On p-summing operators that factor through L^p-spaces of a vector measure

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5th Work-Shop on Functional Analysis.
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1 Preliminaries and motivation

- The L^p-space of a vector measure
- The class of p-th power factorable operators
- The operator ideal of (p, q)-factorable operators

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   - The L^p-space of a vector measure
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2. The class of $\mathcal{F}_{p,q}$-factorable operators
   - Extrapolation from classical theory
   - $\mathcal{F}_{p,q}$-factorable operator
   - Extrapolation from $\mathcal{F}_{p,q}$

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   - Extrapolation from classical theory
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3. Some examples and applications
   - Kernel operators
   - Convolution type operators
   - Application to extrapolation in $\mathcal{L}_{p,q}$

The L^p-space of a vector measure

(Ω, Σ) **Measure space**

$m \colon \Sigma \to E$ **Banach space-valued measure**

$\langle m, x^* \rangle (A) := \langle m(A), x^* \rangle$ **control measure**, $x^* \in E^*$

Let f **real or complex function**. $f \in L^1(m)$ if :

(1) $f \in L^1(|\langle m, x^* \rangle|)$ for every $x^* \in E$,
(2) there exists $x_0 \in E$ such that

$$\langle x_0, x^* \rangle = \int_\Omega |f| \, d\langle m, x^* \rangle \text{ for every } x^* \in E.$$

$L^1(m)$ is **Banach** and

$$\|f\|_{L^1(m)} := \sup_{x^* \in B_{E^*}} \int_\Omega |f| \, d\langle m, x^* \rangle.$$

The L^p-space of a vector measure

Let μ **finite** scalar measure and $\mathbf{0} < \mathbf{p} < \infty$.

$Z(\mu)$ **B.f.s.** (Banach function space) as **Lindenstrauss-Tzafriri**.

$Z(\mu)_{[p]}$ p-**th power space** of $Z(\mu)$, i.e. its $1/p$-**covexification**.

quasi-norm: $\|f\|_{Z(\mu)_{[p]}} := \||f|^{1/p}\|_{Z(\mu)}$. Also $Z(\mu) \subseteq Z(\mu)_{[p]}$ $(p \geq 1)$.

$L^1(m)$ is a **B.f.s. over** $\langle m, x^* \rangle$, and $L^p(m) := L^1(m)_{[1/p]}$.

$S\colon Z(\mu) \to Y$ **linear operator**, $Z(\mu)$ **o.c.** (order continuous).

S **always** factors through $L^1(m_S)$, where $\mathbf{m_S(A) := S(\chi_A)}$.

And **sometimes** S factors through $L^p(m_T)$.

The class of p-th power factorable operators

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Property 1

S p-**th power factorable**, then so is for $\mathbf{q}$ s.t. $\mathbf{1} \leq \mathbf{q} \leq \mathbf{p}$.

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Property 2

(Okada, Ricker & Sánchez-Pérez, 2008)

S p-**th power factorable** $\Leftrightarrow$ $S\colon Z(\mu) \overset{\text{inclusion/quotien}}{\hookrightarrow} L^p(m_S) \overset{I_{m_S}^{(p)}}{\to} Y.$

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Corollary S p-**th power factorable** $\Leftrightarrow$

$$\begin{array}{ccc} Z(\mu) & \xrightarrow{\quad S \quad} & Y \\ {\scriptstyle J_S^{(p)}} \downarrow & & \uparrow {\scriptstyle I_{m_S}} \\ L^p(m_S) & \hookrightarrow & L^1(m_S) \end{array}$$

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$\mathcal{F}_{\mathbf{p}}(\mathbf{Z}(\mu), \mathbf{Y}) := \{p\text{-th power factorable operators}\}$
$\mathcal{F}_{\mathbf{p}}^{\text{dual}}(\mathbf{X}, \mathbf{Z}(\mu)) := \{\text{oper. with } p\text{-th power factorable adjoint}\}$

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Examples

Laplace transform: (Galdames-Bravo, 2017)

$1 < q \leq 2 \leq p < \infty$. $L^p(0,\infty) \subseteq L^p(m_{\mathcal{L}}) \xrightarrow{\mathcal{L}} L^q(hdx)$, $\mathcal{L}$ is r-th power factorable for $r \in [p/2, p)$.

