The Coburn-Simonenko theorem for Toeplitz operators acting between Hardy type subspaces of different Banach function spaces

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Classical Hardy spaces

Let $\mathbb{T}$ be the unit circle in the complex plane $\mathbb{C}$ equipped with the normalized Lebesgue measure $dm(t) = |dt|/(2\pi)$. For a complex-valued function $f \in L^1(\mathbb{T})$, let

$$\widehat{f}(n) := \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} d\theta, \quad n \in \mathbb{Z},$$

be the sequence of its Fourier coefficients.

For $1 \leq p \leq \infty$, consider the Hardy space

$$H^p(\mathbb{T}) := \{f \in L^p(\mathbb{T}) \ : \ \widehat{f}(n) = 0 \text{ for } n < 0\}.$$

The Riesz projection

The Riesz projection is the operator P which is defined on $\mathcal{P}$ by

$$P : \sum_{k=-n}^{n} \alpha_k t^k \mapsto \sum_{k=0}^{n} \alpha_k t^k, \quad t \in \mathbb{T}.$$

Theorem (Marcel Riesz, 1925)

If $1 < p < \infty$, then P extends to a bounded operator

$$P : L^p(\mathbb{T}) \to L^p(\mathbb{T}).$$

Toeplitz operators

For $a \in L^\infty(\mathbb{T})$, consider the Toeplitz operator

$$T(a) : H^2(\mathbb{T}) \to H^2(\mathbb{T}) \quad (\text{or} \quad T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T}), \quad 1 < p < \infty)$$

with symbol a defined by

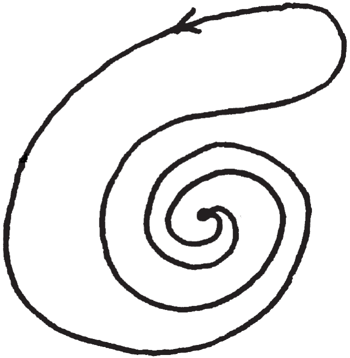
$$T(a)f = P(af).$$

We are going to extend the study of Toeplitz operators in three directions:

- replace the unit circle by a Jordan rectifiable curve
- replace Lebesgue spaces by arbitrary Banach function spaces
- consider Toeplitz operators between different Hardy type subspaces of Banach function spaces

Jordan rectifiable curves

Let Γ be a Jordan curve, that is, a curve that homeomorphic to a circle. We suppose that Γ is rectifiable and equip it with the Lebesgue length measure $|d\tau|$ and the counter-clockwise orientation:



Banach function norm I

Let

- $\mathcal{M}(\Gamma)$ be the set of all measurable complex-valued functions on Γ,
- $\mathcal{M}^+(\Gamma)$ be the subset of functions in $\mathcal{M}(\Gamma)$ whose values lie in $[0, \infty]$,
- χ_E be the characteristic function of a measurable set $E \subset \Gamma$.

A mapping

$$\rho : \mathcal{M}^+(\Gamma) \to [0, \infty]$$

is called a Banach function norm if,

- for all functions $f, g, f_n \in \mathcal{M}^+(\Gamma)$ with $n \in \mathbb{N}$,
- for all constants $a \geq 0$,
- for all measurable subsets E of Γ,

the following properties hold:

Banach function norm II (after W. Luxemburg, 1955)

(A1) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e.,

$\rho(af) = a\rho(f)$,

$\rho(f+g) \le \rho(f) + \rho(g)$,

(A2) $0 \le g \le f$ a.e. $\Rightarrow \rho(g) \le \rho(f)$ (the lattice property),

(A3) $0 \le f_n \uparrow f$ a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$ (the Fatou property),

(A4) $\rho(\chi_E) < \infty$,

(A5) $\int_E f(\tau)|d\tau| \le C_E \rho(f)$

with the constant $C_E \in (0, \infty)$ that may depend on E and ρ, but is independent of f.

Banach function spaces

When functions differing only on a set of measure zero are identified, the set $X(\Gamma)$ of all functions $f \in \mathcal{M}(\Gamma)$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X(\Gamma)$, the norm of f is defined by

$$\|f\|_{X(\Gamma)} := \rho(|f|).$$

The set $X(\Gamma)$ under the natural linear space operations and under this norm becomes a Banach space and

$$L^\infty(\Gamma) \hookrightarrow X(\Gamma) \hookrightarrow L^1(\Gamma).$$

Pointwise multipliers between Banach function spaces

For Banach function spaces $X(\Gamma)$ and $Y(\Gamma)$, let $M(X, Y)$ denote the space of pointwise multipliers from $X(\Gamma)$ to $Y(\Gamma)$ defined by

$$M(X, Y) := \{f \in \mathcal{M}(\Gamma) \ : \ fg \in Y(\Gamma) \text{ for all } g \in X(\Gamma)\}.$$

