Algebras of hypercyclic vectors

Universitat de València

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Conference on Non Linear Functional Analysis
Workshop on Infinite Dimensional Analysis Valencia 2017
(Joint work with Karl Grosse-Erdmann)

Overview

1. Introduction

   Defining our framework. Some basic definitions (hyperciclycity and dense orbits)

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Hypecyclicity

Definition (Linear dynamical system)**.**

A linear dynamical system is a pair (X, T) consisting of a separable Banach (Fréchet) space X and a continuous linear operator $T : X \to X$.

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Definition (hypercyclic operator)**.**
An operator $T : X \to X$ is called *hypercyclic* if there is some $x \in X$ whose orbit under T is dense in X. In such a case, x is called a *hypercyclic vector* for T. The set of hypercyclic vectors for T is denoted by $HC(T)$.

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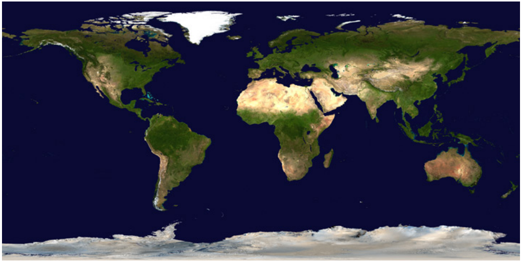

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We're here!

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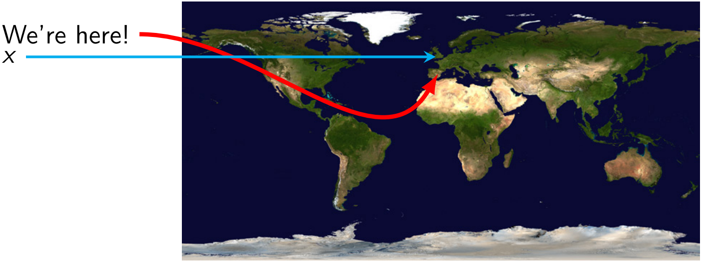

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(As far as we know, photo is public domain)

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High Five Interchange, Dallas, Texas

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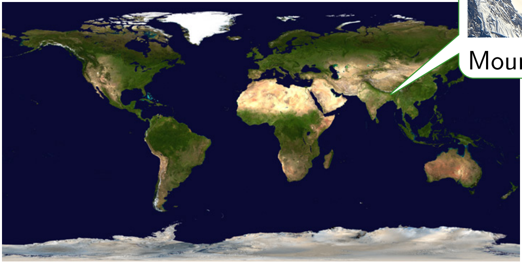

Mount Everest

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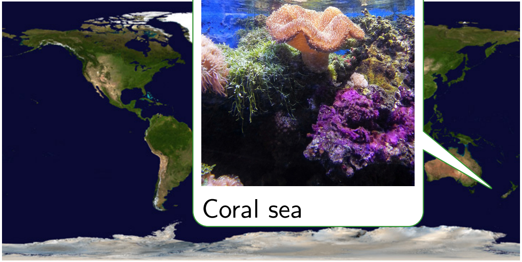

Coral sea

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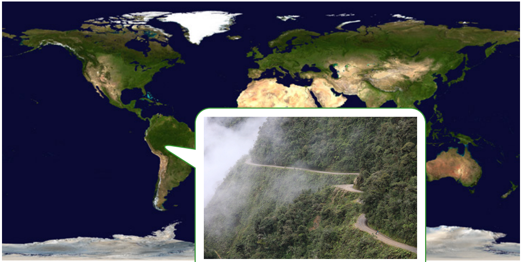

Bolivia

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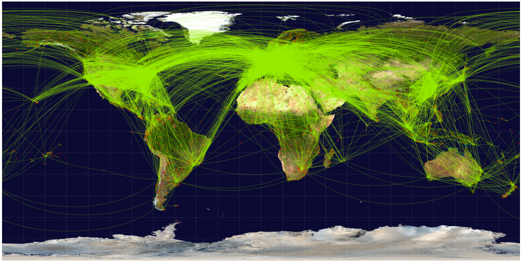

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$$T_a f(z) = f(z+a), \quad a \neq 0.$$

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Structures in $HC(T)$

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- Rolewicz's operators **doesn't have** any hypercyclic subspace.

