Segre cone of Banach spaces and Σ-operators

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To study multilinear mappings with an eye on linear operators.

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What to do with other "bounded" conditions?

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If T is multilinear?

Segre cone and Σ-operators:

A geometrical framework to study multilinear mappings.

The domain

If $X_1, \ldots, X_n$ are vector spaces, we denote the set of **decomposable tensors** as

$$\Sigma_{X_1,\ldots,X_n} := \{x_1 \otimes \cdots \otimes x_n;\ x_i \in X_i\} \subset X_1 \otimes \cdots \otimes X_n.$$

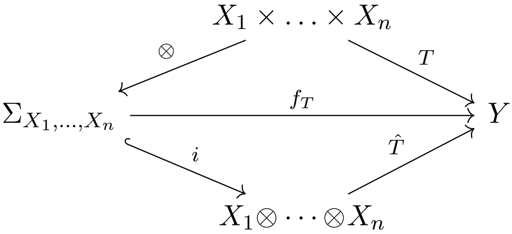
- The Segre variety is the image of the Segre embedding:

$$\begin{array}{rcl} \Sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m & \longrightarrow & \mathbb{P}^{(m+1)(n+1)-1} \\ ((x_0 : \ldots : x_n), (y_0 : \ldots : y_m)) & \mapsto & (\ldots : x_i y_j : \ldots) \end{array}$$

- We will use the same notation for the cone: Σ.

Mappings

Given $T \in L(X_1, \ldots, X_n; Y)$, let $\hat{T} \in L(X_1 \otimes \cdots \otimes X_n; Y)$ its associated linear mapping and $f_T := \hat{T}_{|\Sigma_{X_1,\ldots,X_n}}$. Then,



Multilinear mappings and/vs Σ-operators.

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1. Change , by $\otimes$ ($(x_1, \ldots, x_n)$ vs $x_1 \otimes \cdots \otimes x_n$) and T by f_T.
2. Solve in terms of $\otimes$ and f_T. The work to do is here*.
3. Change $\otimes$ by , and f_T by T.

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Benefit: a geometric insight.

- $\Sigma_{X_1,\ldots,X_n}$ is richer than $X_1 \times \ldots \times X_n$.
- The metric, the tensor and the multilinear structures merge in $f_T : \Sigma_{X_1,\ldots,X_n} \to Y$.

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* The problem now lies within a geometric environment.

$(S^r_{X_1,\dots,X_n}, d_\beta)$

$X_1, \dots, X_n$ vector spaces, the *rank* of $z \in X_1 \otimes \cdots \otimes X_n$ is:

$$r_z := \min\{r \in \mathbb{N};\ z = \sum_{i=1}^{r} x_1^i \otimes \cdots \otimes x_n^i;\ x_j^i \in X_j\}.$$

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Theorem (Lipschitz equivalence of tensor metrics in S^r)

Let X_i be B.s. and α, β are r.c. norms on $X_1 \otimes \cdots \otimes X_n$, then:

$$(\mathcal{S}^r_{X_1,\dots,X_n}, d_\alpha) \overset{Id}{\simeq} (\mathcal{S}^r_{X_1,\dots,X_n}, d_\beta).$$

$d_\alpha(w, z) \leq (2r)^{n-1} d_\beta(w, z)$ *and* $\|z\|_\alpha \leq r^{n-1} \|z\|_\beta$.

If X_i B.s. and α is a r.c. norm on $X_1 \otimes \cdots \otimes X_n$, the completion of $(\mathcal{S}^r_{X_1,\dots,X_n}, d_\alpha)$ is the closure

$$\overline{\mathcal{S}^r_{X_1,\dots,X_n}}^{\alpha} \subset X_1 \hat{\otimes}_\alpha \cdots \hat{\otimes}_\alpha X_n.$$

Corollary (The closure is independent of the r.c.norm)

$$\overline{\mathcal{S}^r_{X_1,\dots,X_n}}^{\alpha} = \overline{\mathcal{S}^r_{X_1,\dots,X_n}}^{\pi}$$

Thus, $(\mathcal{S}^r_{X_1,\dots,X_n}, d_\alpha)$ *is complete iff* $(\mathcal{S}^r_{X_1,\dots,X_n}, d_\beta)$ *is complete.*

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Thus, $(\mathcal{S}^r_{X_1,\dots,X_n}, d_\alpha)$ is complete iff $(\mathcal{S}^r_{X_1,\dots,X_n}, d_\beta)$ is complete.

Corollary (A unique topolgy)

All r.c norms on $X_1 \otimes \cdots \otimes X_n$ induce the same topology on $\mathcal{S}^r_{X_1,\dots,X_n}$.

Closedness of $\mathcal{S}^r_{X_1,\dots,X_n}$

Let $X_1, \dots, X_n$ be Banach spaces. Then,

- $r = 1,\ n \geq 1 \implies S^1_{X_1,\dots,X_n} (= \Sigma_{X_1,\dots,X_n})$ is closed.
- $r \geq 1,\ n = 2 \implies S^r_{X_1,X_2}$ is closed.
- $r \geq 2,\ n \geq 3 \implies S^r_{X_1,\dots,X_n}$ is not* closed.

