Growth Rates of Frequently Hypercyclic Harmonic Functions

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(With E. Saksman and H.-O. Tylli)

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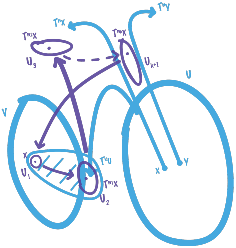

Space of Harmonic Functions

Setting

- $\mathcal{H}(\mathbb{R}^N)$ space of harmonic functions on $\mathbb{R}^N$, $N \geq 2$.
- Partial differentiation operator

$$\frac{\partial}{\partial x_j} : \mathcal{H}(\mathbb{R}^N) \to \mathcal{H}(\mathbb{R}^N)$$

$1 \leq j \leq N$.

Question

What is the *minimal growth* of a harmonic function that is *frequently hypercyclic* for $\frac{\partial}{\partial x_j}$?

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Hypercyclicity

- X separable Fréchet space.
- $T\colon X \to X$, continuous linear operator.

Definition

If there exists $x \in X$ such that

$$\overline{\{x, Tx, T^2x, T^3x, \dots\}} = X$$

then T is *hypercyclic*.

- Purely infinite dimensional phenomenon.

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Many Natural Examples

- Birkhoff (1929): translation operator

$$f(z) \mapsto f(z+a)$$

for $a \neq 0$ on the space of entire functions $H(\mathbb{C})$.

- MacLane (1952): differentiation operator on $H(\mathbb{C})$

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If there exists $x \in X$ such that for any nonempty, open $U \subset X$ we have

$$\liminf_{N\to\infty} \frac{\#\{n : T^n x \in U, 1 \leq n \leq N\}}{N} > 0$$

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- $x \in X$ a *frequently hypercyclic vector* for T.
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Growth of Harmonic Functions

L^2-norm on spheres

- $S(r)$ the sphere of radius r centred at the origin of $\mathbb{R}^N$.
- For $h \in \mathcal{H}(\mathbb{R}^N)$ and $r > 0$

$$M_2(h, r) = \left(\int_{S(r)} |h|^2 \, \mathrm{d}\sigma_r \right)^{1/2}$$

- σ_r normalised $(N-1)$-dimensional surface measure on $S(r)$.
- Growth on $S(r)$ as $r \to \infty$.

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Entire Function Case

$D\colon f \mapsto f'$

- Growth in L^p-norm, $1 \le p \le \infty$.

Hypercyclic case

- Initial estimates: MacLane (1952).
- Sharp growth: Grosse-Erdmann (1990), Shkarin (1993).

Frequently hypercyclic case

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Growth of Hypercyclic Harmonic Functions

Aldred and Armitage (1998)

1. For any $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(r) \to \infty$ as $r \to \infty$,

   $\exists h \in \mathcal{H}(\mathbb{R}^N)$, hypercyclic for $\dfrac{\partial}{\partial x_j}$ such that

   $$M_2(h, r) \leq \varphi(r) \frac{e^r}{r^{(N-1)/2}}$$

   for $r > 0$ sufficiently large.

2. $\nexists h \in \mathcal{H}(\mathbb{R}^N)$, hypercyclic for $\dfrac{\partial}{\partial x_j}$, such that

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Blasco, Bonilla and Grosse-Erdmann (2010)

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2. Let $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\psi(r) \to 0$ as $r \to \infty$.

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- Can φ be replaced with a constant in the growth rate?

Theorem (G., Saksman, Tylli)

Let $N \geq 2$ and $1 \leq j \leq N$. For any $C > 0$, there exists $h \in \mathcal{H}(\mathbb{R}^N)$, frequently hypercyclic for $\frac{\partial}{\partial x_j}$ with

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Aldred and Armitage (1998)

- For a harmonic polynomial H and $n \in \mathbb{N}$, define the n^{th} *primitive* of H
$$H \mapsto P_n(H)$$
- $P_n(H)$ a harmonic polynomial with
$$\frac{\partial^n}{\partial x_j^n} P_n(H) = H$$
- Suitable upper bounds for calculating growth.
- Brelot and Choquet 1950s.
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Construction of h

- $\mathcal{H}(\mathbb{R}^N)$ separable under topology of local uniform convergence.
- Fix a countably dense sequence of harmonic polynomials

$$(F_k) \subset \mathcal{H}(\mathbb{R}^N)$$

- Aim: construct $h \in \mathcal{H}(\mathbb{R}^N)$ to frequently approximate each F_k.
- Associate with each F_k an $\ell_k \in \mathbb{N}$.
- Sequence (ℓ_k) strictly increasing.
- Final choice for (ℓ_k) later.