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Examples

Fourier transform: (Okada, Ricker & Sánchez-Pérez, 2008)
$1 < p < 2$. $F_p\colon L^p(G) \to c_0(\Gamma)$ is r-th power factorable for $r \in [1, p]$.

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Examples

Hardy operator adjoint: (Galdames-Bravo & Sánchez-Pérez)
$1 \leq p \leq q < \infty$. $H\colon L^p[0,1] \to L^q[0,1]$, H^* is r-th power factorable for $r \in [1, 2q')$.

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Examples

Convolution operator adjoint:
$1 < p < \infty$. $C_h\colon L^p(G) \to L^p(G)$, C_h^* is p'/r-th power factorable for $r \in (1, p')$.

The operator ideal of (p, q)-factorable operators

They come from p-**summing operator** theory. As a generalization of p-**integral operators**, which are always p-summing.

Characterization: $\mathbf{1/p + 1/q \geq 1}$

$$\begin{array}{ccccc} X & \xrightarrow{T} & Y & \hookrightarrow & Y^{**} \\ {\scriptstyle R}\downarrow & & & & \uparrow{\scriptstyle S} \\ L^{q'}(\mu) & \xhookrightarrow{I} & & & L^p(\mu) \end{array}$$

$\mathcal{L}_{p,q}(X, Y)$ **Banach operator ideal**

with $\alpha_{p,q}(T) := \inf \|S\|\|I\|\|R\|$.

Properties

(1) $\mathcal{L}_{p,q}(X, Y) \subseteq \mathcal{L}_{r,s}(X, Y)$ for $p \leq r < \infty$ and $q \leq s < \infty$.

(2) $\mathcal{L}_{p,1}(X, Y) = \mathcal{I}_p(X, Y) \subseteq \Pi_p(X, Y)$.

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We need some **extrapolation** for the indexes of $\mathcal{L}_{\mathbf{p,q}}$.

Extrapolation from classical theory

Obviously: for suitable X and Y,
there are p-**summing** operators in $\mathcal{L}_{\mathbf{p},\mathbf{q}}(\mathbf{X},\mathbf{Y})$

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Example 1

Maurey-Rosenthal's Factorization Theorem
X and Y **Banach lattices**.
$T \in \mathcal{L}(X,Y)$ p-convex (q-concave) and Y p-concave (X q-convex)
$\Rightarrow T\colon X \to L^p(\mu) \to Y$ ($T\colon X \to L^q(\mu) \to Y$).

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Corollary

If $1 \le r < p$, $1 \le s < q$, X r'-**convex** and Y s-**concave**. Then

$$\Rightarrow \mathcal{L}^+_{p,q}(X,Y) = \mathcal{L}^+_{r,s}(X,Y).$$

Extrapolation from classical theory

Example 2

Maurey's Extrapolation Theorem, '74

$\Pi_p(X, \ell^p) = \Pi_r(X, \ell^p)$ for $1 \leq r < p < \infty \Rightarrow \Pi_p(X, Y) = \Pi_r(X, Y)$

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Corollary

If $1 < r < p < \infty$, $1 < s < q < \infty$, $\Pi_{r'}(X, \ell^{r'}) = \Pi_{p'}(X, \ell^{r'})$ and $\Pi_{s'}(X, \ell^{s'}) = \Pi_{q'}(X, \ell^{s'})$. Thanks to Kwapień representation ('72)

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$$\Rightarrow \Big(\mathcal{L}_{p,q} = \mathcal{D}^*_{p',q'} = (\Pi^{\text{dual}}_{q'} \circ \Pi_{p'})^* = (\Pi^{\text{dual}}_{s'} \circ \Pi_{r'})^* = \mathcal{L}_{r,s}\Big)(X, Y).$$

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$\mathbf{R} \in \mathcal{F}_{\mathbf{q}}^{\text{dual}}(\mathbf{X}, \mathbf{Z}(\mu))$ and $\mathbf{S} \in \mathcal{F}_{\mathbf{p}}(\mathbf{Z}(\mu), \mathbf{Y}^{**})$ such that

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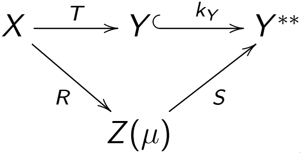
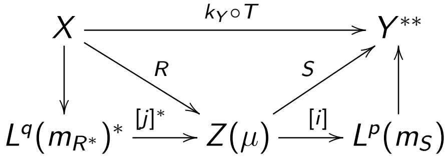
$$\mathbf{k_Y} \circ \mathbf{T} = \mathbf{S} \circ \mathbf{R}.$$

i.e.