Algebraic structures

Definition.

Recall a *Fréchet algebra* is an algebra X over the complex numbers that at the same time is a (locally convex) Fréchet space whose topology is induced by an increasing family $(p_q)_{q\geq 1}$ of seminorms that are submultiplicative, i.e.,

$$p_q(xy) \leq p_q(x)p_q(y).$$

Algebraic structures

When the underlying vector space X is a Banach or Fréchet algebra, it is natural to ask whether the set of hypercyclic vectors for a given hypercyclic operator $T \in HC(X)$ also contains a non-trivial algebra (except zero). Such an algebra is called a *hypercyclic algebra* for T.

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- Birkhoff translation operator $T_a \Longrightarrow$ Does NOT contain a hypercyclic algebra.

Theorem (Aron, Conejero, Peris and Seoane-Sepúlveda)**.**

Let p be a positive integer, and let $f \in H(\mathbb{C}) \setminus \{0\}$. Also, let T be a non-trivial translation operator on $H(\mathbb{C})$. If a non-constant function $g \in H(\mathbb{C})$ belongs to the closure of $Orb(f^p, T)$ then the order of each zero of g is a multiple of p.

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Shkarin (2010)

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Theorem (Aron, Conejero, Peris and Seoane-Sepúlveda)**.**

There exists $f \in H(\mathbb{C})$ such that $f^k \in HC(D)$ for every $k \in \mathbb{N}$. Moreover, this behavior is generic, i.e. the following set is residual $\{f \in H(\mathbb{C}) : f^k \in HC(D)$ for every $k \in \mathbb{N}\}$.

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Constructive approach

Bayard and Matheron

Using Baire's theorem

Product structures

A *sequence space* is a subspace of

$$\omega := \{x = (x_n)_{n \geq 1} : x_n \in \mathbb{C}, n \in \mathbb{N}\}$$

When we have an additional structure of a Fréchet algebra such that the canonical embedding into ω is continuous we speak of a *Fréchet sequence algebra*.

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Example (Fréchet algebras)**.**

- $(\ell^p, \cdot)$ or $(c_0, \cdot)$ and $(\ell^p, *)$ or $(c_0, *)$,
- The space $H(\mathbb{C})$ of entire functions considered as a sequence space via Taylor coefficients with the family of seminorms

$$p_q((a_n)_{n\geq 0}) = \sum_{n=0}^{\infty} |a_n| q^n, \quad q \geq 1.$$

Weighted backward shift

A weighted backward shift on X is an operator B_w given by

$$B_w(x_1, x_2, x_3, ...) = (w_2x_2, w_3x_3, w_4x_4, ...), \quad x \in \omega,$$

where $w = (w_n)$ is a sequence of non-zero complex numbers, called a weight sequence.

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The forward shift associated to the weight w is the operator given by

$$F_w(x_1, x_2, x_3, ...) = (0, w_2 x_1, w_3 x_2, w_4 x_3, ...), \quad x \in \omega.$$

Naturally we have that $B_w F_{w^{-1}} = Id$ where Id is the identity map on X and $w^{-1} = (w_n^{-1})_n$.

coordinatewise multiplication

Theorem (J. F. – K. G. Grosse-Erdmann).

Let $(X, (\|\cdot\|_q)_q)$ be a Fréchet sequence algebra under coordinatewise multiplication in which $(e_n)_n$ is a basis with Property A. Let B_w be a hypercyclic weighted backward shift on X. If there exists an increasing sequence $(p_k)_k$ of natural numbers such that

$$\text{for any } n \geq 0, \quad \prod_{\nu=0}^{p_k+n} w_\nu^{-1} \to 0, \quad v_{p_k+n}^{-1} e_{p_k+n} \to 0 \quad \text{as } k \to \infty,$$

then there exists a point $x \in HC(B_w)$ such that the algebra generated by x, except zero, is contained in $HC(B_w)$.

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What is *Property A*?

Definition.

