The Bishop-Phelps-Bollobás property on closed bounded convex sets

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Bishop-Phelps Theorem

Theorem (E. Bishop and R.R. Phelps (1961))

Let C be a closed bounded convex set in a real Banach space X. Then the set of linear functionals that attain their maximum on C is dense in X^*.

In particular, the set of all norm-attaining linear functionals on a Banach space X is dense in the dual space X^*.

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V. Lomonosov (2000)

The Bishop-Phelps theorem cannot be extended to general complex Banach spaces by constructing a closed bounded convex set with no support points.

Norm Attaining Mappings

$X, \ Y =$ Real or Complex Banach Space
Let S_X and B_X be the unit sphere and closed unit ball of X, respectively.

$T \in L(X, Y)$ attains its norm if there is $x_0 \in S_X$ such that
$\|T(x_0)\| = \|T\|$.

$NA(L(X, Y)) =$ Set of all norm-attaining linear mappings from X into Y.

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Question.

Is the set $NA(L(X, Y))$ dense in $L(X, Y)$?

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(Lindenstrauss, 1963)
Counterexamlpe: X= c_0, Y = Equivalently Renormed Space c_0 to be Strictly Convex.

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A Banach space X has property (A) if $NA(L(X, Y))$ is dense in $L(X, Y)$ for every Banach space Y.

A Banach space Y has property (B) if $NA(L(X, Y))$ is dense in $L(X, Y)$ for every Banach space X.

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A Banach space Y has property (B) if $NA(L(X, Y))$ is dense in $L(X, Y)$ for every Banach space X.

Question. (The Most Irritating Open Problem)

Does the 2-dimensional Euclidean space $\mathbb{R}^2$ have property (B) ?

Theorem (J. Bourgain (1977))

A Banach space X has the Radon-Nikodym Property if and only if every Banach space isomorphic to X has property (A).

Examples with RNP : (1) Reflexive spaces (2) Separable Duals (3) WCG Duals (4) Locally Uniformly Convex Space (5) $l_1(I)$, I, any set

Nonlinear Version of Bourgain's Result

Theorem (C. Stegall (1978))

Let X be a Banach space with RNP, D be a bounded closed convex subset of X and $f : D \to \mathbb{R}$ be an upper semicontinuous bounded above function. Then for $\epsilon > 0$, there exists $x^* \in X^*$ such that $\|x^*\| < \epsilon$ and $f + x^*$, $f + |x^*|$ strongly expose D.

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Applying this result to a vector-valued case, he showed the following.

Theorem (C. Stegall (1978))

Let X be a Banach space with RNP, D be a bounded closed convex subset of X, and Y be a Banach space. Suppose that $\varphi : D \to Y$ is a uniformly bounded function such that the function $x \to \|\varphi(x)\|$ is upper semicontinuous. Then, for $\delta > 0$, there exist $T : X \to Y$ a bounded linear operator of rank one, $\|T\| < \delta$ such that $\varphi + T$ attains its supremum in norm on D and does so at most two points

Bishop-Phelps-Bollobás Property

The Bishop-Phelps theorem (Bishop and Phelps, 1961)

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Theorem (Bollobás, 1970)

For $\epsilon > 0$, if $x \in B_X$ and $x^* \in S_{X^*}$ satisfy $|1 - x^*(x)| < \frac{\epsilon^2}{4}$, then there are $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|y - x\| < \epsilon$ and $\|y^* - x^*\| < \epsilon$.

Bronsted-Rockafellar Theorem

Theorem (Bronstead-Rockafellar Theorem, PAMS, 1965)

Suppose that f is a convex proper lower semicontinuous function on the Banach space X. The given any point $x_0 \in dom(f)$, $\epsilon > 0$, $\lambda > 0$ and any $x_0^* \in \partial_\epsilon f(x_0)$, there exist $x \in dom(f)$ and $x^* \in X^*$ such that

$$x^* \in \partial(f), \ \|x - x_0\| \leq \frac{\epsilon}{\lambda}, \ \textit{and} \ \|x^* - x_0^*\| \leq \lambda.$$

In particular, the domain of ∂f is dense in $dom(f)$.

Definition (Acoasta, Aron, García and Maestre, JFA 2008)

We say that the couple (X, Y) satisfies the Bishop-Phelps-Bollobás property for operators (BPBp for short), if given $\epsilon > 0$ there exists $\eta(\epsilon) > 0$ such that for $T \in S_{\mathcal{L}(X,Y)}$, if $x_0 \in S_X$ is such that $\|Tx_0\| > 1 - \eta(\epsilon)$, then there exist a point $u_0 \in S_X$ and an operator $S \in S_{\mathcal{L}(X,Y)}$ that satisfy the following conditions :

$$\|Su_0\| = 1, \|x_0 - u_0\| < \epsilon \text{ and } \|T - S\| < \epsilon.$$

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They showed that if a Banach space Y has property (β), then the couple (X, Y) has the *BPBp* for every Banach space X.