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- For n odd and $n = 0$: $Q_n \equiv 0$.
- If n even: associate a fixed F_k.
- If $n < 10\ell_k$: $Q_n \equiv 0$.
- If $n \geq 10\ell_k$:

$$Q_n = P_{n^2+\ell_k}(F_k) + P_{n^2+2\ell_k}(F_k) + \cdots + P_{n^2+n}(F_k)$$

- Degrees of the primitives disjointly supported:

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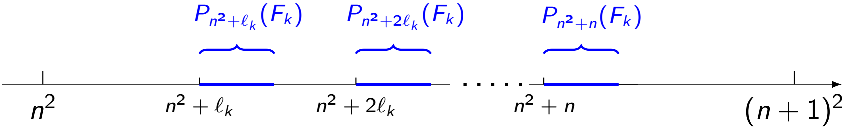

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The blocks Q_n

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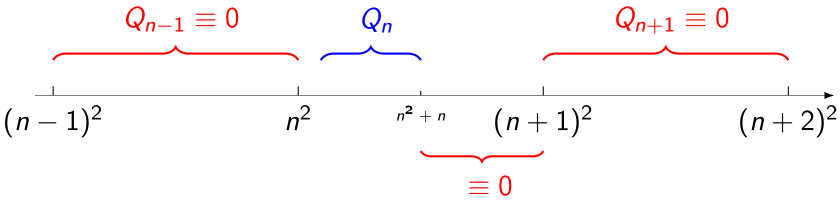

The function h

- For fixed F_k, repeat corresponding Q_n's often enough to give frequent hypercyclicity.
- Do this for every $k \geq 1$.
- Define h as

$$h = \sum_{n=1}^{\infty} Q_n$$

- Frequently hypercyclic by construction.
- Growth: for $r > 0$

$$M_2^2(h, r) = \sum_{n=1}^{\infty} M_2^2(Q_n, r)$$

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Estimates for the Primitives

Aldred and Armitage (1998)

- H a homogeneous, harmonic polynomial, $\deg H = m$.
- For $P_n(H)$

$$M_2^2\left(P_n(H), 1\right) \leq c_{n,m,N} \cdot M_2^2\left(H, 1\right)$$

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$$\left(c_{n,m,N}\right)^{1/2} \leq \frac{c_m}{(n+m)!(n+m+1)^{N/2-1}}$$

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Illustration of the Growth

- For each $k \geq 1$, require suitable upper bounds for sums of the form

$$\sum_{j=2\ell_k}^{\infty} \frac{r^{2j\ell_k}}{(j\ell_k)!^2\,(j\ell_k+1)^{N-2}}$$

and of course

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Maximum attained close to the point $x = r$

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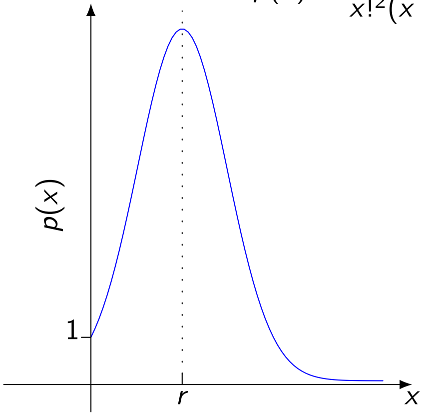

Growth of h

Barnes (1906)

$\exists C > 0$ such that $\forall r > 0$

$$\sum_{j=0}^{\infty} \frac{r^{2j}}{j!^2 (j+1)^{N-2}} < C \frac{e^{2r}}{r^{N-3/2}}$$

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$\exists C' > 0$ (independent of ℓ_k) such that $\forall r > 0$

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Growth of h

Summing up for every $k \geq 1$

$$M_2^2(h,r) = \sum_{n=1}^{\infty} M_2^2(Q_n,r) \leq \frac{e^{2r}}{r^{N-3/2}} \left(C' \sum_{k=1}^{\infty} \frac{c_{m_k} M_2^2(F_k,1)}{\ell_k} \right)$$

- Choose integers in (ℓ_k) sufficiently large to give

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Thank you for your attention ☺

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