Let X be a Fréchet sequence space that contains the finite sequences. We say that $(e_n)_n$ satisfies *Property A* if there exists an increasing sequence $(p_q)_q$ of seminorms defining the topology of X such that, for any integer $m \geq 1$ and any $q \geq 1$ there is some $r \geq 1$ and some $M > 0$ such that, for all $n \geq 1$,

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(a) If $(e_n)_n$ is a bounded then

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(b) The basis $(e_n)_n$ in the space $H(\mathbb{C})$ of entire functions, considered as a sequence space via Taylor coefficients, also has Property A. If we consider the seminorms

$$p_q((a_n)_{n\geq 0}) = \sum_{n=0}^{\infty} |a_n| q^n, \quad q \geq 1,$$

then we have that

$$p_q(e_n)^m = p_{q^m}(e_n), \quad n \geq 0.$$

Why do we need Property A?

Proposition

Let X be a Fréchet sequence space in which $(e_n)_n$ is a basis. Suppose that the weighted shift B_w is an operator on X. Then B_w is hypercyclic if and only if there exists an increasing sequence $(n_k)_k$ of positive integers such that,

$$v_{n_k}^{-1} \; e_{n_k} \quad \to 0$$

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$$v_{n_k+j}^{-1} e_{n_k+j} \to 0$$

in X as $k \to \infty$.

Corollary.

Let X be a Fréchet sequence space in which $(e_n)_n$ is a basis and has Property A. Suppose that the weighted shift B_w is a hypercyclic operator on X. Then there exists an increasing sequence $(n_k)_k$ of positive integers such that, for each $j \geq 1$ and each integer $m \geq 1$,

$$v_{n_k+j}^{-\frac{1}{m}} e_{n_k+j} \to 0$$

in X as $k \to \infty$, were $a^{-\frac{1}{m}}$ is any mth root of a^{-1} in $\mathbb{C}$.

Why do we need Property A?

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$$p_q(v_{n_k+j}^{-\frac{1}{m}} e_{n_k+j}) = |v_{n_k+j}|^{-\frac{1}{m}} p_q(e_{n_k+j})$$
$$= \left(|v_{n_k+j}|^{-1} p_q(e_{n_k+j})^m\right)^{\frac{1}{m}}$$
$$\leq \left(|v_{n_k+j}|^{-1} \; M p_r(e_{n_k+j})\right)^{\frac{1}{m}}$$
$$= \left(M p_r(v_{n_k+j}^{-1} e_{n_k+j})\right)^{\frac{1}{m}} \xrightarrow{k\to\infty} 0$$

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$a_1 = 3$

$\_\ \_\ \_\ \underline{4^{\frac{1}{m}}}\underline{2^{\frac{1}{m}}}\underline{8^{\frac{1}{m}}}\underline{.1^{\frac{1}{m}}}\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_$

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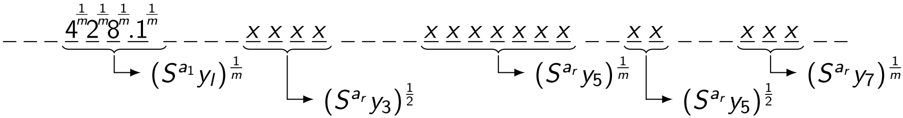

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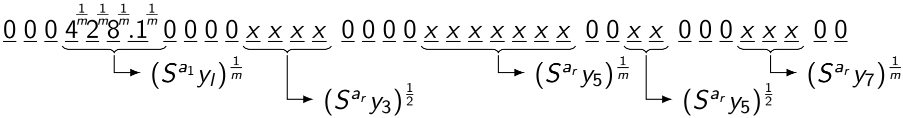

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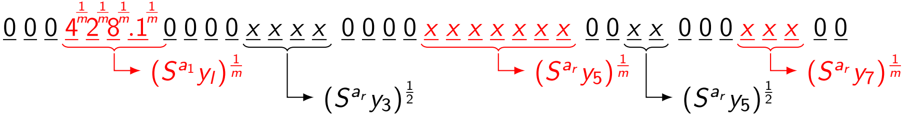
The increasing sequence of natural numbers $(a_r)_r$ such that, if $r = (m, l) \in \mathbb{N}_m$, then

- $\|(S^{a_r} y_l)^{\frac{1}{m}}\| \leq 2^{-r}$
- $\|T^{a_i}\left((S^{a_r} y_l)^{\frac{j}{m}}\right)\| \leq 2^{-r}$ for $i = 1, \ldots, r-1,\ j = 1, \ldots, d_{r-1}$,
- $a_r - a_{r-1} \geq s_{l'}$,

where $d_r = \max_{(m,l)<r} m$ and $r - 1 = (m', l')$.



