A remark on smooth images of Banach spaces

Petr Hájek, Michal Johanis

Smooth surjections

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Theorem 3 (Sean Michael Bates, 1997)

If a Banach space X has property $\mathcal{B}$, then for any separable Banach space Y there exists a C^∞-smooth mapping from X onto Y.

Smooth surjections

Theorem 4 (Petr Hájek, 1999)

Let X be a Banach space such that there exists a non-compact operator $T \in \mathcal{L}(X;\ell_p)$, $1 \leq p < \infty$. Then for any separable Banach space Y there exists a $\lceil p \rceil$-homogeneous polynomial surjection $P\colon X \to Y$.

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Theorem 4 (Petr Hájek, 1999)

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P. Hájek (1998):
There is no C^2-smooth surjection $f\colon c_0 \to \ell_2$.

Smooth surjections: the non-separable case

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- CH: There is a C^∞-smooth surjection from $c_0(\omega_1)$ onto ℓ_2. (easy)
- MA_{ω_1}: There is no C^2-smooth surjection from $c_0(\omega_1)$ onto ℓ_2 (follows from the results of P. Hájek (1998), formulated in [Guirao, Hájek, Montesinos, 2010]).

Smooth surjections: the non-separable case

Theorem 5 (Robert G. Bartle and Lawrence M. Graves, 1952)

Let X, Y be Banach spaces and let $T \in \mathcal{L}(X; Y)$ be onto. Then there is a subspace $Z \subset X$ with $\operatorname{dens} Z = \operatorname{dens} Y$ such that $T\restriction_Z$ is still surjective.

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Theorem 6 (Richard M. Aron, Jesús A. Jaramillo, and Enrico Le Donne, 2017)

Let X, Y be Banach spaces and let $f \in C^1(X; Y)$ be onto and such that the set of critical values of f has cardinality at most $\operatorname{dens} Y$. Then there is a subspace $Z \subset X$ with $\operatorname{dens} Z = \operatorname{dens} Y$ such that $f\restriction_Z$ is still surjective.

Smooth surjections: density continuum

Theorem 7 (Richard M. Aron, Jesús A. Jaramillo, and Thomas Ransford, 2013)

Let Γ be a set of cardinality at least continuum and suppose there exists a bounded linear operator $T\colon X \to c_0(\Gamma)$ such that $T(X)$ contains the canonical basis of $c_0(\Gamma)$. Then for any separable Banach space Y of dimension at least two there exists a C^∞-smooth surjective mapping $f\colon X \to Y$ such that the restriction of f onto any separable subspace of X fails to be surjective.

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The theorem holds in particular for $X = \ell_p(\Gamma)$, $\operatorname{card}\Gamma \geq \mathfrak{c}$.
Question: Does it hold also for $X = \ell_p(\Gamma)$, $\operatorname{card}\Gamma = \omega_1$? Does it involve axioms of set theory?

Smooth surjections: density ω_1

Theorem 8

Let X be a Banach space for which there is $T \in \mathcal{L}(X; \ell_p(\Gamma))$ for some uncountable Γ and $1 \leq p < \infty$ such that $T(B_X)$ contains the canonical basis of $\ell_p(\Gamma)$.

Then for any separable Banach space Y with $\dim Y \geq 2$ there is $f \in C^\infty(X; Y)$ such that $f(X) = Y$ but $f(Z) \neq Y$ for any separable subset $Z \subset X$.

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Theorem 8

Let X be a Banach space for which there is $T \in \mathcal{L}(X; \ell_p(\Gamma))$ for some uncountable Γ and $1 \leq p < \infty$ such that $T(B_X)$ contains the canonical basis of $\ell_p(\Gamma)$. (This holds in particular if X is a non-separable super-reflexive space.) Then for any separable Banach space Y with $\dim Y \geq 2$ there is $f \in C^\infty(X; Y)$ such that $f(X) = Y$ but $f(Z) \neq Y$ for any separable subset $Z \subset X$.

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Theorem 9 (Felix Hausdorff, 1936)

Every uncountable Polish space is a union of an increasing ω_1-sequence of G_δ sets.

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Theorem 10

Let X be an infinite-dimensional Banach space that admits a C^k-smooth bump, $k \in \mathbb{N} \cup \{\infty\}$, with each derivative bounded on X. Let Y be a separable Banach space, let $C \subset Y$ be convex, $y_1 \in C$, and $C \subset A \subset \overline{C}$ an analytic set. Then there is $f \in C^k(X; Y)$ with $\operatorname{supp}_\circ f \subset B_X$ such that $f(X) = [0, y_1] \cup A$.

