Frequently hypercyclic bilateral shifts

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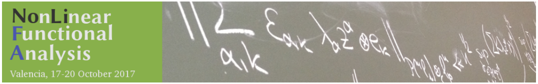

How not to solve a problem that is not due to Bayart and Grivaux

Linear dynamics

In linear dynamics, the three central notions are those of

- hypercyclic operators
- chaotic operators
- frequently hypercyclic operators

An operator $T : X \to X$ on a separable Banach space X is hypercyclic if there is some $x \in X$ (called hypercyclic vector) such that

$$\mathrm{orb}(x, T) = \{x, Tx, T^2x, \dots\} \quad \text{is dense in } X.$$

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And then the set of hypercyclic vectors is residual.
All of this thanks to Baire's theorem.

Now,

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Corollary

If T is invertible, then

$$T \text{ hypercyclic} \implies T^{-1} \text{ hypercyclic.}$$

This is a consequence of Baire.

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Corollary

If T is invertible, then

$$T \text{ chaotic} \implies T^{-1} \text{ chaotic.}$$

Another triviality...

Frequent hypercyclicity

Frequent hypercyclicity

Definition (Bayart, Grivaux 2004)

An operator $T : X \to X$ is called frequently hypercyclic if there is some $x \in X$ (called frequently hypercyclic vector) such that

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Recall that, for $A \subset \mathbb{N}_0$,

$$\underline{\text{dens}}\, A = \liminf_{N \to \infty} \frac{\#\{n \leq N : n \in A\}}{N+1}.$$

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Problem (Bayart, Grivaux 2006)

If T is invertible, do we have

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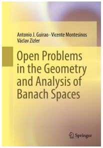
This is **Problem 44** in
A. J. Guirao, V. Montesinos, V. Zizler:
Open problems in the geometry and analysis of Banach spaces (2016).



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It is natural to look for bilateral weighted shifts as possible candidates, that is, operators of the form

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This implies that Bayart-Grivaux has a positive solution for the B_w on $\ell^p(\mathbb{Z})$.

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Theorem (Bayart, Ruzsa 2015)

An invertible B_w is frequently hypercyclic on $c_0(\mathbb{Z})$ $\Longleftrightarrow$ there is a sequence $(A_p)_{p\geq 1}$ of pairwise disjoint sets of positive lower density such that

(sym) $\forall p, q \geq 1$, $\min_{\substack{n\in A_p, m\in A_q\\ n\neq m}} |n-m| > p+q$;

$\forall p, q \geq 1$, $m \in A_q$, $n \in A_p$:

$$w_{n-m+1}\cdots w_0 < \frac{1}{2^{p+q}} \quad (n<m),$$

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(my) Conjecture

There is an invertible weighted backward shift B_w on $c_0(\mathbb{Z})$ that is frequently hypercyclic, but its inverse is not.

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Definition (Shkarin 2009)

An operator $T : X \to X$ is called upper frequently hypercyclic if there is some $x \in X$ (called upper frequently hypercyclic vector) such that

$$\forall U \neq \varnothing \text{ open}, \quad \overline{\text{dens}}\{n \geq 0\ ;\ T^n x \in U\} > 0.$$

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But upper frequent hypercyclicity is more interesting than it seems!

Theorem (Bayart, Ruzsa 2015)

If T is upper frequently hypercyclic then the set of upper frequently hypercyclic vectors for T is residual.

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Theorem (Bonilla, G-E, 2016)

Let T be an operator. The following assertions are equivalent:

(a) *T is upper frequently hypercyclic;*

(b) $\forall V \neq \varnothing$ *open*, $\exists \delta > 0, \forall U \neq \varnothing$ *open*, $\exists x \in U$

$$\overline{dens}\,\{n \geq 0 : T^n x \in V\} > \delta.$$

The proof uses, naturally, Baire's theorem.

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The proof uses, naturally, Baire's theorem.

But the symmetry of the Birkhoff transitivity theorem is lost. So we again have the following.

Problem

If T is invertible, do we have

$$T \text{ upper frequently hypercyclic } \implies T^{-1} \text{ upper frequently hypercyclic?}$$

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Again, one would first look at weighted backward shifts B_w.

Theorem (Bayart, Ruzsa 2015)

On $\ell^p(\mathbb{Z})$, $1 \leq p < \infty$,

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So the problem has a positive solution for the B_w on $\ell^p(\mathbb{Z})$.

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If B_w be an invertible weighted backward shift on $c_0(\mathbb{Z})$. Then

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Thus, for upper frequent hypercyclicity one has to look for other underlying spaces or other operators to construct counter-examples (if there are any).

Idea of the proof

The proof contains a constructive part and a Baire argument.

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Idea of the proof

Proposition 1

Let B_w be invertible on $c_0(\mathbb{Z})$. TFAE:
(a) *there is a sequence $(A_p)_{p\geq 1}$ of pairwise disjoint sets of positive upper density such that*

(sym) $$\forall p,q \geq 1,\ \min_{\substack{n\in A_p, m\in A_q\\ n\neq m}} |n-m| > p+q;$$

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(b) $B_w : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ *is upper frequently hypercyclic for* $c_0(\mathbb{Z})$.

The proof is constructive.

Proposition 2

Let B_w be invertible on $c_0(\mathbb{Z})$. TFAE:
(b) $B_w : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ *is upper frequently hypercyclic for* $c_0(\mathbb{Z})$.
(c) $B_w : c_0(\mathbb{Z}) \to c_0(\mathbb{Z})$ *is upper frequently hypercyclic.*

The proof uses Baire.

Proposition 2

Let B_w be invertible on $c_0(\mathbb{Z})$. TFAE:
(b) $B_w : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ *is upper frequently hypercyclic for* $c_0(\mathbb{Z})$.
(c) $B_w : c_0(\mathbb{Z}) \to c_0(\mathbb{Z})$ *is upper frequently hypercyclic.*

The proof uses Baire.

Thus, the upper frequent hypercyclicity of B_w on $c_0(\mathbb{Z})$ is characterized by **(Sym)**, which implies that B_w^{-1} is also upper frequently hypercyclic. □

Thank you!

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