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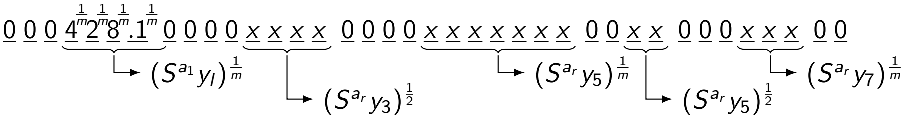

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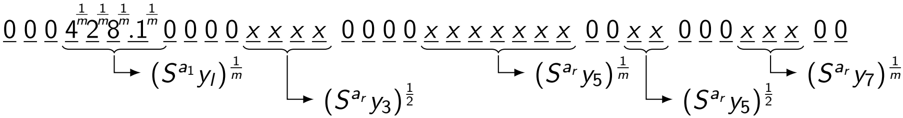

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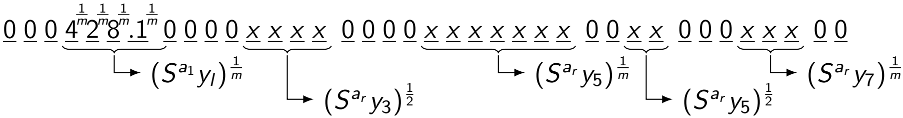

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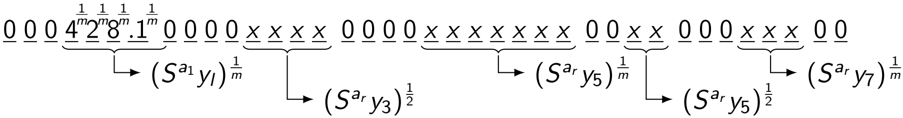
gives us that for $x = \sum_{v=j}^{N} \alpha_v x_0^v$ form some $\alpha_j, \ldots, \alpha_N \in \mathbb{C}$ with $\alpha_j \neq 0$

$$\| T^{a_k}(x) - y_{l_0} \| \xrightarrow{i \to \infty} 0$$

where $(k = (j, l_i))_i$ goes to infinity and $y_{l_i} = y_{l_0}$.



Theorem (J. F. – K. G. Grosse-Erdmann).

Let $(X, (\|\cdot\|_q)_q)$ be a Fréchet sequence algebra under coordinatewise multiplication in which $(e_n)_n$ is a basis with Property A. Let B_w be a hypercyclic weighted backward shift on X. If there exists an increasing sequence $(p_k)_k$ of natural numbers satisfying (2) such that

$$\text{for any } n \geq 0, \quad \prod_{\nu=0}^{p_k+n} w_\nu^{-1} \to 0, \quad v_{p_k+n}^{-1} e_{p_k+n} \to 0 \quad \text{as } k \to \infty,$$

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A stronger result for Rolewicz's operators.

Proposition.

Set $X = \ell_p$, $1 \leq p < \infty$ or $X = c_0$. Let fix $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, a natural number j and a point $x_0 \in X$. If $x_0^j \in HC(\lambda B)$, then for all $N \in \mathbb{N}$, $N > j$, and all $\alpha_{j+1}, \ldots, \alpha_N \in \mathbb{C}$ the point $x_0^j + \sum_{v=j+1}^{N} \alpha_v x_0^v \in HC(\lambda B)$.

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What about the Cauchy product?

Theorem (J. F. – K. G. Grosse-Erdmann)**.**

Let $(X, (\|\cdot\|_q)_q)$ be a Fréchet sequence algebra under the *Cauchy product* in which $(e_n)_n$ is a basis with Property B, and let B_w be a **mixing** weighted backward shift on X. Then there exists a point $x \in HC(B_w)$ such that the algebra generated by x, except zero, is contained in $HC(B_w)$.

