/Praciano Pereira/Dimant-Sevilla Peris

Theorem (Abstract form)

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$q = 2$, $r_k = p_k$ and the inclusion theorem for summing maps.

Inclusion theorems

Let $T : X \to Y$ be linear. If T is (r, p)-summing, then it is also (s, q)-summing provided $q \geq p$ and $\frac{1}{s} - \frac{1}{q} = \frac{1}{r} - \frac{1}{p}$.

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Theorem (Perez-Garcia (2004))

Let $T : X_1 \times \cdots \times X_m \to Y$ be m-linear. If T is multiple (p,p)-summing, for $p \in [1,2)$, then it is also multiple (q,q)-summing for $q \in [p,2)$.

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Theorem (Perez-Garcia (2004))

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Theorem

Let $T : X_1 \times \cdots \times X_m \to Y$ be m-linear, let $r, s \in [1,+\infty)$, $\mathbf{p}, \mathbf{q} \in [1,+\infty)^m$. Assume that T is multiple $(r,\mathbf{p})$-summing and that $q_k \geq p_k$ for all $k = 1, \ldots, m$. Then T is multiple $(s,\mathbf{q})$-summing, with

$$\frac{1}{s} - \sum_{j=1}^{m} \frac{1}{q_j} = \frac{1}{r} - \sum_{j=1}^{m} \frac{1}{p_j}.$$

Proved independently by Pellegrino et al when all p_k are equal.

Sidon sets

Let G be a compact abelian group with dual group Γ. A subset Λ of Γ is called p-Sidon ($1 \leq p < 2$) if there is a constant $\kappa > 0$ such that each $f \in \mathcal{C}(G)$ with $\hat{f}$ supported on Λ satisfies

$$\|\hat{f}\|_{\ell_p} \leq \kappa \|f\|_\infty.$$

For instance, $\{3^k;\ k \geq 1\}$ is a 1-Sidon set in $\mathbb{Z}$, the dual of $\mathbb{T} = \{e^{i\theta}:\ \theta \in \mathbb{R}\}$.

Theorem (Edward-Ross (1974), Johnson-Woodward (1974))

The direct product of m 1-Sidon sets is $2m/(m+1)$-Sidon.

A subset Λ of Γ is called a $\Lambda(2)$-set if for one $q \in [1,2)$ (equivalently, for all $q \in [1,2)$), there exists $\kappa > 0$ such that, for all $f \in \mathcal{C}(G)$ with $\hat{f}$ supported on Λ,

$$\|f\|_{L^q(G)} \leq \kappa \|f\|_{L^2(G)}.$$

Theorem

Let $G_1, \dots, G_m$, $m \geq 2$, be compact abelian groups with respective dual groups $\Gamma_1, \dots, \Gamma_m$. For $1 \leq j \leq m$, let $\Lambda_j \subset \Gamma_j$ be a p_j-Sidon and $\Lambda(2)$-set. Then $\Lambda_1 \times \cdots \times \Lambda_m$ is a p-Sidon set in $\Gamma_1 \times \cdots \times \Gamma_m$ for

$$p = \frac{2R}{R+1} \text{ and } R = \sum_{k=1}^{m} \frac{p_k}{2 - p_k}.$$

Moreover, this value of p is optimal.

Multiple summing

First tool: coefficients of nonnegative linear forms

Proposition

Let $m\geq 1$, $1\leq p_1,\dots,p_m\leq+\infty$ and $A:\ell_{p_1}\times\cdots\times\ell_{p_m}\to\mathbb{C}$ be a nonnegative m-linear form. Then

$$\left(\sum_{\mathbf{i}\in\mathbb{N}^m} A(e_{\mathbf{i}})^\rho\right)^{1/\rho}\leq\|A\|$$

provided $\rho^{-1}=1-\sum_{j=1}^m p_j^{-1}>0$.

