Multiple Dirichlet series

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3 The algebras $\mathcal{H}_\infty(\mathbb{C}_+^k)$

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Dirichlet series

Definition

An ordinary Dirichlet series is a series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$ is the sequence of coefficients of the series, and $s \in \mathbb{C}$ is a complex variable.

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Theorem

If a Dirichlet series is convergent at $s_0 \in \mathbb{C}$, then it is convergent in $[\mathrm{Re} > \mathrm{Re}\, s_0]$.

Uniform convergence in angular regions

Theorem

Let $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *a Dirichlet series which converges at* s_0. *Then* $D(s)$ *converges uniformly on the angular region*
$\mathcal{S}_{s_0,a} = \{s \in \mathbb{C} : |\,\mathrm{Arg}(s - s_0)| \leq a < \frac{\pi}{2}\}$.

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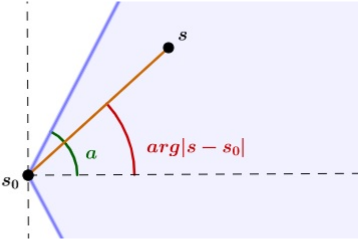

Abscissas of convergence

$$\sigma_c(D) = \inf\{\sigma \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ is convergent in } [\operatorname{Re} s > \sigma]\},$$

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$$\sigma_c(D) \leq \sigma_u(D) \leq \sigma_a(D), \qquad \sup_D \sigma_a(D) - \sigma_c(D) = 1.$$

The algebra $\mathcal{H}_\infty(\mathbb{C}_+)$

Definition

We denote by $\mathcal{H}_\infty(\mathbb{C}_+)$ the space of all Dirichlet D series which are convergent on $\mathbb{C}_+$ and define a bounded holomorphic function there.

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Definition

We denote by $\mathcal{H}_\infty(\mathbb{C}_+)$ the space of all Dirichlet D series which are convergent on $\mathbb{C}_+$ and define a bounded holomorphic function there.

Endowed with the norm $||D||_\infty = \sup_{\mathrm{Re}\, s>0} \left|\sum_{n=1}^\infty \frac{a_n}{n^s}\right|$, the space $\mathcal{H}_\infty(\mathbb{C}_+)$ is a Banach algebra.

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Theorem

For every Dirichlet series D, $\sigma_u(D) = \sigma_b(D)$.

Bohr's fundamental result

Theorem

Let $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *be a Dirichlet series and* $\sigma > 0$*. Suppose that* D *is convergent in* $[\mathrm{Re} > \sigma]$ *to an analytic function* f*, which is bounded on* $\mathbb{C}_+$ *and* $f(s) = D(s)$ *in* $[\mathrm{Re} > \sigma]$*. Then* $D \in \mathcal{H}_\infty(\mathbb{C}_+)$ *and, for every* $\delta > 0$*,* D *converges uniformly to* f *in* $[\mathrm{Re} > \delta]$*. Furthermore, for every* $\delta > 0$ *there exists* $c_\delta > 0$ *only dependent on* δ *such that*

$$\sup_{\mathrm{Re}\, s > \delta} \left| f(s) - \sum_{n=1}^{M} \frac{a_n}{n^s} \right| \leq c_\delta \frac{\log M}{M^\delta} ||f||_\infty .$$

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Convergence of multiple series

Definition

A k-multiple Dirichlet series is a series of the form

$$\sum_{m_1,\ldots,m_k=1}^{\infty} \frac{a_{m_1,\ldots,m_k}}{m_1^{s_1}\cdots m_k^{s_k}},$$

where $\{a_{m_1,\ldots,m_k}\} \subset \mathbb{C}$ is the k-multiple sequence of coefficients of the series, and $s_1,\ldots,s_k \in \mathbb{C}$ are complex variables.