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Let $(X, (\|\cdot\|_q)_q)$ be a Fréchet sequence space that contains the finite sequences. We say that $(e_n)_n$ has *Property B* if the following conditions hold:

- there is some $q \geq 1$ such that $\|e_n\|_q > 0$ for all $n \geq 0$;
- for any $r \geq 1$ there is some $q \geq 1$ and some $C_1 > 0$ such that, for all $n, k \geq 0$,

$$\|e_n\|_r \cdot \|e_k\|_r \leq C_1 \|e_{n+k}\|_q;$$

- for any $m \geq 2$, $M \geq 1$, $r \geq 1$ there is some $\rho \geq 1$ such that for any $t \geq 1$ there is some $\tau \geq 1$ and some $C_2 > 0$ such that, for any $0 \leq k \leq M$, $n \geq M$,

$$\|e_{mn}\|_t \cdot \|e_{n-k}\|_r \leq C_2 \|e_{mn}\|_\tau^{\frac{1}{m}} \cdot \|e_{mn-k}\|_\rho.$$

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For a fixed element $y \in X_{00}$ of length s and a natural number m bigger than one there exists sequences of natural numbers $(\gamma_n)_n$, $(\eta_n)_n$ and sequences of complex numbers $(\alpha_{(n,j)})_n$, $j = 1, \ldots, s$ and $(\beta_n)_n$ such that the point

$$p_{m,n} = q_{m,n} + \beta_n e_{\gamma_n}$$

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$$q_{m,n} = \sum_{j=1}^{s} \alpha_{(n,j)} y(j) e_{\eta_n + j}$$

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For a fixed element $y \in X_{00}$ of length s and a natural number m bigger than one there exists sequences of natural numbers $(\gamma_n)_n$, $(\eta_n)_n$ and sequences of complex numbers $(\alpha_{(n,j)})_n$, $j = 1, \ldots, s$ and $(\beta_n)_n$ such that the point

$$p_{m,n} = q_{m,n} + \beta_n e_{\gamma_n}$$

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An easier and much shorter proof...

Find $x \in$ 🌍 with

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Find $x \in$  with

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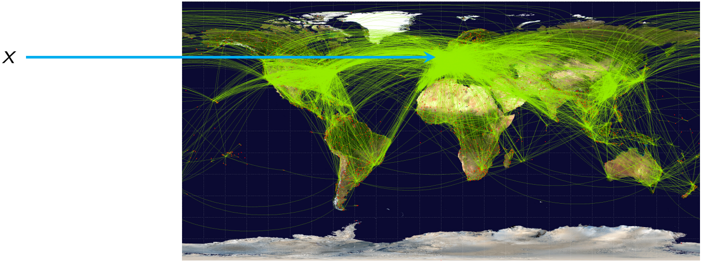

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Find $x \in$  with

$x =$ 

Phileas Fogg

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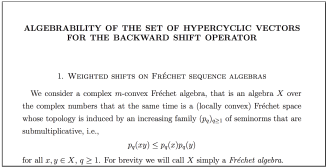
The itinerary (as originally planned)

London, UK to Suez, Egypt	rail to Brindisi, Italy and steamer (the *Mongolia*) across the Mediterranean Sea	7 days
Suez to Bombay, India	steamer (the *Mongolia*) across the Red Sea and the Indian Ocean	13 days
Bombay to Calcutta, India	rail	3 days
Calcutta to Victoria, Hong Kong	steamer (the *Rangoon*) across the South China Sea	13 days
Hong Kong to Yokohama, Japan	steamer (the *Carnatic*) across the South China Sea, East China Sea, and the Pacific Ocean	6 days
Yokohama to San Francisco, US	steamer (the *General Grant*) across the Pacific Ocean	22 days
San Francisco to New York City, US	rail	7 days
New York to London	steamer (the *China*) across the Atlantic Ocean to Liverpool and rail	9 days
Total		**80 days**

Map of the trip

What can we do now?