The proof is done by using the fact that a nonnegative m-linear map $B:\ell_{p_1}\times\cdots\times\ell_{p_m}\to\ell_q$ factors through ℓ_s, with

$$\frac{1}{s}=\frac{1}{q}-\left(\frac{1}{p_1}+\cdots+\frac{1}{p_m}\right).$$

Second tool: an abstract Hardy-Littlewood method

Lemma

Let $p_1, p_2, q \in [1, +\infty)$, $(a_{i,j})_{i,j\in\mathbb{N}}$ a sequence of nonnegative real numbers. Assume that there exists $\kappa > 0$ and $0 < \alpha, \beta \leq q$ such that

- *for all $u \in B_{\ell_{p_1}}$,* $\left(\sum_{j\in\mathbb{N}}\left(\sum_{i\in\mathbb{N}} u_i^q a_{i,j}^q\right)^{\alpha/q}\right)^{1/\alpha} \leq \kappa$;
- *for all $v \in B_{\ell_{p_2}}$,* $\left(\sum_{i\in\mathbb{N}}\left(\sum_{j\in\mathbb{N}} v_j^q a_{i,j}^q\right)^{\beta/q}\right)^{1/\beta} \leq \kappa$.

Then $\left(\sum_{j\in\mathbb{N}}\left(\sum_{i\in\mathbb{N}} a_{i,j}^q\right)^{\gamma/q}\right)^{1/\gamma} \leq \kappa$ *where* $\frac{1}{\gamma} = \frac{1}{\alpha} - \frac{1}{p_1}\left(\frac{1-\frac{q}{\alpha}-\frac{q}{p_2}}{1-\frac{q}{\beta}-\frac{q}{p_1}}\right)$ *provided* $\gamma > 0$, $\frac{\alpha}{p_1} \leq 1$ *and* $\frac{\beta}{p_2} \leq 1$.

Why?

Third tool: a mixed norm inequality

$$\left(\sum_{i,j}|a_{i,j}|^{4/3}\right)^{3/4} \leq \left(\sum_{i=1}^N\left(\sum_{j=1}^N|a_{i,j}|^2\right)^{1/2}\right)^{1/2}\left(\sum_{j=1}^N\left(\sum_{i=1}^N|a_{i,j}|^2\right)^{1/2}\right)^{1/2}.$$

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In other words,

$$\left(\int |f(x,y)|^{\frac{4}{3}}d\mu_1(x)d\mu_2(y)\right)^{\frac{3}{4}} \leq \left(\int\left(\int |f(x,y)|^2 d\mu(y)\right)^{\frac{1}{2}}d\mu(x)\right)^{\frac{1}{2}} \times\left(\int\left(\int |f(x,y)|^2 d\mu(x)\right)^{\frac{1}{2}}d\mu(y)\right)^{\frac{1}{2}}$$

Third tool: a mixed norm inequality

Let (M_j, μ_j) be σ-finite measure spaces for $j = 1, \dots, n$ and introduce the product measure spaces (M^n, μ^n) and (M_j^n, μ_j^n) by
$M^n = \prod_{k=1}^n M_k,\ \mu^n = \prod_{k=1}^n \mu_k,\ M_j^n = \prod_{\substack{k=1\\k\neq j}}^n M_k,\ \mu_j^n = \prod_{\substack{k=1\\k\neq j}}^n \mu_k.$

Proposition (Popa, Sinnamon, after Benedek, Panzone and Blei)

Let $q > 0$, $n \geq 2$ and $r_1, \dots, r_n \in (0, q)$. If $h \geq 0$ is μ^n-measurable, then

$$\left(\int_{M_n} h^Q d\mu^n\right)^{\frac{1}{Q}} \leq \prod_{j=1}^n \left(\int_{M_j} \left(\int_{M_j^n} h^q d\mu_j^n\right)^{\frac{r_j}{q}} d\mu_j\right)^{\frac{1}{R(q-r_j)}}$$

where $R = \sum_{j=1}^n \frac{r_j}{q-r_j}$ and $Q = \frac{qR}{1+R}$.

PoppaSinnamon!

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PoppaSinnamon!

$q = 2, r_1 = r_2 = 2$

Muchas Gracias!