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Definition

We say that a k-multiple series is convergent to S if for every $\varepsilon > 0$ we can find $M_0 \in \mathbb{N}$ such that $\min\{m_1,\ldots,m_k\} \geq M_0$ implies $|s_{m_1,\ldots,m_k} - S| < \varepsilon$, where $s_{m_1,\ldots,m_k}$ denotes the partial sum of the series and S is the sum of the series.

Cauchy condition for multiple series

Condition 1

For every $\varepsilon > 0$ we can find $M_0 \in \mathbb{N}$: $\min\{n_j\} \geq M_0$ implies

$$\left| \sum_{m_1=n_1}^{p_1} \cdots \sum_{m_k=n_k}^{p_k} a_{m_1,\ldots,m_k} \right| < \delta, \quad p_j \geq n_j, \quad 1 \leq j \leq k.$$

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1	1	1	1	1	1	$\cdots$
1	0	0	0	0	0	$\cdots$
1	0	0	0	0	0	$\cdots$
1	0	0	0	0	0	$\cdots$
1	0	0	0	0	0	$\cdots$
1	0	0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Counterexamples

The double series defined by:

0	0	1	2	3	4	$\cdots$
0	0	-1	-2	-3	-4	$\cdots$
1	-1	0	0	0	0	$\cdots$
2	-2	0	0	0	0	$\cdots$
3	-3	0	0	0	0	$\cdots$
4	-4	0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

is convergent but neither $\{a_{m,n}\}$ nor $\{s_{m,n}\}$ is bounded.

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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

is convergent but neither $\{a_{m,n}\}$ nor $\{s_{m,n}\}$ is bounded. Moreover, the row-subseries and column-subseries are not convergent.

Regular convergence and restricted convergence

Definition

We say that a k-multiple series is *regularly convergent* if it is convergent and all of its j-dimensional subseries are convergent, where a j-dimensional subseries is a series of the same multiple sequence in which we take the sum over j indexes $m_{i_1}, \ldots, m_{i_j}$, where the other indexes $m_{l_1}, \ldots, m_{l_{k-j}}$ remain fixed.

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Definition

We say that a k-multiple series *converges in a restricted sense* if for every $\varepsilon > 0$ there exists $M_0 > 0 : \max\{n_j\} \geq M_0$ implies

$$\left| \sum_{m_1=n_1}^{p_1} \cdots \sum_{m_k=n_k}^{p_k} a_{m_1,\ldots,m_k} \right| < \varepsilon \quad p_j \geq n_j, \quad 1 \leq j \leq k.$$

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Theorem

A k-multiple series converges regularly if and only if it converges in a restricted sense.

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A k-multiple series converges regularly if and only if it converges in a restricted sense.

Problems solved:

- Boundedness of terms and partial sums.
- Existence of row and column subseries.
- $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} = \sum_{m,n=1}^{\infty} a_{m,n} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{m,n}$.

Convergence of multiple Dirichlt series

Theorem

Let $D(s_1,\ldots,s_k)=\sum_{m_1,\ldots,m_k}\frac{a_{m_1,\ldots,m_k}}{m_1^{s_1},\ldots,m_k^{s_k}}$ *be a* k*-multiple Dirichlet series which converges regularly at* $(\bar{s_1},\ldots,\bar{s_k})$*. Then it converges regularly and uniformly on the angular region*
$\mathcal{S}_a^k=\{(s_1,\ldots,s_k): |\operatorname{Arg}(s_i-\bar{s_i})|\le a<\frac{\pi}{2}, 1\le i\le k\}$.

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$\mathcal{S}_a^k = \{(s_1, \ldots, s_k) : |\operatorname{Arg}(s_i - \bar{s_i})| \leq a < \frac{\pi}{2}, 1 \leq i \leq k\}$.

Corollary

Let $D(s_1, \ldots, s_k) = \sum_{m_1,\ldots,m_k} \frac{a_{m_1,\ldots,m_k}}{m_1^{s_1},\ldots,m_k^{s_k}}$ *be a k-multiple Dirichlet series which converges regularly at* $(\bar{s_1}, \ldots, \bar{s_k})$*. Then it converges regularly on the product of complex half-planes* $[\operatorname{Re} s_1 > \operatorname{Re} \bar{s_1}] \times \cdots \times [\operatorname{Re} s_k > \operatorname{Re} \bar{s_k}]$.