ALGEBRABILITY OF THE SET OF HYPERCYCLIC VECTORS FOR THE BACKWARD SHIFT OPERATOR

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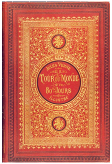

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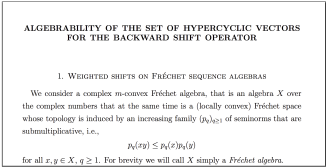

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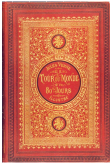

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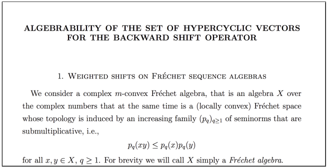

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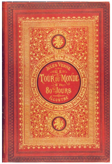

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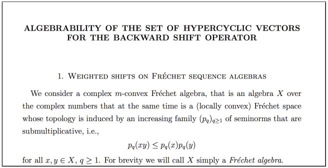

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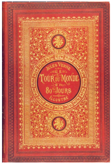

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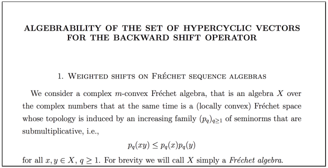

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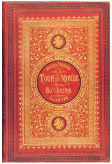

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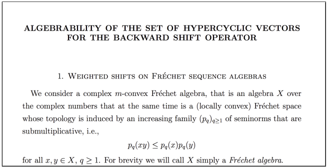

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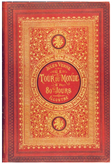

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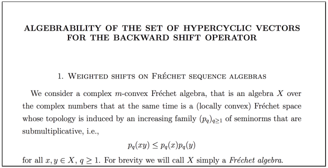

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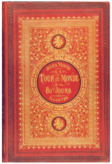

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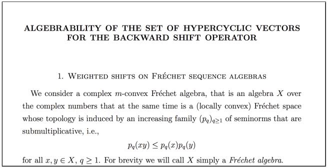

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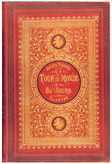

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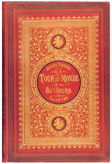

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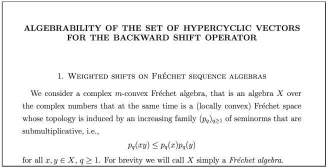

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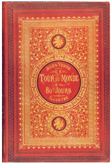

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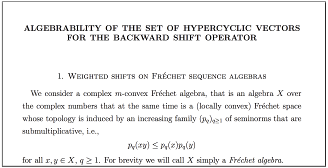

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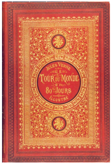

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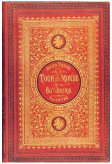

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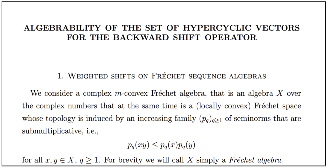

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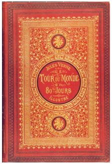

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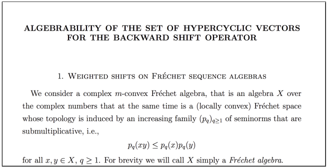

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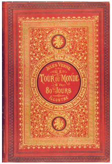

- 252 pages
- Only studies one specific space 
- He studies the orbit of one point
- Focuses on the first 80 days
  - The first version sold 108,000 copies,
  - Was translated into English, Russian, Italian, and Spanish as soon as it was published,
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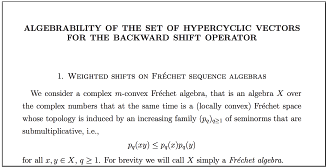
What can we do now?



ALGEBRABILITY OF THE SET OF HYPERCYCLIC VECTORS FOR THE BACKWARD SHIFT OPERATOR

1. Weighted shifts on Fréchet sequence algebras

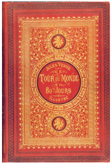
We consider a complex m-convex Fréchet algebra, that is an algebra X over the complex numbers that at the same time is a (locally convex) Fréchet space whose topology is induced by an increasing family $(p_q)_{q\geq 1}$ of seminorms that are submultiplicative, i.e.,

$$p_q(xy) \leq p_q(x)p_q(y)$$

for all $x, y \in X$, $q \geq 1$. For brevity we will call X simply a *Fréchet algebra.*

- 21 pages (more than 10 times shorter!)
- We work on Fréchet sequence algebras with "reasonable" conditions
- We provide many hypercyclic vectors
- Provide the complete orbit of the vectors

Expected impact!



