Factorization of Multilinear Operators via Σ-Operators

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Σ-Operators

The Segre Cone Σ

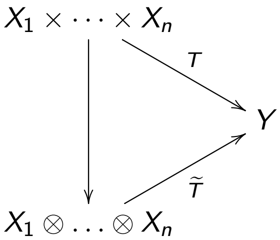
Throughout the presentation $n \in \mathbb{N}$ and $X_1, \ldots, X_n$, Y are Banach spaces.

Consider the algebraic tensor product $X_1 \otimes \ldots \otimes X_n$. Define

$$\Sigma_{X_1 \ldots X_n} := \{\, x^1 \otimes \ldots \otimes x^n \mid x^i \in X_i \,\}\,.$$

Linearization of Multilinear Operators

Let T be a multilinear operator and $\widetilde{T}$ its linearization

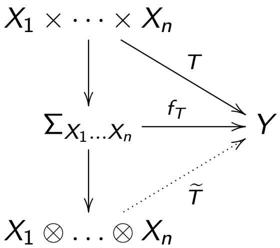


$$\begin{aligned} \widetilde{T} : X_1 \otimes \ldots \otimes X_n &\to Y \\ x^1 \otimes \ldots \otimes x^n &\mapsto T(x^1, \ldots, x^n) \end{aligned}$$

The linear operator $\widetilde{T}$ is unique.

Σ-Operators

Let T be a multilinear operator. Define



$$\begin{array}{rcl} f_T : \Sigma_{X_1 \dots X_n} & \to & Y \\ x^1 \otimes \dots \otimes x^n & \mapsto & T(x^1, \dots, x^n) \end{array}$$

In this situation, f_T **and** T **are called associated**.

Σ-Operators: Tensorial Representation

The collection of Σ-operators admits a tensorial representation as follows: For every Banach spaces

$$\begin{array}{rcl} (X_1 \otimes \ldots \otimes X_n \otimes Y)^* & \to & L(\Sigma_{X_1 \ldots X_n}; Y^*) \\ \varphi & \mapsto & f_\varphi \end{array}$$

is a linear isomorphism, where

$$\begin{array}{rcl} f_\varphi(x^1 \otimes \ldots \otimes x^n) : Y & \to & \mathbb{K} \\ y & \mapsto & \varphi(x^1 \otimes \ldots \otimes x^n \otimes y). \end{array}$$

Bounded Σ-Operators

Induced Topologies

Each reasonable-cross norm β on $X_1 \otimes \ldots \otimes X_n$ defines two topologies on $\Sigma_{X_1 \ldots X_n}$:

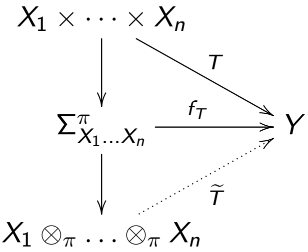
1. Strong: Unique metric topology, no matter β
2. Weak: It depends on each β

The symbol $\Sigma^{\beta}_{X_1 \ldots X_n}$ denotes the metric space and recalls the isometry

$$\Sigma^{\beta}_{X_1 \ldots X_n} \to (X_1 \otimes \ldots \otimes X_n, \beta) .$$

Bounded Σ-Operators

The following are equivalent:



1. T is bounded
2. f_T **is Lipschtz**
3. $\widetilde{T}$ is bounded

In this situation, $\|T\| = Lip^{\pi}(f_T) = \|\widetilde{T}\|$.

Bounded Σ-Operators

The Σ-operator $f_T : \Sigma^{\beta}_{X_1 \dots X_n} \to Y$ is called **bounded** if the associated multilinear operator $T : X_1 \times \cdots \times X_n \to Y$ **is bounded**.

The symbol $\mathcal{L}\left(\Sigma^{\beta}_{X_1 \dots X_n}, Y\right)$ denotes the collection of Σ-operators with the Lipschitz norm Lip^{β}.

Bounded Σ-Operators: The Norm π^β

Define π^β on $X_1 \otimes \ldots \otimes X_n \otimes Y$ by

$$\pi^\beta(u) = \inf \left\{ \sum_i \beta(p_i - q_i) \left\| y_i \right\| \,\middle|\, u = \sum_i (p_i - q_i) \otimes y_i \right\}.$$

Then

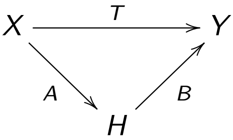
$$\begin{array}{ccc} \left(X_1 \otimes \ldots \otimes X_n \otimes Y, \pi^\beta \right)^* & \to & \mathcal{L}\left(\Sigma^\beta_{X_1 \ldots X_n}, Y^* \right) \\ \varphi & \mapsto & f_\varphi \end{array}$$

holds linearly and isometrically.

Factorization through a Hilbert Space

Factor through a Hilbert space: Linear Definition

The linear operator $T : X \to Y$ **factors through a Hilbert space** if:



1. H Hilbert space
2. $A : X \to H$ bounded
3. $B : H \to Y$ bounded

Define $\Gamma(T) := \text{ínf} \, \|A\| \, \|B\|$.