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Definitions and elemental consequences

Definition

For every $k \in \mathbb{N}$, $\mathcal{H}_\infty(\mathbb{C}_+^k)$ is the space of all k-multiple Dirichlet series that are regularly convergent on $\mathbb{C}_+^k$ to a bounded holomorphic function $f \in H_\infty(\mathbb{C}_+^k)$.

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- For every $m \in \mathbb{N}$ and every $t \in \mathbb{C}_+$,

$$\sum_{n=1}^{\infty} \frac{a_{mn}}{m^s n^t} = \frac{1}{m^s} \sum_{n=1}^{\infty} \frac{a_{mn}}{n^t} = \frac{1}{m^s} \alpha_m(t) \text{ converges.}$$

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- For every $n \in \mathbb{N}$ and every $s \in \mathbb{C}_+$,

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The vector-valued perspective

- $$D_t(s) = \sum_{m=1}^{\infty} \frac{\alpha_m(t)}{m^s} = \sum_{m,n=1}^{\infty} \frac{a_{mn}}{m^s n^t} = \sum_{n=1}^{\infty} \frac{\beta_n(s)}{n^t} = D_s(t)$$

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- $||\alpha_m||_\infty \leq ||D||_\infty, ||\beta_n||_\infty \leq ||D||_\infty, \; \forall m, n \in \mathbb{N},$

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- $||\alpha_m||_\infty \leq ||D||_\infty, ||\beta_n||_\infty \leq ||D||_\infty, \ \forall m, n \in \mathbb{N},$
- $||D_s||_\infty \leq ||D||_\infty, ||D_t||_\infty \leq ||D||_\infty; \ \forall s, t \in \mathbb{C}_+.$

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- $||\alpha_m||_\infty \leq ||D||_\infty, ||\beta_n||_\infty \leq ||D||_\infty, \ \forall m, n \in \mathbb{N},$
- $||D_s||_\infty \leq ||D||_\infty, ||D_t||_\infty \leq ||D||_\infty; \ \forall s, t \in \mathbb{C}_+.$

Therefore

$$\alpha_m, \beta_n, D_s, D_t \in \mathcal{H}_\infty(\mathbb{C}_+) \quad \forall m, n \in \mathbb{N}, \ \forall s, t \in \mathbb{C}_+.$$

The vector-valued perspective

- $$D_t(s) = \sum_{m=1}^{\infty} \frac{\alpha_m(t)}{m^s} = \sum_{m,n=1}^{\infty} \frac{a_{mn}}{m^s n^t} = \sum_{n=1}^{\infty} \frac{\beta_n(s)}{n^t} = D_s(t)$$
- $||\alpha_m||_\infty \leq ||D||_\infty, ||\beta_n||_\infty \leq ||D||_\infty, \ \forall m, n \in \mathbb{N},$
- $||D_s||_\infty \leq ||D||_\infty, ||D_t||_\infty \leq ||D||_\infty; \ \forall s, t \in \mathbb{C}_+.$

Therefore

$$\alpha_m, \beta_n, D_s, D_t \in \mathcal{H}_\infty(\mathbb{C}_+) \quad \forall m, n \in \mathbb{N}, \ \forall s, t \in \mathbb{C}_+.$$

$$\Psi : \mathcal{H}_\infty(\mathbb{C}_+^2) \qquad \rightarrow \qquad \mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+))$$

is an isometry into.