Extrapolation from $\mathcal{F}_{p,q}$

From definition and characterization



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$$
\begin{array}{ccccccccc}
 & & X & & \xrightarrow{\quad k_Y \circ T \quad} & & Y^{**} & & \\
 & {\scriptstyle I^*_{m_{R^*}} \circ k_X} \swarrow & \downarrow & \searrow {\scriptstyle R} & & \nearrow {\scriptstyle S} & \uparrow & \nwarrow {\scriptstyle I_{m_S}} & \\
L^1(m_{R^*})^* & \xhookrightarrow{\gamma_q^*} & L^q(m_{R^*})^* & \xrightarrow{[j]^*} & Z(\mu) & \xrightarrow{[i]} & L^p(m_S) & \xhookrightarrow{\gamma_p} & L^1(m_S)
\end{array}
$$

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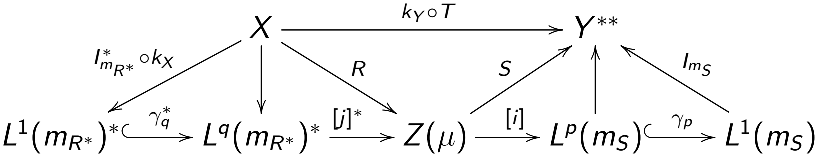

Properties

(1) $\mathcal{F}_{\mathbf{p},\mathbf{q}}(\mathbf{X},\mathbf{Y}) \subseteq \mathcal{F}_{\mathbf{r},\mathbf{s}}(\mathbf{X},\mathbf{Y})$ for $\mathbf{1} \leq \mathbf{r} \leq \mathbf{p}$ and $\mathbf{1} \leq \mathbf{s} \leq \mathbf{q}$.

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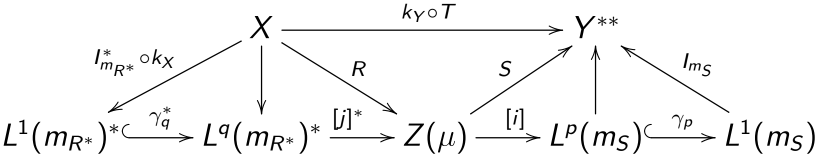

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(2) $\mathbf{1/pr} + \mathbf{1/qr} \geq \mathbf{1} \Rightarrow \mathcal{L}_{p,q}(X,Y) \subseteq \mathcal{F}_{r,s}(X,Y)$

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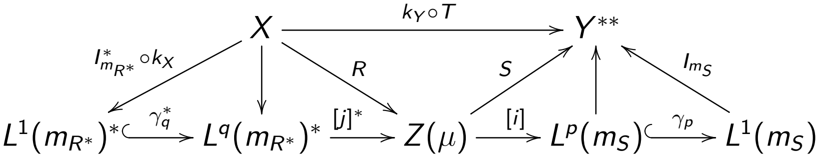

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(2) $\mathbf{1/pr} + \mathbf{1/qr} \geq \mathbf{1} \Rightarrow \mathcal{L}_{p,q}(X,Y) \subseteq \mathcal{F}_{r,s}(X,Y)$

A natural (and open) question:
When $\mathbf{L^q(m_{R^*})} = \mathbf{L^s}(\nu)$ or R **factors** through $L^s(\nu)$?

Extrapolation from $\mathcal{F}_{p,q}$

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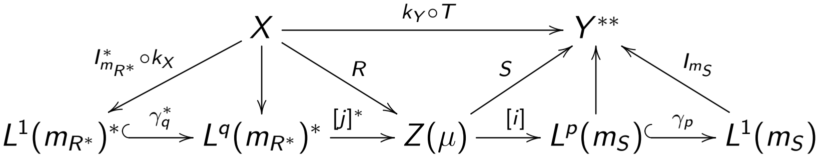

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For example: (S. Okada, W. J. Ricker, L. Rodríguez-Piazza, 2002) $\mathbf{L^q(m_{R^*}) = L^q(|m_{R^*}|)}$ when R is **compact**.