Factor through a Hilbert Space: Sequence Characterization

Given (x_i) and (y_i) finite sequences in X, the symbol $(x_i) \leq (y_i)$ means

$$\sum_i |x^*(x_i)|^2 \leq \sum_i |x^*(y_i)|^2$$

holds for all $x^* \in X^*$.

Factor through a Hilbert Space: Sequence Characterization

The linear operator $T : X \to Y$ factors through a Hilbert space iff there exist a constant $C > 0$ such that

$$(x_i) \leq (y_i) \Rightarrow \sum_i \|T(x_i)\|^2 \leq C^2 \sum_i \|(y_i)\|^2.$$

In this case $\Gamma(T) = \text{ínf}\, C$.

Factor through a Hilbert space: Tensorial Representation

Define on $X \otimes Y$

$$\gamma_2(u) = \text{ínf}\left\{ \|(x_i)\|_2 \, \|(y_i)\|_2 \;\Big|\; u = \sum_i x_i \otimes y_i \right\}.$$

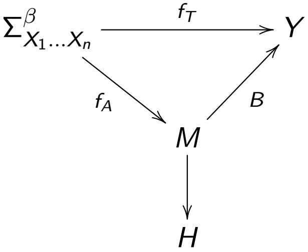
Then

$$\begin{array}{rcl} (X \otimes_{\gamma_2} Y)^* & \to & \Gamma_2\,(X; Y^*) \\ \varphi & \mapsto & T_\varphi \end{array}$$

holds linearly and isometrically.

Factor through a Hilbert Space: Factorization

The Σ-operators $f_T : \Sigma^{\beta}_{X_1 \ldots X_n} \to Y$ **factors through a Hilbert space** if:



1. H Hilbert space
2. $f_A : \Sigma^{\beta}_{X_1 \ldots X_n} \to H$ bounded
3. $B : M \to Y$ Lipschitz

We define $\Gamma(f) := \text{ínf}\, Lip^{\beta}(f_A)\, Lip(B)$

Factor through a Hilbert Space: Sequence Characterization

Given (p_i), (q_i), (a_i), (b_i) finite sequences in $\Sigma^{\beta}_{X_1 \ldots X_n}$, we write $(p_i, q_i) \leq_{\beta} (a_i, b_i)$ if

$$\sum_i |\varphi(p_i - q_i)|^2 \leq \sum_i |\varphi(a_i - b_i)|^2$$

holds for all $\varphi \in (X_1 \otimes \ldots \otimes X_n, \beta)^*$.

Factor through a Hilbert Space: Sequence Characterization

The Σ-operator $f : \Sigma^{\beta}_{X_1 \ldots X_n} \to Y$ FTH iff there exist $C > 0$ such that

$$(p_i, q_i) \leq_\beta (a_i, b_i) \Rightarrow \sum_i \|f(p_i) - f(q_i)\|^2 \leq C^2 \sum_i \beta(a_i - b_i)^2.$$

Under these circumstanses $\Gamma(f) = \text{ínf}\, C$.

Factor through a Hilber Space: The Norm γ_2^β

Define γ_2^β on $X_1 \otimes \ldots \otimes X_n \otimes Y$ by $\gamma_2^\beta(u)$

$$\text{ínf}\left\{ \|(a_j - b_j)\|_2 \, \|(y_i)\|_2 \;\middle|\; u = \sum_i (p_i - q_i) \otimes y_i,\ (p_i, q_i) \leq_\beta (a_j, b_j) \right\}.$$

Then

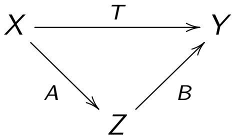
$$\begin{array}{ccc} \left(X_1 \otimes \ldots \otimes X_n \otimes Y, \gamma_2^\beta\right)^* & \to & \Gamma\left(\Sigma^\beta_{X_1 \ldots X_n}; Y^*\right) \\ \varphi & \mapsto & f_\varphi \end{array}$$

holds linearly and isometrically.

2-Dominated Σ-operators

2-Dominated Operators: Definition by Factorization

The linear operator $T : X \to Y$ **is 2-dominated** if T factors as follows:



1. Z Banach space
2. $A : X \to Z$, 2-summing
3. $B : Z \to Y$, B^* is 2-summing

Define $D_2(T) = \text{ínf}\, \pi_2(A)\, \pi_2(B^*)$.

2-Dominated Operators: Domination

The linear operator $T : X \to Y$ is 2-dominated iff there exists a RBP measure μ on $K := B_{X^*} \times B_{Y^{**}}$ and $C > 0$ such that

$$| \langle y^*, T(x) \rangle | \leq C \left(\int_K |x^*(x)|^2 d\mu \right)^{\frac{1}{2}} \left(\int_K |y^{**}(y^*)|^2 d\mu \right)^{\frac{1}{2}}$$

for all $x \in X$ and $y^* \in Y^*$.

In this case $D_2(T) = \text{ínf}\, C$.