Vector-valued results

Theorem

Let X be a Banach space. Let $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ be a Dirichlet series with coefficients $\{a_n\} \subset X$ and $\sigma > 0$. Suppose that D is convergent on $[\mathrm{Re} > \sigma]$ to a function $f : \mathbb{C}_+ \to X$, $f \in H_\infty(\mathbb{C}_+, X)$, such that $f(s) = D(s)$ in $[\mathrm{Re} > \sigma]$. Then $D \in \mathcal{H}_\infty(\mathbb{C}_+, X)$ and, for every $\delta > 0$, D converges uniformly to f in $[\mathrm{Re} > \delta]$. Furthermore, for each $\delta > 0$ there exists $c_\delta > 0$ only dependent on δ such that

$$\sup_{\mathrm{Re}\, s > \delta} \left|\left| \sum_{n=1}^{N} \frac{a_n}{n^s} - f(s) \right|\right|_X \leq c_\delta \frac{\log N}{N^\delta} ||f||_\infty .$$

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$$X = \mathcal{H}_\infty(\mathbb{C}_+), \qquad D(s) = \sum_{m=1}^{\infty} \frac{\alpha_m}{m^s}.$$

Vector-valued results

Lemma

Let $F \in \mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+))$*. Write*

$$F(s)(t) = \sum_{m=1}^{\infty} \frac{1}{m^s} \left(\sum_{n=1}^{\infty} \frac{a_{mn}}{n^t} \right), \quad \alpha_m(t) = \sum_{n=1}^{\infty} \frac{a_{mn}}{n^t} \in \mathcal{H}_\infty(\mathbb{C}_+)$$

and $F(s) = \sum_{m=1}^{\infty} \frac{\alpha_m}{m^s}$*. Then for every* $\delta > 0$

$$\sum_{m=1}^{M} \sum_{n=1}^{N} \frac{a_{mn}}{m^s n^t} \text{ converges to } F(s)(t)$$

uniformly on $[\mathrm{Re} > \delta]^2$*.*

The isometry is bijective

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and $F(s) = \sum_{m=1}^{\infty} \frac{\alpha_m}{m^s}$. *If* $\beta_n(s) = \sum_{n=1}^{\infty} \frac{a_{m,n}}{m^s}$, *then* $\beta_n \in \mathcal{H}_\infty(\mathbb{C}_+)$.

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Lemma

Let $F \in \mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+))$. *Write*

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and $F(s) = \sum_{m=1}^{\infty} \frac{\alpha_m}{m^s}$. *If* $\beta_n(s) = \sum_{n=1}^{\infty} \frac{a_{m,n}}{m^s}$, *then* $\beta_n \in \mathcal{H}_\infty(\mathbb{C}_+)$.

Theorem

$$\Psi : \mathcal{H}_\infty(\mathbb{C}_+^2) \quad \rightarrow \quad \mathcal{H}_\infty(\mathbb{C}_+, \mathcal{H}_\infty(\mathbb{C}_+))$$

is a bijective isometry.

$\mathcal{H}_\infty(\mathbb{C}_+^2)$ is Banach.

Corollary

If a double Dirichlet series D is absolutely convergent in $[\mathrm{Re} > \sigma_1] \times [\mathrm{Re} > \sigma_2]$ *to a function* $f \in H_\infty(\mathbb{C}_+^2)$*, then it converges uniformly to* f *in* $[\mathrm{Re} > \delta]^2$ *and* $D \in \mathcal{H}(\mathbb{C}_+^2)$*.*

$\mathcal{H}_\infty(\mathbb{C}^2_+)$ is Banach.

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If a double Dirichlet series D is absolutely convergent in $[\mathrm{Re} > \sigma_1] \times [\mathrm{Re} > \sigma_2]$ *to a function* $f \in H_\infty(\mathbb{C}^2_+)$*, then it converges uniformly to* f *in* $[\mathrm{Re} > \delta]^2$ *and* $D \in \mathcal{H}(\mathbb{C}^2_+)$*.*

Theorem

$\mathcal{H}_\infty(\mathbb{C}^2_+)$ *is a Banach space.*

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The spaces $\mathcal{A}(\mathbb{C}_+^k)$

Definition

We denote by $\mathcal{A}(\mathbb{C}_+^k)$ the set of all k-multiple Dirichlet series which are regularly convergent on $\mathbb{C}_+^k$ and define uniformly continuous functions on $\mathbb{C}_+^k$.