It is a Banach space with respect to the operator norm

$$\|f\|_{M(X,Y)} = \sup\{\|fg\|_{Y(\Gamma)} \ : \ g \in X(\Gamma), \ \|g\|_{X(\Gamma)} \le 1\}.$$

In particular,

$$M(X, X) \equiv L^\infty(\Gamma).$$

Pointwise multipliers for Lebesgue spaces

Warining: it may happen that the space $M(X, Y)$ contains only the zero function: if $1 \leq p < q < \infty$, then

$$M(L^p, L^q) = \{0\}.$$

If $1 \leq q \leq p \leq \infty$, then $L^p(\Gamma) \hookrightarrow L^q(\Gamma)$ and

$$M(L^p, L^q) \equiv L^r(\Gamma),$$

where $1/r = 1/q - 1/p$.

Brief (and incomplete) history

Properties of $M(X, Y)$
for general and particular spaces $X(\Gamma)$ and $Y(\Gamma)$ were systematically studied by

- Zabreiko and Rutickii (1976)
- Reisner (1981)
- Maligranda and Persson (1989)
- Nakai (1995)
- Calabuig, Delgado, Sánchez Pérez (2008)
- Maligranda and Nakai (2010)
- Kolwicz, Leśnik, Maligranda (2013-14)
- Leśnik and Tomaszewski (2016)
- Nakai (2016)

Nontrivialty of the space of pointwise multipliers

The continuous embedding

$$L^\infty(\Gamma) \hookrightarrow M(X, Y)$$

holds if and only if

$$X(\Gamma) \hookrightarrow Y(\Gamma).$$

The Cauchy singular integral operator

The Cauchy singular integral of a measurable function $f : \Gamma \to \mathbb{C}$ is defined by

$$(Sf)(t) := \lim_{\varepsilon \to 0} \frac{1}{\pi i} \int_{\Gamma \setminus \Gamma(t,\varepsilon)} \frac{f(\tau)}{\tau - t} d\tau, \quad t \in \Gamma,$$

where the "portion" $\Gamma(t, \varepsilon)$ is

$$\Gamma(t, \varepsilon) := \{\tau \in \Gamma : |\tau - t| < \varepsilon\}, \quad \varepsilon > 0.$$

It is well known that $(Sf)(t)$ exists a.e. on Γ whenever f is integrable.

Abstract Hardy spaces built upon Banach function spaces over rectifiable Jordan curves

Lemma (OK, 2003)

If $X(\Gamma)$ is a reflexive Banach function space over a rectifiable Jordan curve Γ and the operator S is bounded on $X(\Gamma)$, then the operator

$$P := (I + S)/2$$

is a bounded projection on $X(\Gamma)$, that is, $P^2 = P$.

Consider the Hardy type subspace

$$PX := PX(\Gamma)$$

built upon a Banach function space $X(\Gamma)$.

Note that for $1 < p < \infty$,

$$H^p(\mathbb{T}) = PL^p(\mathbb{T}).$$

Toeplitz operators between different abstract Hardy spaces

The idea to consider $T(a) : H^p(\mathbb{T}) \to H^q(\mathbb{T})$ goes back to Vadim Tolokonnikov (1987). It was further developed by Karol Leśnik (2016-2017), who suggested to replace $L^p(\mathbb{T})$ and $L^q(\mathbb{T})$ by rearrangement-invariant spaces $X(\mathbb{T})$ and $Y(\mathbb{T})$.

Let Γ be a rectifiable Jordan curve. Assume that $X(\Gamma)$ and $Y(\Gamma)$ are reflexive Banach function spaces and S is bounded on both $X(\Gamma)$ and $Y(\Gamma)$. For $a \in M(X, Y)$, define the Toeplitz operator

$$T(a) : PX \to PY$$

with symbol a by

$$T(a)f = P(af), \quad f \in PX.$$

It is clear that $T(a) \in \mathcal{L}(PX, PY)$ and

$$\|T(a)\|_{\mathcal{L}(PX,PY)} \leq \|P\|_{\mathcal{L}(Y)} \|a\|_{M(X,Y)}.$$

Main result

Theorem (OK, November of 2016)

Let $X(\Gamma)$ and $Y(\Gamma)$ be reflexive Banach function spaces over a rectifiable Jordan curve Γ. Suppose $X \hookrightarrow Y$ and the Cauchy singular integral operator S is bounded on X and on Y. If $a \in M(X, Y) \setminus \{0\}$, then

$$T(a) \in \mathcal{L}(PX, PY)$$

has a trivial kernel in PX or a dense image in PY.

- For $X(\mathbb{T}) = Y(\mathbb{T}) = L^2(\mathbb{T})$ this result was obtained by Lewis Coburn (1966),
- for $X(\Gamma) = Y(\Gamma) = L^p(\Gamma)$, where $1 < p < \infty$ and Γ is a sufficiently smooth Jordan curve, this result was obtained by Igor Simonenko (1968).

