The class of p-compact mappings in the operator space setting

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The basics

What is a p-**compact mapping**?

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What is a p-**compact mapping**?

Where?

Banach Space setting

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Banach Space setting $\rightsquigarrow$ operator space framework

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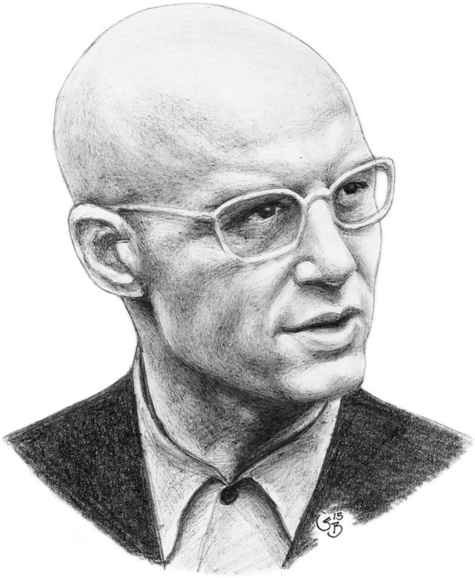

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$$K \subset \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : \sum_{n=1}^{\infty} |\alpha_n| \leq 1 \right\} \text{ and } \lim_{n\to\infty} \|x_n\| = 0.$$

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$$m_\infty(K;X) := \sup_{x\in K} \|x\| = \inf \left\{ \|(x_n)_n\|_{c_0(X)} \right\}.$$

p-compact sets

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"Measure the size" of a p-compact set $K \subset X$:

$$m_p(K;X) := \inf \left\{ \|(x_n)_n\|_{\ell_p(X)} \right\}.$$

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Therefore, p-compactness reveals "finer and subtle" structures on compact sets.

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$\mathcal{K}_p$ is a Banach ideal of operators (in the sense of Pietsch).

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Questions

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Plan:

$\mathcal{K}_p \leftrightsquigarrow$ study it from an operator ideal/tensor norm perspective.

Right p-nuclear mappings

Recall, the Chevet-Saphar tensor norms d_p are related with the ideals of absolutely summing operators in the following way:

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The class $\mathcal{N}^p$

A right p-nuclear mapping between the Banach spaces X and Y is exactly an operator which is in the range of

$$J^p : X' \widehat{\otimes}_{d_p} Y \to X' \widehat{\otimes}_{\varepsilon} Y,$$

and its norm coincides with the quotient norm inherited from the inclusion.

Easier way to understand $\mathcal{N}^p$

The class $\mathcal{N}^p$

$\Theta : X \to Y$ is right p-nuclear if there is a factorization

$$\begin{array}{ccc} X & \xrightarrow{\Theta} & Y \\ {\scriptstyle U}\downarrow & & \uparrow{\scriptstyle V} \\ \ell_{p'} & \xrightarrow[D_\lambda]{} & \ell_1, \end{array}$$

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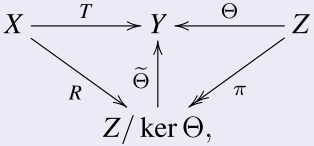
$$\|\Theta\|_{\mathcal{N}^p} = \inf\{\|U\|\|D_\lambda\|\|V\|\},$$

where the infimum runs over all factorizations as above.

Characterization of $\mathcal{K}_p$

The following are equivalent:

- $T : X \to Y$ is <u>p-compact</u>.
- There is a <u>right p-nuclear mapping</u> $\Theta \in \mathcal{N}^p(Z, Y)$ and a bounded mapping $R \in \mathcal{L}(X, Z/\ker\Theta)$ with $\|R\| \leq 1$ such that the following diagram commutes



where π stands for the natural quotient mapping and $\widetilde{\Theta}$ is given by $\widetilde{\Theta}(\pi(z)) = \Theta(z)$.

$$\|T\|_{\mathcal{K}_p} := \inf\{\|\Theta\|_{\mathcal{N}^p}\}.$$

How certain classical ideals relate with $\mathcal{K}_p$

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Operators whose adjoint are p-summing correspond to the one that map compact sets to relatively p-compact sets.

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Idea:

See if these two characterizations have a counterpart in the context of operator spaces.

