Factorization of (p, σ)-continuous operators

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Abstract

We give the Pietsch factorization theorem for the (p, σ)-absolutely continuous linear operators. Although this result is essentially already known (it was proved by Matter in 1989), we write a new direct proof that highlights the role of the spaces $C(B_{X^*})$ and $L^p(\eta)$, where η is a regular Borel probability measure on B_{X^*}.

As an application we show that (p, σ)-absolutely continuous linear operators are compact under some requirements and we prove a Dvoretzky-Rogers type theorem for this class of operators.

Preliminary

p-summing linear operators

Definition

(Pietsch 1967) Let $1 \leq p < \infty$. We say that $T \in \mathcal{L}(X, Y)$ is p-summing if there exists $C > 0$ such that that for every $(x_i)_{i=1}^{n}$ in X,

$$\left(\sum_{i=1}^{n} \| T(x_i) \|^p \right)^{\frac{1}{p}} \leq C \sup_{\|\xi\|_{X^*} \leq 1} \left(\sum_{i=1}^{n} |\xi(x_i)|^p \right)^{\frac{1}{p}}, \qquad (1)$$

The set of all p-summing operators $\Pi_p(X, Y)$ is Banach space with the norm.

$$\pi_p(T) := \inf \{ C, \text{ for all } C \text{ verifying the inequality } (1) \}.$$

Preliminary

(p, σ)-absolutely continuous linear operators

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Definition

(Matter 1987) *We say that $T \in \mathcal{L}(X, Y)$ is a (p, σ)-absolutely continuous operator, in symbols $T \in \Pi_{p,\sigma}(X, Y)$, if there exist a Banach space G and an operator $S \in \Pi_p(X, G)$ such that*

$$\|Tx\| \leq \|x\|^{\sigma} \|Sx\|^{1-\sigma}, \quad x \in X. \tag{2}$$

In such case, we put $\pi_{p,\sigma}(T) = \inf \pi_p(S)^{1-\sigma}$, taking the infimum over all Banach spaces G and $S \in \Pi_p(X, G)$ such that (2) holds.

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Preliminary

The subsequent characterizations of $\Pi_{p,\sigma}$ are derived from the corresponding properties of Π_p.
The Pietsch domination theorem concerning the (p, σ)-absolutely continuous linear operators was proved by Matter.
As usual, the unit ball B_{X^*}, of X^* is considered as a compact space with respect to the weak star topology.

Preliminary

Theorem

The following statements are equivalent.

(i) $T \in \Pi_{p,\sigma}(X, Y)$

(ii) There is a regular Borel probability measure μ *on* B_{X^*}*, such that*

$$\|T(x)\| \leq C\left(\int_{B_{X^*}} \left(|\langle x, x^*\rangle|^{1-\sigma}\|x\|^{\sigma}\right)^{\frac{p}{1-\sigma}} d\mu(x^*)\right)^{\frac{1-\sigma}{p}}, \quad x \in X.$$

(iii) For every finite sequence $(x_i)_{i=1}^n$ *in* X,

$$\left\|\left(T(x_i)\right)_{i=1}^n\right\|_{\frac{p}{1-\sigma}} \leq C. \sup_{\|\xi\|_{X^*}\leq 1}\left(\sum_{i=1}^n \left(|\xi(x_i)|^{1-\sigma}\|x_i\|^{\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}.$$

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Preliminary

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Proposition

The inclusion $\Pi_{\frac{p}{1-\sigma}} \subset \Pi_{p,\sigma}$ holds. Consequently, every p-summing operator is (p, σ)-absolutely continuous .

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New results for the class of (p, σ) linear operators

In the following proposition we prove a Dvoretzky-Rogers type theorem for (p, σ)-absolutely continuous linear operators.

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Proposition

Let $1 \leq p < \infty$ and $0 \leq \sigma < 1$. A Banach space X is finite dimensional if and only if the identity mapping $id_X : X \to X$ is (p, σ)-absolutely continuous.

New results for the class of (p, σ) linear operators

Proposition

Let $T \in \mathcal{L}(X, Y)$. Then T is (p, σ)-absolutely continuous if and only if its second adjoint, $T^{**} : X^{**} \longrightarrow Y^{**}$, is (p, σ)-absolutely continuous. In this case

$$\pi_{p,\sigma}(T) = \pi_{p,\sigma}(T^{**}).$$

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New results for the class of (p, σ) linear operators

(Factorization Theorem)

Let X be a Banach space, $p \geq 1$, $0 \leq \sigma < 1$ and let η be a regular Borel probability measure on B_{X^*} (with the weak star topology). We denote by i_X the isometric embedding

$$X \longrightarrow C(B_{X^*}), \qquad i_X(x) = \langle x, . \rangle$$

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For $f \in i_X(X) \subset C(B_{X^*})$, we define the semi norm,

$$\|f\|_{p,\sigma} = \inf \left\{ \sum_{k=1}^{n} \|f_k\|_{i_X(X)}^{\sigma} \cdot \left(\int_{B_{X^*}} |f_k|^p \, d\eta \right)^{\frac{1-\sigma}{p}}, \; f = \sum_{k=1}^{n} f_k, \; (f_k)_{k=1}^{n} \subset i_X(X) \right\}$$

(Factorization Theorem)

Let S be the subspace of $i_X(X)$ given by

$$S = \left\{ f \in i_X(X), \|f\|_{p,\sigma} = 0 \right\}$$

We write $L_{p,\sigma}(\eta)$ for the completion of the quotient space $i_X(X)/S$ with the norm

$$\|[f]\|_{p,\sigma} = \|f\|_{p,\sigma} \, ,$$

where $[f]$ is the equivalence class of $f \in i_X(X)$.

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(Factorization Theorem)

Notice that with this notation in the case $\sigma = 0$ the space $L_{p,0}(\eta)$ do not coincide with $L_p(\eta)$ but with the subspace of $L_p(\eta)$ that allows the factorization theorem for p-summing operators.

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Lemma

The canonical mapping

$$J_{p,\sigma} : i_X(X) \longrightarrow L_{p,\sigma}(\eta), \qquad J_{p,\sigma}(\langle x, \cdot \rangle) = [\langle x, \cdot \rangle]$$

is (p, σ)-absolutely continuous, and $\pi_{p,\sigma}(J_{p,\sigma}) \leq 1$.

(Factorization Theorem)

Theorem

For every linear operator $T : X \to Y$, the following statements are equivalent.

(i) T is (p, σ)-absolutely continuous.

(ii) There exist a regular Borel probability measure μ on B_{X^*} (with the weak star topology) and a linear continuous operator $\widetilde{T} \in \mathcal{L}(L_{p,\sigma}(\mu), Y)$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & Y \\ {\scriptstyle i_X}\downarrow & & \uparrow{\scriptstyle \widetilde{T}} \\ i_X(X) & \xrightarrow{\quad J_{p,\sigma} \quad} & L_{p,\sigma}(\mu). \end{array}$$

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(An application)

As an application we show that (p, σ)-absolutely continuous linear operators are compact under some requirements. For this we present the next proposition.

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Proposition

Let $0 \leq \sigma < 1$, $1 \leq p < \infty$ and X be a Banach space. The mapping

$$i : X \longrightarrow L_{p,\sigma}(\eta), \qquad i(x) = [\langle x, . \rangle]$$

is completely continuous.

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Corollary

Let X, Y be Banach space, X in addition reflexive, and let $0 \leq \sigma < 1$, $1 \leq p < \infty$. If $T \in \Pi_{p,\sigma}(X, Y)$, then T is compact.

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Thank You Very Much

For Your Attention