The Bishop-Phelps-Bollobás version of (Lindenstrauss) properties A and B

Recall that J. Lindenstrauss introduced the following two properties.

A Banach space X is said to have *Lindenstrauss property A* if $\overline{NA(X,Z)} = L(X,Z)$ for every Banach space Z.

A Banach space Y is said to have *Lindenstrauss property B* if $\overline{NA(Z,Y)} = L(Z,Y)$ for every Banach space Z.

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Definition

Let X and Y be Banach spaces.
We say that X is a universal BPB domain space if for every Banach space Z, the pair (X,Z) has the BPBp.

We say that Y is a universal BPB range space if for every Banach space Z, the pair (Z,Y) has the BPBp.

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THEOREM (R. ARON, C, S.K. KIM, H.J. LEE AND M. MARTIN, TRANSACTIONS AMS 2015)

Every Banach space isomorphic to X is a universal BPB domain space if and only if X is the basic field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

Operators from $\ell_p(c_0) \to Y$

Theorem (Acoasta, Aron, García and Maestre, JFA 2008)

The couple (ℓ_1, Y) satisfies the Bishop-Phelps-Bollobás property for operators if and only if Y has the AHSP.

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The following Banach spaces have the *AHSP*:

(a) a finite dimensional space, (b)a real or complex space $L_1(\mu)$ for a σ-finite measure μ,
(c)a real or complex space $C(K)$ for a compact Hausdorff space K, and
(d) a uniformly convex space.

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Theorem (AGGM (TAMS, 2012), H.J Lee and S. K. Kim (Canadian J. Math. 2013))

Let X be a uniformly convex Banach space. Then the couple (X, Y) has the BPBp for every Banach space Y.
More precisely, given $0 < \epsilon < 1$, let $0 < \eta < \frac{\epsilon}{8+2\epsilon}\delta(\epsilon)$. If $T \in S_{\mathcal{L}(X,Y)}$ and $x \in S_X$ satisfy

$$\|Tx_0\| > 1 - \eta,$$

then there exist $S \in S_{\mathcal{L}(X,Y)}$ and $u_0 \in S_X$ such that $\|Su_0\| = 1$, $\|S - T\| < \epsilon$ and $\|x_0 - u_0\| < \epsilon$.

Operators from $\ell_p(c_0) \to Y$

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Answer : Yes [Sun Kwang Kim, Israel J. Math. 2013].

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Theorem (Aron, Cascales, Kozhushkina, PAMS 2011)

Let L be a locally compact space. Then $(X, C_0(L))$ has the BPBp if X is Asplund.
In particular, $(c_0, C_0(L))$ has the BPBp.

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Theorem (Cascales, Guirao, and Kadets, Advances in Math. 2012)

Let A be a uniform algebra. Then (X, A) has the BPBp if X is Asplund.
In particular, (c_0, A) has the BPBp.

Definition X, Y = Banach space

$D \subseteq X$ closed, bounded convex

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We say that (X,Y) has the BPBp on D if for every $\varepsilon > 0$ there exists $\eta_D(\varepsilon) > 0$ s.t. for every $T \in L(X,Y)$, $\|T\|_D = 1$ and for every $x \in D$ satisfying

$$\|Tx\| \geq 1 - \eta_D(\varepsilon),$$

there exist $S \in L(X,Y)$ and $z \in D$ s.t.

$$\|S\|_D = 1 = \|Sz\|, \quad \|x - z\| < \varepsilon \text{ and } \|T - S\| < \varepsilon.$$

$$\|T\|_D = \sup\{\|Tx\| : x \in D\}$$

The BPBp holds for bounded linear functionals on arbitarary bounded convex sets of a real Banach space.

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Ekeland Variation Principle (JMAA, 1974)

Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper, lower-semicontinuous and bounded below function on a real Banach space X. Then given $\varepsilon > 0$ and $\delta > 0$, there exists $x_1 \in X$ s.t. $f(x_1) < f(x) + \varepsilon \|x - x_1\|$ for every $x \in X$, $x \neq x_1$. Moreover if $f(x_0) < b + \delta/2$ where $b = \inf\{f(x) : x \in X\}$ then x_1 can be chosen s. that $\|x_1 - x_0\| < \delta/\varepsilon$.