Smooth surjections: density ω_1

Proposition 11

Let (X, ρ) be a separable metric space, $U \subset X$, $A \subset \overline{U}$ a non-empty Suslin set . Then there is an ω-branching tree T of height ω with a least element and a family $\{x_t\}_{t \in T} \subset U$ such that $A = \{\lim_{n\to\infty} x_{b_n};\ b \in \mathcal{B}(T)\}$.

Smooth surjections: density ω_1

Proposition 11

Let (X,ρ) be a separable metric space, $U \subset X$, $A \subset \overline{U}$ a non-empty Suslin set, and let $\{\varepsilon_n\}_{n=1}^{\infty} \subset (0,+\infty)$. Then there is an ω-branching tree T of height ω with a least element and a family $\{x_t\}_{t\in T} \subset U$ such that $A = \left\{\lim_{n\to\infty} x_{b_n};\ b \in \mathcal{B}(T)\right\}$ and $\rho(x_u, x_t) < \varepsilon_n$ for each $u \in t^+$, $t \in T_n$, $n \in \mathbb{N}$.

Smooth surjections: density ω_1

Theorem 12

Let X be a Banach space for which there is $T \in \mathcal{L}(X; \ell_p(\Gamma))$ for some infinite Γ and $1 \leq p < \infty$ such that $T(B_X)$ contains the canonical basis of $\ell_p(\Gamma)$. Then for every Banach space Y of density at most $\operatorname{card}\Gamma$ there exists a $\lceil p \rceil$-homogeneous polynomial surjection $P\colon X \to Y$.

Canonical basis of $\ell_p(\Gamma)$ in a linear image

Theorem 13

Let X be a Banach space, $\mu > \omega$ a regular cardinal, Γ a set, $1 < p < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$. Consider the following statements:

(i) *X is WCG with $\operatorname{dens} X \geq \mu$ and X^* is w^*-$\ell_q(\Gamma)$-generated.*

(ii) *X contains a non-zero weakly null net $\{x_\alpha\}_{\alpha\in[0,\mu)}$ and there is $T \in \mathcal{L}(X;\ell_p(\Gamma))$ such that $\operatorname{dens}\ker T < \mu$.*

(iii) *There is $T \in \mathcal{L}\big(X;\ell_p([0,\mu))\big)$ such that $\{e_\gamma\}_{\gamma\in[0,\mu)} \subset T(B_X)$.*

Then (i)⇒(ii)⇒(iii).

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Note that a Banach space contains a normalised weakly null sequence if and only if it is not a Schur space. In particular, if X is an infinite-dimensional Banach space such that X^* has the Banach-Saks property (or more generally if X^* is not a Schur space and has the weak Banach-Saks property), then X has property $\mathcal{B}$.

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Proposition 14

Let X be a Banach space with a sub-symmetric Schauder basis. Then X has property $\mathcal{B}$ if and only if X is not isomorphic to c_0.

Stuff

A tree is a partially ordered set $(T, \preceq)$ with the property that for every $t \in T$ the subset $\{s \in T;\ s \preceq t\}$ is well-ordered.
For $t \in T$ we denote by t^+ the set of all immediate successors of t, i.e.
$t^+ = \{u \in T;\ s \prec u \text{ if and only if } s \preceq t\}$.
The height of $t \in T$ is a unique ordinal $\mathrm{ht}(t)$ with the same order type as $\{s \in T;\ s \prec t\}$. The height of the tree T is defined by $\sup\{\mathrm{ht}(t) + 1;\ t \in T\}$.
A branch of T is a maximal linearly ordered subset and we denote by $\mathcal{B}(T)$ the set of all branches of T. For an ordinal α we denote by $T_\alpha = \{t \in T;\ \mathrm{ht}(t) = \alpha\}$ the αth level of the tree T.
For a branch $b \in \mathcal{B}(T)$ we denote $b_\alpha = b \cap T_\alpha$. Let μ be a cardinal. We say that T is μ-branching if $\operatorname{card} T_0 \leq \mu$ and $\operatorname{card} t^+ < \mu$ for each $t \in T$.

Stuff

Let μ be a cardinal. We say that a subset S of a topological space X is μ-Suslin in X if there is a μ-branching tree T of height ω and closed sets $F_t \subset X$, $t \in T$ such that $S = \bigcup_{b \in \mathcal{B}(T)} \bigcap_{n=1}^{\infty} F_{b_n}$.
We remark that ω-Suslin sets are called simply Suslin in the classical terminology and that a classical result states that in Polish spaces Suslin sets (our ω-Suslin sets) are precisely the analytic sets.