Geometric Properties of Cones

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Notation and terminology

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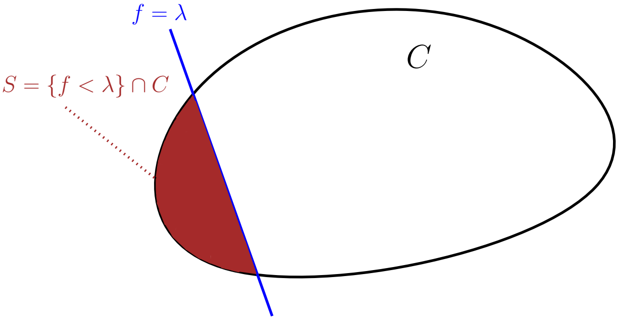
- An slice of a set C is a non empty intersection of C with an open half space of X



Denting points

Definition

Let C be a subset of X, $c \in C$ is said to be a denting point of C if

$$c \notin \overline{\text{conv}}(C \setminus B_\varepsilon(c)), \ \forall \varepsilon > 0.$$

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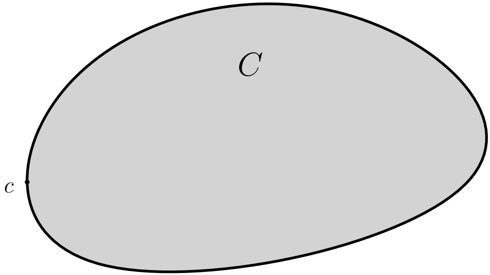

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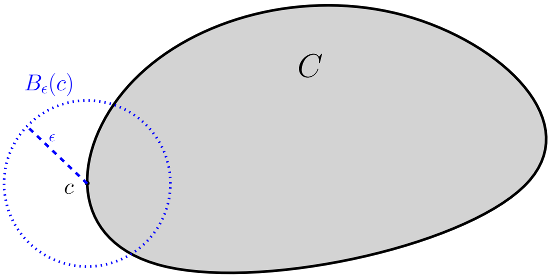

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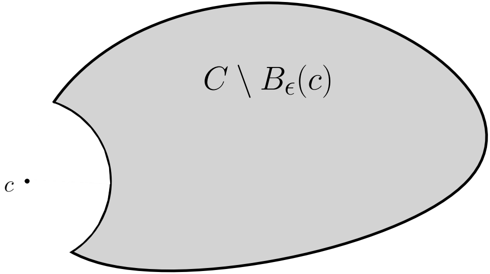

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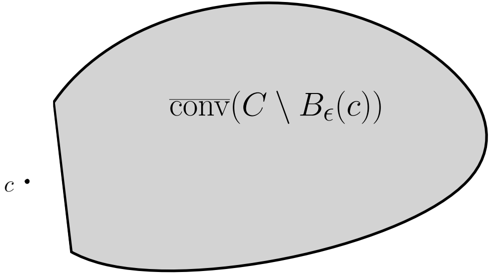

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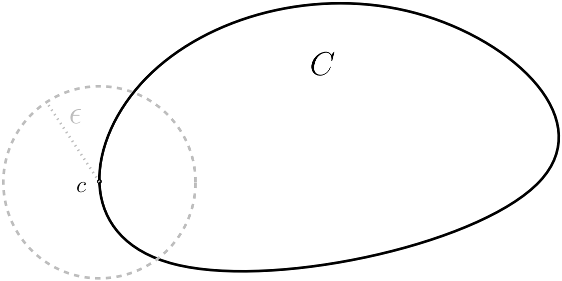

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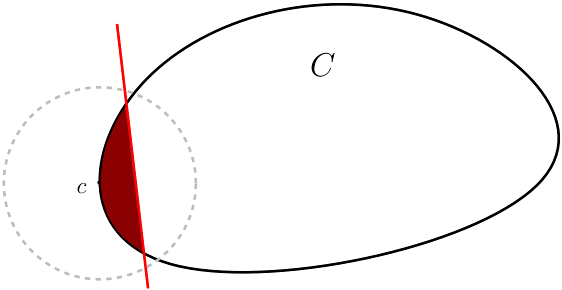

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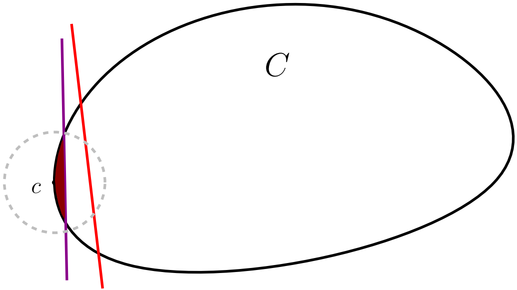

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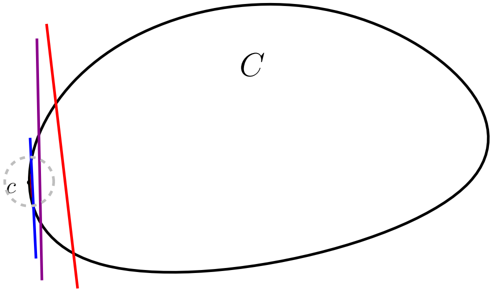

Denting points

Dentability is applied to study:

- Radon-Nikodým property
- LUR renorming
- Optimization
- Operators theory

Points of continuity

Definition

Let C be a subset of X, $c \in C$ is said to be a point of continuity for C if the identity map $(C, \text{weak}) \to (C, \| \ \|)$ is continuous at c.