Theorem (Cho/C, JLMS 2016)

Let D be a bounded convex closed subset of a real Banach space X. Given $\varepsilon > 0$ and $\delta > 0$, if $f \in X^*$ and $x_0 \in D$ s.t.

$$f(x_0) > \sup\{f(x) : x \in D\} - \delta/2,$$

then there exist $g \in X^*$ and $x_1 \in D$ satisfying $g(x_1) = \sup\{g(x) : x \in D\}$, $\|f - g\| \leq \varepsilon$ and $\|x_1 - x_0\| \leq \delta/\varepsilon$.

Theorem. Let D be a bounded closed convex set in a real Banach space. Given $0<\varepsilon<\frac{1}{4}$ and $f\in X^*$, there exists $x^*\in X^*$ and $x_0\in D$ s.t. both $f+x^*$ and $f+|x^*|$ attain their suprema simultaneously at x_0 and $\|x^*\|<\varepsilon$. Moreover,

$$(f+x^*)(x_0)=(f+|x^*|)(x_0).$$

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$$(f+x^*)(x_0)=(f+|x^*|)(x_0).$$

<u>Sketch of Proof</u> May assume $D\subseteq B_X$ & $\|f\|_D=1$.

B-P theorem $\Rightarrow \exists\, x^*$, $\|x^*\|<\varepsilon/2$ s.t.

$f+x^*$ attains its max. at $x_0\in D$.

$$(f+x^*)(x_0)\geq(f+|x^*|)(x),\quad \forall x\in D$$

Otherwise, $\exists\, y \in D$ s.t. $f(y) + |x^*(y)| > f(x_0) + x^*(x_0)$.

clearly $x^*(y) < 0$, and

$$f(y) - x^*(y) > f(x_0) + x^*(x_0).$$

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Let $s = \sup\{f(x) - x^*(x) : x \in D\}$ &

$$\alpha = s - (f(x_0) + x^*(x_0)) < (1 + \varepsilon/2) - (1 - \varepsilon/2) = \varepsilon.$$

Choose $y_0 \in D$ so that $f(y_0) - x^*(y_0) > s - \dfrac{\alpha^2 \varepsilon^2}{2}$.

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Choose $y_0 \in D$ so that $f(y_0) - x^*(y_0) > S - \frac{\alpha^2 \varepsilon^2}{2}$.

$\exists\, x_1^*$ s.t. $(f - x^*) + x_1^*$ attains its max.
at $z_0 \in D$, $\|x_1^*\| \leq \alpha\varepsilon$ and $\|y_0 - z_0\| \leq \alpha\varepsilon$.

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$\exists\, x_1^*$ s.t. $(f - x^*) + x_1^*$ attains its max.
at $z_0 \in D$, $\|x_1^*\| \leq \alpha\varepsilon$ and $\|y_0 - z_0\| \leq \alpha\varepsilon$.

Set $x_2^* = x^* - x_1^*$. Then $\|x_2^*\| < \varepsilon$ &
$f + |x_2^*|$ attains its max. at $z_0 \in D$

$f(z_0) + x_2^*(z_0)$

Further, we can show that

$$\{f : |f| \text{ attains its max. in } D\} \underset{\text{dense}}{\subset} X^*.$$

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$\{f : |f| \text{ attains its max. in } D\} \subset X^*$ (dense).

Theorem. X = real Banach space

$D \subseteq X$ bounded, closed, convex.

Given $f \in X^*$ and $\varepsilon > 0$, $\exists\, x^* \in X^*$ s.t. $|f + x^*|$ attains its max. in D and $\|x^*\| < \varepsilon$. Moreover, if D is symmetric and $f(x_0) > \|f\|_D - \delta/2$ for some $x_0 \in D$ and $\delta > 0$, then x^* and $x_1 \in D$ can be chosen s.t. $\|x^*\| < \varepsilon$, $\|x_1 - x_0\| < \delta/\varepsilon$, $|f + x^*|$ attains its max. at x_1 in D.