* If the spaces are finite dimensional, sometimes it is closed (depends on r and the dimensions of X_j).

The Segre cone

For $X_1, \ldots, X_n$ B.s. and α, β are r.c.n. on $X_1 \otimes \cdots \otimes X_n$,

$$(\Sigma_{X_1,\ldots,X_n}, d_\alpha) \overset{Id}{\simeq} (\Sigma_{X_1,\ldots,X_n}, d_\beta)$$

with $d_\alpha(w,z) \leq d_\pi(w,z) \leq 2^{n-1} d_\alpha(w,z)$. With this,

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- $(\Sigma_{X_1,\ldots,X_n}, d_\pi)$ is complete.
- If $Y_i \subset X_i$, then $\Sigma_{Y_1,\ldots,Y_n}$ is closed in $\Sigma_{X_1,\ldots,X_n}$.
- If $Y \subset \Sigma_{X_1,\ldots,X_n}$ is a subspace of $\dim Y \geq 2$, $Y \subset x_1 \otimes \cdots \otimes x_{i_0-1} \otimes X_{i_0} \otimes x_{i_0+1} \otimes \cdots \otimes x_n$.

Σ-operators

Definition

A mapping $f : \Sigma_{X_1,\ldots,X_n} \to Y$ is a (bounded) **Σ-operator** if there exists a (bounded) multilinear mapping $T \in L(X_1, \ldots, X_n; Y)$ such that $f = \hat{T}_{|_{\Sigma_{X_1,\ldots,X_n}}}$.

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* No sense in terms of $\{, , T\}$ instead of $\{\otimes, f\}$.

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Problem:

Given a Banach-space condition on linear mappings, how to state an analogous (appropriate? natural? satisfactory?...) condition on multilinear mappings.

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Given a boundedness condition on linear operators,

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write the analogous boundedness (Lipschitz) conditions on Σ-operators*

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p-summability (with respect to a r.c.norm β)*

Definition

A linear mapping $T : X \to Y$ is absolutely p-summing if $\exists c > 0$

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A Σ-operator $f : \Sigma_{X_1,\ldots,X_n} \to Y$ is p-summing if $\exists c > 0$ s.t.

$$\sum_{i=1}^{k} \|f(u_i) - f(v_i)\|^p \leq c^p \cdot \sup_{\varphi \in B_{\mathcal{L}_{\Sigma\beta}}} \left\{ \sum_{i=1}^{k} |\varphi(u_i) - \varphi(v_i)|^p \right\}$$

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$\mathcal{L}_{\Sigma\beta}$ ={Scalar valued Σ-operators s.t $\hat{T}_f \in (X_1 \hat{\otimes}_\beta \cdots \hat{\otimes}_\beta X_n)^*$}.

* Joint work with J.C Angulo

- Domination Thm.

$$\|f(u) - f(v)\| \leq c \cdot \left(\int_{B_{\mathcal{L}_{\Sigma\beta}}} |\varphi(u) - \varphi(v)|^p \, d\mu(\varphi) \right)^{1/p}.$$

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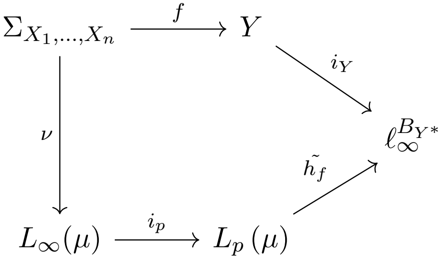
- Factorization Thm.

$$\begin{array}{ccc} \Sigma_{X_1,\ldots,X_n} & \xrightarrow{\quad f \quad} & Y \\ \downarrow i_{\Sigma} & & \uparrow h_f \\ \Sigma^{\beta}_{X_1,\ldots,X_n} & \xrightarrow{\quad j_{p|_{\Sigma}} \quad} & \Sigma^{\beta}_{p} \\ \cap & & \cap \\ C(B_{\mathcal{L}_{\Sigma\beta}}, w^*) & \xrightarrow{\quad j_p \quad} & L_p(\mu) \end{array}$$

where h_f is a Lipschitz mapping.

In the case of the projective norm

- f is p-summing iff



where ν is a Σ-operator and $\tilde{h_f}$ is a Lipschitz mapping.

$T : X_1 \times \cdots \times X_n \to Y$ mulitlinear. Tfae:

- Local notion:

$$\sum_{i=1}^{k} \|T(u_i) - T(v_i)\|^p \leq c^p \cdot \sup_{\varphi \in B_{\mathcal{L}}} \left\{ \sum_{i=1}^{k} |\varphi(u_i) - \varphi(v_i)| \right\}$$

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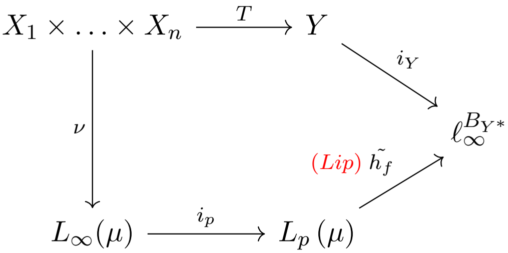

Thank you!