Points of continuity

c is a point of continuity for C if and only if for every open ball $B_\varepsilon(c)$, there exists a weakly open U such that

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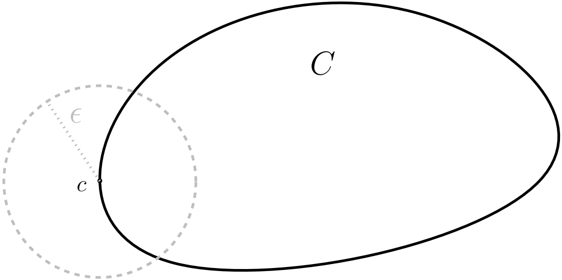

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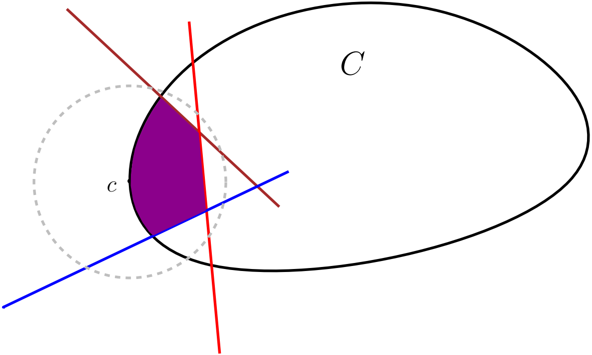

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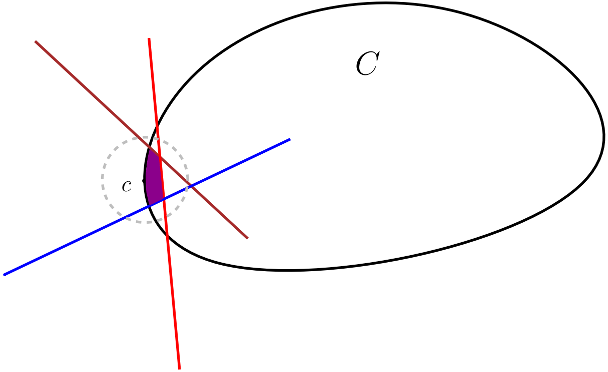

Points of continuity

The notion of point of continuity is applied to:

- Provide a geometric proof to a fixed point theorem
- Geometric properties related to Radon-Nikodým property
- Optimization

Denting points and points of continuity

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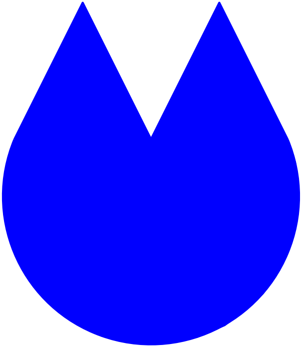

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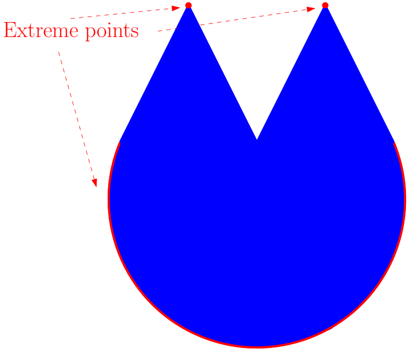

Denting points and points of continuity

Theorem (Lin–Lin–Troyanski, 1985)

Let c be an extreme point of a closed, convex, and bounded subset C of a Banach space. If c is a point of continuity for C, then it is a denting point.

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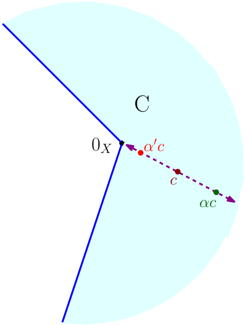
What about cones?