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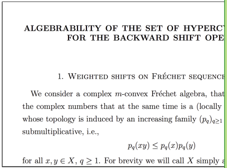

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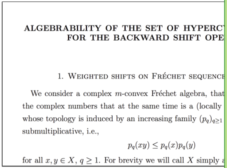

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Pushing harder

Frequently hypercyclic operators

The lower density of a subset $A \subset \mathbb{N}_0$ is defined as

$$\underline{dens}(A) = \liminf_{N\to\infty} \frac{card\{0 \le n \le N; n \in A\}}{N+1}.$$

Definition.

An operator T on a Fréchet space X is called *frequently hypercyclic* if there is some $x \in X$ so that, for any non-empty open subset U of X,

$$\underline{dens}\{n \in \mathbb{N}_0; T^n x \in U\} > 0.$$

In this case, x is called a *frequently hypercyclic vector* for T. The set of frequently hypercyclic vectors for T is denoted by $FHC(T)$.

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Birkhoff, MacLane and Rolewicz operators are examples of frequently hypercyclic operators.

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In general no!

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Proposition.

Let consider the Banach algebra $X = \ell_p$, $1 \leq p < \infty$, or $X = c_0$ endowed with coordinatewise multiplication. Then, for any $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, if $x_0 \in FHC(\lambda B)$, then there exists a natural number M_0 such that $x_0^M \notin HC(\lambda B)$ for any $M \geq M_0$.

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Let consider the Banach algebra $X = \ell_p$, $1 \leq p < \infty$, or $X = c_0$ endowed with coordinatewise multiplication. If $x_0 \in X$ with $x_0^j \in FHC(B_w)$, then for all $N \in \mathbb{N}$, $N > j$, and all $\alpha_{j+1}, \ldots, \alpha_N \in \mathbb{C}$ the point $x_0^j + \sum_{v=j+1}^{N} \alpha_v x_0^v \in FHC(B_w)$.

A TRUE positive example

Theorem.

Let consider the Banach algebra $(\ell_1, \times)$. For any $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ there exists an algebraically independent sequence of vectors $(x_n)_n \subset \ell_1$ such that $\mathcal{B} = \mathbb{C}[x_1, x_2, \ldots]$ but zero is in $FHC(\lambda B)$. Furthermore, the vectorial space

$$\mathcal{B}_0 := \{\sum_{j=1}^{k} \alpha_j x_j : k \in \mathbb{N}, \alpha_1, \ldots, \alpha_k \in \mathbb{C}\}$$

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Given two sequences x, y of ℓ_1 we consider the product $x \times y = z$ given by

$$z(t) = \begin{cases} x(1)y(1) & \text{if } j = 1 \\ x(1)y(t-1) & \text{if } n_k + 2 \leq t \leq n_k + s_l \text{ for some } n_k \in A(l) \\ 0 & \text{otherwise.} \end{cases}$$

where $(n_k)_k$, $(s_l)_l$ are two fixed sequences of natural numbers and $A(l)$ is a set of positive lower density for all $l \in \mathbb{N}$.

Bibliography

Aron, R. M.; Conejero, J. A.; Peris, A.; Seoane-Seplveda, J. B. Powers of hypercyclic functions for some classical hypercyclic operators. Integral Equations Operator Theory 58 (2007), no. 4, 591-596.

Bayart, Frédéric; Matheron, Étienne. Dynamics of linear operators. Cambridge Tracts in Mathematics, 179. Cambridge University Press, Cambridge, 2009. xiv+337 pp. ISBN: 978-0-521-51496-5.

Grosse-Erdmann, Karl-G.; Peris Manguillot, Alfredo. Linear chaos. Universitext. Springer, London, 2011. xii+386 pp. ISBN: 978-1-4471-2169-5.

Shkarin, Stanislav. On the set of hypercyclic vectors for the differentiation operator. Israel J. Math. 180 (2010).

Thank you!