2-Dominated Operators: Tensorial Representation

The tensor norm ω_2 is defined by

$$\omega_2(u) = \text{ínf}\left\{ \|(x_i)\|_2^w \, \|(y_i)\|_2^w \;\middle|\; u = \sum_i x_i \otimes y_i \right\}.$$

Then

$$\begin{aligned} (X \otimes_{\omega_2} Y)^* &= \mathcal{D}_2(X; Y^*) \\ (X \otimes_{\omega_2} Y^*)^* \cap \mathcal{L}(X, Y) &= \mathcal{D}_2(X; Y) \end{aligned}$$

holds linearly and isometrically.

2-Dominated Σ-Operators: The Norm ω_2^β

Define ω_2^β on $X_1 \otimes \ldots \otimes X_n \otimes Y$ by

$$\omega_2^\beta(u) := \text{ínf}\left\{ \|\,(p_i - q_i)\,\|_2^{w\beta} \;\; \|(y_i)\|_2^{w} \;\Big|\; u = \sum_i (p_i - q_i) \otimes y_i \right\}$$

with

$$\sup \left(\sum_i |\varphi(p_i) - \varphi(q_i)|^2 \right)^{\frac{1}{2}}$$

where the supreme is taken over all $\varphi \in (X_1 \otimes \ldots \otimes X_n, \beta)^*$ with $\|\varphi\|_\beta \leq 1$.

2-Dominated Σ-Operators: Functionals ω_2^β-Continuous

The functional $\varphi : \left(X_1 \otimes \ldots \otimes X_n \otimes Y, \omega_2^\beta\right) \to \mathbb{K}$ is bounded iff there exists a RBP measure μ on $K := B_{(X_1 \otimes \ldots \otimes X_n, \beta)^*} \times B_{Y^*}$ and $C > 0$ such that

$$|\varphi((p-q) \otimes y)| \leq C \left(\int_K |\zeta(p) - \zeta(q)|^2 d\mu \right)^{\frac{1}{2}} \left(\int_K |y^*(y)|^2 d\mu \right)^{\frac{1}{2}}$$

for all $p, q \in \Sigma^\beta_{X_1 \ldots X_n}$ and $y \in Y$.

In this case $\|\varphi\| = \text{ínf}\, C$.

2-Dominated Σ-Operators: Tensorial Definition

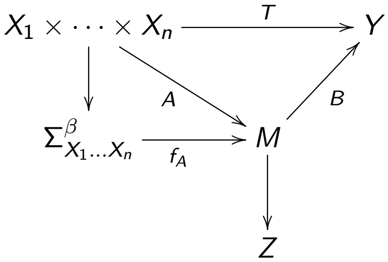
The Σ-operator $f : \Sigma^{\beta}_{X_1 \dots X_n} \to Y$ is called 2-**dominated** if the associated functional

$$\begin{array}{rcl} \varphi_f : (X_1 \otimes \ldots \otimes X_n \otimes Y^*, \omega_2^{\beta}) & \to & \mathbb{K} \\ x^1 \otimes \ldots \otimes x^n \otimes y^* & \mapsto & y^*(f(x^1 \otimes \ldots \otimes x^n)) \end{array}$$

is bounded. Define $D_2(f) = \|\varphi_f\|$.

2-Dominated Σ-Operators: Factorization

The Σ-operator $f_T : \Sigma^{\beta}_{X_1 \ldots X_n} \to Y$ is 2-dominated iff T factors as follows:



1. Z Banach space
2. $f_A : \Sigma^{\beta}_{X_1 \ldots X_n} \to M \subset Z$, 2-summing
3. $B : M \to Y$, B^* is 2-summing

Under these circumstances $D_2(f) = \text{ínf}\, \pi_2(f_A)\, \pi_2(B^*)$.

2-Dominated Σ-Operators: Factorization (Fernández M.)

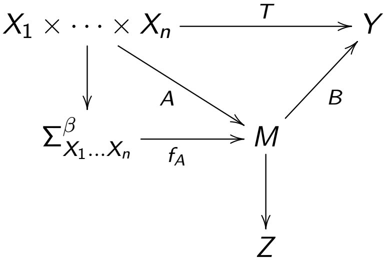
The Σ-operator $f_A : \Sigma^{\beta}_{X_1 \ldots X_n} \to Z$ is said to be 2-**summing** if there exists a constant $C > 0$ such that for all finite sequences (p_i), (q_i) in $\Sigma^{\beta}_{X_1 \ldots X_n}$

$$\sum_i \|f(p_i) - f(q_i)\|^2 \leq C^2 \,\|(p_i - q_i)\|_2^{w\beta}$$

The 2-summing norm $\pi_2(f) := \text{ínf}\, C$.

Conclusions

1. The theory of Σ-operators induces factorizations of the type



2. The theory of Σ-operators provides a new approach for the study of multilinear operators combining ideas of linear and Lipchitz theory.
3. It is possible to develop a theory of ideas of Σ-operators.
4. A new approximation of tensor norms arise (in two different versions).