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Theorem

(Galdames-Bravo & Sánchez-Pérez)
$\mathbf{1/r = 1/p + 1/s}$. $\left\| \|K(\cdot, y)\|_{L^q(\mu)} \right\|_{L^{s/r}(\mu)} < \infty$. Then
$T_K \colon L^p(\mu) \to L^q(\mu)$ is r-**th power factorable**.

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Corollary

F and H kernel functions and $\mathbf{1/r = 1/w + 1/u}$, $\mathbf{1/s = 1/w' + 1/v}$.
$\mathbf{K(x,y) := \int_\Omega F(x,z) H(z,y)\, d\mu(z)}$. $\left\| \|F(\cdot, y)\|_{L^q(\mu)} \right\|_{L^{u/r}(\mu)} < \infty$ and
$\left\| \|H(x, \cdot)\|_{L^{p'}(\mu)} \right\|_{L^{v/s}(\mu)} < \infty$. Then $\mathbf{T_K} \in \mathcal{F}_{\mathbf{r,s}}(\mathbf{L^p}(\mu), \mathbf{L^q}(\mu))$.

Convolution type operators

Theorem

(Okada, Ricker & Sánchez-Pérez)
$\mathbf{1} < \mathbf{r}, \mathbf{u} < \mathbf{p}$, $\mathbf{1}/\mathbf{u} + \mathbf{1}/\mathbf{r} = \mathbf{1}/\mathbf{p} + \mathbf{1}$, $h \in L^r(G) \setminus L^p(G)$
$C_h \colon L^p(G) \to L^p(G)$ is **p/u-th power factorable**.

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Corollary

$\mathbf{1} < \mathbf{u} < \mathbf{q}'$, $\mathbf{u} \leq \mathbf{v}$ s.t. $\mathbf{1}/\mathbf{u} + \mathbf{1}/\mathbf{q} = \mathbf{1}/\mathbf{v} + \mathbf{1}$.
$g \in L^q(G) \setminus L^{u'}(G)$ and $f \in L^1(G)$. Then
$C_{f*g} = C_f \circ C_g \in \mathcal{F}_{1,u'/v'}(L^u(G), L^1(G))$

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$\mathbf{1} < \mathbf{u} < \mathbf{q}'$, $\mathbf{u} \leq \mathbf{v}$ s.t. $\mathbf{1}/\mathbf{u} + \mathbf{1}/\mathbf{q} = \mathbf{1}/\mathbf{v} + \mathbf{1}$.
$g \in L^q(G) \setminus L^{u'}(G)$ and $f \in L^1(G)$. Then
$C_{f*g} = C_f \circ C_g \in \mathcal{F}_{1,u'/v'}(L^u(G), L^1(G))$

Remark $C_{f*g} = T_K$ for $K(x,y) := \int_G f(x-z)g(z-y)\, d\mu(z)$, as in the **previous corollary**.

Conditions for extrapolation in $\mathcal{L}_{p,q}$

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(Galdames-Bravo)
(1) $\mathbf{1/pr + 1/qs \geq 1} \Rightarrow \mathcal{L}_{\mathbf{p,q}}(\ell^{\mathbf{t}}, \ell^{\mathbf{w}}) \subseteq \mathcal{L}_{\mathbf{u,v}}(\ell^{\mathbf{t}}, \ell^{\mathbf{w}})$,
$1 \leq w < 2 < t \leq \infty$, $1 < u \leq r$ and $1 < v \leq s$.

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Corollary

$u \in [1, p]$, $v \in [1, q]$ s.t. $\mathbf{1/pu + 1/qv \geq 1}$, $r \in [u, p]$, $s \in [v, q]$ s.t.
$\mathbf{1/rt + 1/sw = 1}$ and $\mathcal{F}^{\mathbf{c}}_{\mathbf{u,v}}(\mathbf{X}, \mathbf{Y}) \subseteq \mathcal{F}^{\mathbf{rt}}_{\mathbf{t,w}}(\mathbf{X}, \mathbf{Y})$ $(c \in [pu, (qv)'])$

$$\Rightarrow \mathcal{L}_{\mathbf{p,q}}(\mathbf{X}, \mathbf{Y}) \subseteq \mathcal{L}_{\mathbf{r,s}}(\mathbf{X}, \mathbf{Y}).$$

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