Denting points, points of continuity, and cones

Definition

1. A non empty convex subset C of X is called a cone if

$$\alpha C \subset C, \forall \alpha \geq 0$$



Denting points, points of continuity, and cones

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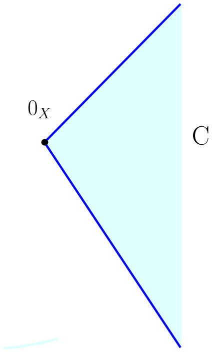
2. A cone C is called pointed if $C \cap (-C) = \{0_X\}$



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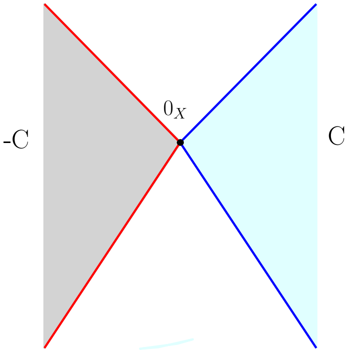
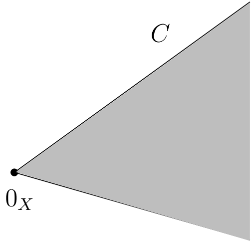
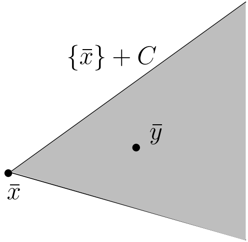


Denting points, points of continuity, and cones

A pointed cone C induces a partial order $\preceq$ on X by

$$\bar{x} \preceq \bar{y} \Leftrightarrow \bar{y} - \bar{x} \in C \Leftrightarrow \bar{y} \in \{\bar{x}\} + C$$



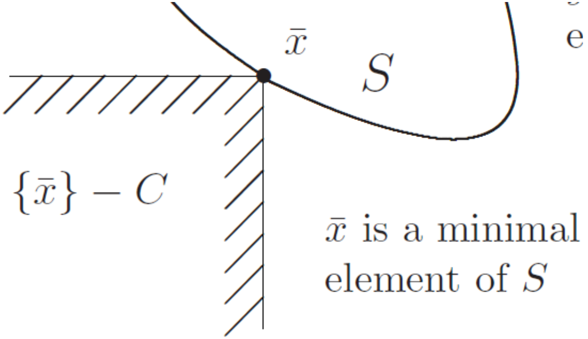


Denting points, points of continuity, and cones

Definition

A point $\bar{x} \in S$ is a minimal point of S if $(\{\bar{x}\} - C) \cap S = \{\bar{x}\}$

$$\bar{x} \in \mathrm{Min}(S, C)$$



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The quasi-interior of C^* is defined as

$$C^{\#} := \{f \in X^* \colon f(c) > 0, \ \forall c \in C \setminus \{0_X\}\}$$

Denting points, points of continuity, and cones

Definition

A point $\bar{x} \in S$ is a positive proper minimal point of S if there exists some $f \in C^{\#}$ such that $f(\bar{x}) \leq f(\bar{y})$, $\forall \bar{y} \in S$.

$$\mathrm{Pos}(S, C) \subset \mathrm{Min}(S, C)$$

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$$\mathrm{Pos}(S, C) \subset \mathrm{Min}(S, C)$$

When is $\mathrm{Pos}(S, C)$ dense in $\mathrm{Min}(S, C)$?

(Density results of Arrow, Barankin and Blackwell's type)

Denting points, points of continuity, and cones

Theorem (Petschke, 1990)

If S is a convex weak compact subset of a normed space X and 0_X is a denting point of a pointed closed cone C, then

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Problem (Gong, 1995)

The property of point of continuity at the origin for a closed and pointed cone in a normed space, is really weaker than the property of denting point at the origin of the cone?

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A negative answer for Banach spaces

Theorem (Daniilidis, 2000)

Let C be a closed and pointed cone in a Banach space X. Then 0_X is a denting point of C if and only if it is a point of continuity for C.

Denting points, points of continuity, and cones

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A positive answer for non closed cones

Example (GC–Melguizo–Montesinos, 2015)

Let us define $X := \mathbb{R}^2$ and $C := \mathbb{R} \times (0, +\infty) \cup \{(0,0)\}$ which is a pointed cone. Then 0_X is point of continuity for C but it is not a denting point.

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Our research focuses on finding assumptions which provide such an equivalence

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$$[x, y] := \{z \in X : x \leq z \leq y\}$$

Theorem (Kountzakis–Polyrakis, 2006)

Let X be a normed space such that $\exists f \in C^*$ such that $X^* = \overline{\cup_{n \geq 1}[-nf, nf]}$. Then 0_X is a denting point of a pointed cone C if and only if it is a point of continuity for C.

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$$\text{Given } C \subset X \Rightarrow \widetilde{C} \text{ denotes the closure of } C \text{ in } (X^{**}, \text{weak}^*)$$

Theorem (GC-Melguizo-Montesinos, 2015)

Let X be a normed space, 0_X is a denting point of a pointed cone C if and only if it is a point of continuity for C and $\widetilde{C} \subset X^{**}$ is pointed.

