Spectra of some algebras of entire functions of bounded type, generated by the sequence of polynomials on a Banach space

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Let X be a complex Banach space.

Let $\mathbb{P} = \{P_1, \ldots, P_n, \ldots\}$ be a sequence of continuous complex-valued polynomials on X such that

1. P_n is an n-homogeneous polynomial;
2. P_n's are algebraically independent, i.e. every polynomial $q : \mathbb{C}^n \to \mathbb{C}$, $n \in \mathbb{N}$, such that

$$q(P_1(x), \ldots, P_n(x)) = 0, \quad \forall x \in X,$$

is equal to zero:

$$q(z_1, \ldots, z_n) = 0, \quad \forall z_1, \ldots, z_n \in \mathbb{C}.$$

Let $A_{\mathbb{P}}(X)$ be the algebra of all polynomials, which are algebraic combinations of $1, P_1, \ldots, P_n, \ldots$:

$$f(x) = a_0 + \sum_{n=1}^{N} \sum_{k_1+2k_2+\ldots+nk_n=n} a_{k_1\ldots k_n} P_1(x)^{k_1} P_2(x)^{k_2} \cdots P_n(x)^{k_n}.$$

Let $H_{\mathbb{P}}(X)$ be a completion of $A_{\mathbb{P}}(X)$ in a metric, generated by a countable family of norms

$$\|f\|_r = \sup_{\|x\| \le r} |f(x)|,$$

where $r > 0$ and $r \in \mathbb{Q}$.

Note that $H_{\mathbb{P}}(X)$ is a Fréchet algebra and $H_{\mathbb{P}}(X)$ is a subalgebra of $H_b(X)$.

For example, algebras $H_{bs}(\ell_1)$ and $H_{bs}(L_\infty)$ are generated by sequences of polynomials

$$F_1(x) = \sum_{k=1}^{\infty} x_k, \ldots, F_n(x) = \sum_{k=1}^{\infty} x_k^n, \ldots$$

and

$$R_1(x) = \int_{[0,1]} x(t)\, dt, \ldots, R_n(x) = \int_{[0,1]} (x(t))^n\, dt, \ldots$$

respectively.

Let $M_{\mathbb{P}}$ be the spectrum (the set of all continuous complex-valued homomorphisms) of the algebra $H_{\mathbb{P}}(X)$.

Theorem 1.

Every $f \in H_{\mathbb{P}}(X)$ can be uniquely represented in the form

$$f(x) = f(0) + \sum_{n=1}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} a_{k_1\ldots k_n} P_1(x)^{k_1} P_2(x)^{k_2} \cdots P_n(x)^{k_n}.$$

Consequently, for every non-trivial $\varphi \in M_{\mathbb{P}}$,

$$\varphi(f) = f(0) + \sum_{n=1}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} a_{k_1\ldots k_n} \varphi(P_1)^{k_1} \varphi(P_2)^{k_2} \cdots \varphi(P_n)^{k_n}.$$

Therefore, every $\varphi \in M_{\mathbb{P}}$ is uniquely determined by the sequence

$$\varphi(P_1), \ldots, \varphi(P_n), \ldots.$$

Let X be a closed subspace of ℓ_∞ such that $X \supset c_{00}$.

Theorem 2.

Let $\mathbb{P}$ be a sequence of continuous polynomials $P_1, \ldots, P_n, \ldots$ such that

1. P_n is an n-homogeneous polynomial;
2. P_n's are algebraically independent;
3. Every P_n depends only on a finite number of coordinates.

Then every function $f \in H_{\mathbb{P}}(X)$ can be uniquely analytically extended to ℓ_∞ and algebras $H_{\mathbb{P}}(X)$ and $H_{\mathbb{P}}(\ell_\infty)$ are isometrically isomorphic.

Theorem 3.

Let $P_n : \ell_\infty \to \mathbb{C}$ be defined by

$$P_n(x) = x_n^n$$

for $x = (x_1, x_2, \ldots) \in \ell_\infty$. Then the spectrum $M_{\mathbb{P}}$ of the algebra $H_{\mathbb{P}}(X)$ coincides with the set of all point-evaluation functionals at points of ℓ_∞.

Let $X = \ell_1$ and $P_n(x) = x_n^n$.

Example.

Let $1 < p < +\infty$. A function

$$f(x) = \sum_{n=1}^{\infty} h(n)^{k(p)} P_1(x) P_2(x) \dots P_n(x),$$

where $\frac{1}{p} < k(p) < 1$ and $h(n) = 1^1 2^2 3^3 \dots n^n$, does not analytically extend to ℓ_p. But, f is well-defined on

$$x = (1, 1/2, \dots, 1/n, \dots).$$

Hypothesis.

The spectrum $M_{\mathbb{P}}$ of the algebra $H_{\mathbb{P}}(X)$ coincides with the set of all point-evaluation functionals at points of the intersection $\bigcap_{p>1} \ell_p$.