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Theorem

For every $k \in \mathbb{N}$, $\mathcal{A}(\mathbb{C}_+^k)$ is a closed subspace of $\mathcal{H}_\infty(\mathbb{C}_+^k)$. Moreover, $f \in \mathcal{A}(\mathbb{C}_+^k)$ if and only if it is the uniform limit of a multiple sequence of Dirichlet polynomials.

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$\mathcal{A}_u(B_{c_0^k})$, the algebra of bounded holomorphic functions that are the uniform limit of a multiple sequence of multiple polynomials defined in the unit ball of c_0.

Bohr's Lemma

$$m = p_1^{\alpha_1} \cdots p_r^{\alpha_r} = \mathfrak{p}^{\alpha}, \quad \text{where} \quad \alpha = (\alpha_1, \ldots, \alpha_r, 0, \ldots) \in \mathbb{N}_0^{\mathbb{N}}.$$

Bohr's Lemma

$m = p_1^{\alpha_1} \cdots p_r^{\alpha_r} = \mathfrak{p}^\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_r, 0, \ldots) \in \mathbb{N}_0^{\mathbb{N}}$.

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$$\sup_{\substack{\operatorname{Re} s_j > 0 \\ 1 \le j \le k}} \left| \sum_{m_1=1}^{M_1} \cdots \sum_{m_k=1}^{M_k} \frac{a_{m_1,\ldots,m_k}}{m_1^{s_1} \cdots m_k^{s_k}} \right|$$

$$= \sup_{\substack{z_j \in \mathbb{D}^{\pi(M_j)} \\ 1 \le j \le k}} \left| \sum_{1 \le \mathfrak{p}^{\alpha_1} \le M_1} \cdots \sum_{1 \le \mathfrak{p}^{\alpha_k} \le M_k} a_{\mathfrak{p}^{\alpha_1}, \ldots, \mathfrak{p}^{\alpha_k}} z_1^{\alpha_1} \cdots z_k^{\alpha_k} \right|.$$

Montel's Theorem for $\mathcal{H}_\infty(\mathbb{C}_+^k)$

Lemma

Let $k \in \mathbb{N}$ and suppose $\{D_n\}$ is a bounded sequence in $\mathcal{H}_\infty(\mathbb{C}_+^k)$. Then there exists a function $D \in \mathcal{H}_\infty(\mathbb{C}_+^k)$ and a subsequence $\{D_{n_d}\}$ that converges uniformly to D on $[\mathrm{Re} > \delta]^k$ for every $\delta > 0$.

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Theorem

The spaces $\mathcal{H}_\infty(\mathbb{C}_+^k)$, $k \in \mathbb{N}$, are all isometrically isomorphic to $H_\infty(B_{c_0})$ and the spaces $\mathcal{A}(\mathbb{C}_+^k)$ are all isometrically isomorphic to $\mathcal{A}_u(B_{c_0})$.

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- Using the characterizations of $\mathcal{A}_u(B_{c_0^k})$ and $\mathcal{A}(\mathbb{C}_+^k)$, $\mathcal{B}_u : \mathcal{A}_u(B_{c_0^k}) \to \mathcal{A}(\mathbb{C}_+^k)$ is a bijective isometry.

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Final remarks

Corollary

The spaces $\mathcal{H}_\infty(\mathbb{C}_+^k)$ are all isometrically isomorphic independently from the dimension, and the same holds for the spaces $\mathcal{A}(\mathbb{C}_+^k)$.

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- $D(s_1, \ldots, s_k) = f(\frac{1}{\mathfrak{p}^{s_1}}, \ldots, \frac{1}{\mathfrak{p}^{s_k}})$.

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Thank you for your attention!