Theorem 1 (GC-Melguizo)

Let X be a normed space and $C \subset X$ a pointed cone. The following are equivalent:

(i) 0_X is a denting point of C.

(ii) There exist $n \in \mathbb{N}$, $\{f_i\}_{i=1}^n \subset C^*$, and $\{\lambda_i\}_{i=1}^n \subset (0, +\infty)$ such that the set, $\cap_{i=1}^n \{f_i < \lambda_i\} \cap C$, is bounded.

(iii) 0_X is a point of continuity for C and $\overline{C^* - C^*} = X^*$ (i.e., C^* is quasi-generating).

(iv) $\exists f \in C^*$ such that $X^* = \cup_{n \geq 1} [-nf, nf]$ (i.e., C^* has an order unit).

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Example 1 (GC-Melguizo)

Let Γ be an abstract nonempty set, consider the vector space

$$c_{00}(\Gamma) := \{(x_\gamma)_{\gamma\in\Gamma} \in l_\infty(\Gamma) \colon \{\gamma \in \Gamma \colon x_\gamma \neq 0\} \text{ is finite } \},$$

the non-complete normed space $(c_{00}(\Gamma), \| \ \|_\infty)$, where

$$\| \ (x_\gamma)_{\gamma\in\Gamma} \ \|_\infty := \sup\{|x_\gamma| \colon \gamma \in \Gamma\},$$

and the order cone

$$c_{00}(\Gamma)^+ := \{(x_\gamma)_{\gamma\in\Gamma} \in c_{00}(\Gamma) \colon x_\gamma \geq 0, \ \forall \gamma \in \Gamma\}.$$

Then the dual cone $(c_{00}(\Gamma)^+)^* \subset (c_{00}(\Gamma), \| \ \|_\infty)^*$ is quasi-generating and the origin is not a point of continuity for $c_{00}(\Gamma)^+$.

Example 2 (GC-Melguizo)

Let us consider the non-complete normed space $(C_{00}(\mathbb{R}), \|\ \|_\infty)$, where $\| f \|_\infty := \sup\{|f(x)| \colon x \in \mathbb{R}\}$ and the order cone

$$C_{00}(\mathbb{R})^+ := \{f \in C_{00}(\mathbb{R}) \colon f(x) \geq 0, \forall x \in \mathbb{R}\}.$$

Then the dual cone $(C_{00}(\mathbb{R})^+)^* \subset (C_{00}(\mathbb{R}), \|\ \|_\infty)^*$ is quasi-generating and the origin is not a point of continuity for $C_{00}(\mathbb{R})^+$.

Example 3 (GC-Melguizo)

Let us fix any $k \geq 1$, consider the vector space $C^k[a,b]$ of all functions on $[a,b]$ that have k continuous derivatives, the non-complete normed space $(C^k[a,b], \| \ \|_\infty)$, where $\| f \|_\infty := \sup\{|f(x)| \colon x \in [a,b]\}$, and the order cone

$$C^k[a,b]^+ := \{f \in C^k[a,b] \colon f(x) \geq 0, \ \forall x \in [a,b]\}.$$

Then the dual cone $(C^k[a,b]^+)^* \subset (C^k[a,b], \| \ \|_\infty)^*$ is quasi-generating and the origin is not a point of continuity for $C^k[a,b]^+$.

Definition

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Corollary 1 (GC-Melguizo)

Let X be a normed space and $C \subset X$ a normal pointed cone. Then 0_X is a point of continuity for C if and only if it is a denting point of C.

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The answer to Gong's question is negative for arbitrary pointed cones in normed vector lattices (framework for convex programming)

Corollary 2 (GC-Melguizo)

Let X be a normed space with a quasi-generating order cone $C \subset X$. If the origin is denting in C, then the following statements hold true:

- **(i)** Every linear and positive operator $T : X^* \to X^*$ is continuous. In addition, if T is not a multiple of the identity, then it has a nontrivial hyperinvariant subspace.
- **(ii)** If a positive contraction $T : X^* \to X^*$ has 1 as an eigenvalue, then there exits an $0 < f \in X^{**}$ such that $T'f = f$.

Corollary 3 (GC-Melguizo)

Let X be a normed space and C a pointed cone. If 0_X is a point of continuity for C and $C^* \subset X^*$ is quasi-generating, then each weakly compact subset of X has super efficient points.

It is known that (even for Banach spaces)

$$C \text{ closed} \not\Rightarrow \overline{C^* - C^*} = X^* \not\Rightarrow C \text{ is closed}$$

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Problem

Is the class of cones with a quasi-generating dual a maximal one for which Gong's question has a negative answer?

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Do our results hold true in the context of locally convex spaces? If so, do they have interesting applications or consequences?

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