Vector-valued case

(ℓ_2^2, Y) has the BPBp on $B_{\ell_2^2}$ for every Y, but there is a Banach space Z s.t. (ℓ_2^2, Z) does not have the BPBp on

$$D = \{(x, y) \mid |x| + |y| \leq 1\}.$$

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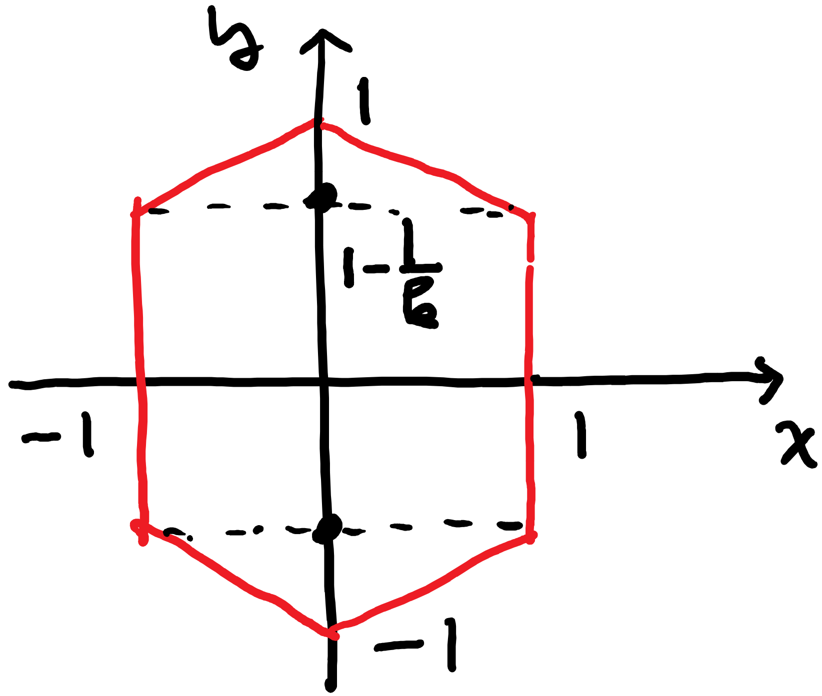
$$D = \{(x,y) \mid |x|+|y| \le 1\}.$$

$$Z = \left(\bigoplus_{k=1}^{\infty} Z_k\right)_{\ell_\infty}$$

B_{Z_k}



Positive results

① X, Y = finite dim'l space
$D \subseteq X$ bounded closed convex

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$D \subseteq X$ bounded closed convex

② Y = Banach space with property (β).

$D \subseteq X$ symmetric

bounded, closed, convex

Recall The convexity of modulus

For $0 < \varepsilon < 1$,

$$\delta(\varepsilon) = \inf\left\{ 1 - \frac{\|x+y\|}{2} : x, y \in B_X, \|x-y\| \geq \varepsilon \right\}.$$

Recall The convexity of modulus

For $0<\varepsilon<1$,

$$\delta(\varepsilon)=\inf\left\{1-\frac{\|x+y\|}{2} : x,y\in B_X, \|x-y\|\geq\varepsilon\right\}.$$

For $0<\varepsilon<1$, define

$$\delta_D(\varepsilon)=\inf\left\{\frac{1}{2}\rho_D(x)+\frac{1}{2}\rho_D(y)-\rho_D\left(\frac{x+y}{2}\right)\right\},$$

where the infimum is taken over all $x,y\in D$ with $\rho_D(x-y)\geq\varepsilon$.

Theorem X, Y = real (complex) Banach space.

$D \subseteq B_X$ absorbing, closed, convex & $\delta_D(\varepsilon) > 0$ for every ε, $0 < \varepsilon < \frac{1}{2}$.

If $T \in L(X, Y)$ and $x_1 \in D$ satisfy

$$\|Tx_1\| > \|T\|_D - \varepsilon^3 \delta_D(\varepsilon)$$

for sufficiently small ε relative to $\|T\|_D$, then there exist $S \in L(X, Y)$ and $z \in D$ s.t. $\|Sz\| = \|S\|_D$,

$$\|S - T\| < 4\varepsilon^2/1-\varepsilon \text{ and } \|x_1 - z\| \leq \rho_D(x_1 - z) \leq \varepsilon/1-\varepsilon.$$

Aron, Cascales & Kozhushkina, PAMS 2011

BPBp holds on B_X for an Asplund operator from X into $C_0(L)$.

Aron, Cascales & Kozhushkina, PAMS 2011

BPBp holds on B_X for an Asplund operator from X into $C_0(L)$.

<u>Theorem</u> For a symmetric bounded closed convex subset $D \subseteq B_X$ and locally compact Hausdorff space L, let $T : X \to C_0(L)$ be an Asplund operator with $\|T\|_D = 1$ & $\|T\| = M \geq 1$. Given $0 < \varepsilon < M/2$, if $x_0 \in D$ satisfies that $\|Tx_0\| > 1 - \frac{\varepsilon^2}{4M}$, then there exist an Asplund operator S and $u_0 \in D$ s.t. $\|S\|_D = 1 = \|Su_0\|$, $\|x_0 - u_0\| < \varepsilon$ and $\|T - S\| < 4\varepsilon$.

Thank you for your attention !!