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Therefore, $M_n(E) \subset B(H^n) \rightsquigarrow$ this provides a norm for every matrix level $M_n(E)$.

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Let $T : E \to F$ a linear mapping between o.s.
For $n \in \mathbb{N}$, this defines an operator $T_n : M_n(E) \to M_n(F)$ (the n-amplification) given by

$$\begin{bmatrix} x_{11} & \dots & x_{1,n} \\ \vdots & \ddots & \\ x_{n1} & & x_{nn} \end{bmatrix} \mapsto \begin{bmatrix} T(x_{11}) & \dots & T(x_{1,n}) \\ \vdots & \ddots & \\ T(x_{n1}) & & T(x_{nn}) \end{bmatrix}$$

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Completely bounded operators are relevant morphisms in this context.

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An operator $T : E \to F$ is "compact" if it maps the matrix unit ball into a "compact a matrix set".

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So... we need a good definition of "compacteness for matrix sets".

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- $\mathcal{K}_p^{o.s.}$ endowed with the norm $\|T\|_{\mathcal{K}_p^{o.s.}} = m_p\left(\left(T_n B_{M_n(E)}\right)_{n\in\mathbb{N}}; F\right)$.

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Non-commutative Chevet-Saphar tensor norm

In the Banach space setting we have:

$$(X\widehat{\otimes}_{d_p}Y)' = \Pi_{p'}(X, Y').$$

A. Chávez Domínguez (Houston J. Math., 2016):

Constructed an operator space version of the Chevet-Saphar tensor norm d_p, denoted by $d_p^{o.s.}$ such that:

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This tensor norm induces a notion of completely right p-nuclear mappings.

Completely right p-nuclear mappings

The class $\mathcal{N}^p$

A right p-nuclear mapping between the Banach spaces X and Y is exactly an operator which is in the range of

$$J^p : X' \widehat{\otimes}_{d_p} Y \to X' \widehat{\otimes}_{\varepsilon} Y,$$

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- Let $1 \leq p \leq \infty$, we say that a linear mapping $T : E \to F$ is completely right p-nuclear if it corresponds to an element in the range of the canonical inclusion

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$$J^p_{o.s.} : E' \widehat{\otimes}_{d_p^{o.s.}} F \to E' \widehat{\otimes}_{\min} F.$$
- $\mathcal{N}^p_{o.s.}(E; F)$ and we endow it with the quotient o.s. structure $(E' \widehat{\otimes}_{d_p^{o.s.}} F) / \ker J^p \rightsquigarrow$ mapping ideal.

Completely right p-nuclear mappings in terms of certain factorizations

Chávez Dominguez, Dimant, G.

The following are equivalent:

(a) $T : E \to F$ is completely right p-nuclear.

(b) There exist $a, b \in S_{2p}$ such that T admits a factorization

$$\begin{array}{ccc} E & \xrightarrow{\quad T \quad} & F \\ {\scriptstyle U}\downarrow & & \uparrow{\scriptstyle V} \\ S_{p'} & \xrightarrow[M(a,b)]{} & S_1 \end{array}$$

Moreover, in this case

$$\|T\|_{\mathcal{N}^p_{o.s.}} = \inf \left\{ \|U\|_{c.b.} \|V\|_{c.b.} \|a\|_{S_{2p}} \|b\|_{S_{2p}} \right\}$$

where the infimum is taken over all factorizations as in (b).

The relation of $\mathcal{K}_p^{o.s.}$ with $\mathcal{N}_{o.s.}^p$

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- $T \in \mathcal{K}_p^{o.s.}(E, F)$.
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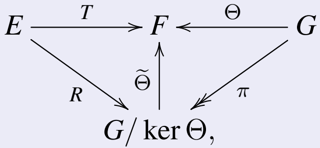

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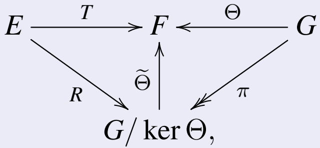

Moreover, $\|T\|_{\mathcal{K}_p^{o.s.}} = \inf\{\|\Theta\|_{\mathcal{N}_{o.s.}^p}\}$.

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THANK YOU VERY MUCH!!!