Nakano spaces (variable Lebesgue spaces)

Given a rectifiable Jordan curve Γ , let $\mathcal{P}(\Gamma)$ be the set of all measurable functions $p : \Gamma \to [1, \infty]$. For $p \in \mathcal{P}(\Gamma)$ put

$$\Gamma_\infty^{p(\cdot)} := \{t \in \Gamma : p(t) = \infty\}.$$

For a measurable function $f : \Gamma \to \mathbb{C}$, consider

$$\varrho_{p(\cdot)}(f) := \int_{\Gamma \setminus \Gamma_\infty^{p(\cdot)}} |f(t)|^{p(t)} |dt| + \|f\|_{L^\infty(\Gamma_\infty^{p(\cdot)})}.$$

The Nakano space (= variable Lebesgue space) $L^{p(\cdot)}(\Gamma)$ is defined as the set of all measurable functions $f : \Gamma \to \mathbb{C}$ such that $\varrho_{p(\cdot)}(f/\lambda) < \infty$ for some $\lambda > 0$. This space is a Banach function space with respect to the Luxemburg-Nakano norm given by

$$\|f\|_{L^{p(\cdot)}(\Gamma)} := \inf\{\lambda > 0 : \varrho_{p(\cdot)}(f/\lambda) \leq 1\}.$$

If $p \in \mathcal{P}(\Gamma)$ is constant, then $L^{p(\cdot)}(\Gamma)$ is nothing but the standard Lebesgue space $L^p(\Gamma)$.

Pointwise multipliers between Nakano spaces

Theorem (Nakai, 2016 (under some additional condition) and OK, January of 2017)

Let Γ *be a rectifiable Jordan curve. Suppose* $p, q, r \in \mathcal{P}(\Gamma)$ *are related by*

$$\frac{1}{q(t)} = \frac{1}{p(t)} + \frac{1}{r(t)}, \quad t \in \Gamma.$$

Then

$$M(L^{p(\cdot)}, L^{q(\cdot)}) = L^{r(\cdot)}(\Gamma).$$

Nakai additionally assumed that

$$\sup_{t \in \Gamma \setminus \Gamma_\infty^{r(\cdot)}} r(t) < \infty.$$

The Cauchy singular integral operator on Nakano spaces

Let Γ be a rectifiable Jordan curve. We say that an exponent $p \in \mathcal{P}(\Gamma)$ is locally log-Hölder continuous if

$$1 < p_- := \operatorname*{ess\,inf}_{t \in \Gamma} p(t) \leq p_+ := \operatorname*{ess\,sup}_{t \in \Gamma} p(t) < \infty$$

and there exists a constant $C_{p(\cdot),\Gamma} \in (0, \infty)$ such that

$$|p(t) - p(\tau)| \leq \frac{C_{p(\cdot),\Gamma}}{-\log|t - \tau|} \quad \text{for all} \quad t, \tau \in \Gamma \text{ satisfying } |t - \tau| < 1/2.$$

The class of all locally log-Hölder continuous exponent will be denoted by $LH(\Gamma)$.

Theorem (Kokilashvili, Paatashvili, Samko, 20006)

Let Γ be a rectifiable Jordan curve and $p \in LH(\Gamma)$. Then the Cauchy singular integral operator S is bounded on $L^{p(\cdot)}$ if and only if Γ is a Carleson curve, that is,

$$\sup_{t \in \Gamma} \sup_{\varepsilon > 0} \frac{|\Gamma(t, \varepsilon)|}{\varepsilon} < \infty.$$

The Coburn-Simonenko theorem for Nakano spaces

Theorem (OK, January of 2017)

Let Γ *be a Carleson Jordan curve. Suppose variable exponents* $p, q \in LH(\Gamma)$ *and* $r \in \mathcal{P}(\Gamma)$ *are related by*

$$\frac{1}{q(t)} = \frac{1}{p(t)} + \frac{1}{r(t)}, \quad t \in \Gamma.$$

If $a \in L^{r(\cdot)}(\Gamma) \setminus \{0\}$*, then the Toeplitz operator*

$$T(a) \in \mathcal{L}(PL^{p(\cdot)}, PL^{q(\cdot)})$$

has a trivial kernel in $PL^{p(\cdot)}$ *or a dense image in* $PL^{q(\cdot)}$.

Toeplitz operators between distinct Hardy spaces

Corollary (OK, November of 2016)

Let $1 < q \le p < \infty$ *and* $1/r = 1/q - 1/p$. *If* $a \in L^r(\mathbb{T}) \setminus \{0\}$, *then the Toeplitz operator*

$$T(a) \in \mathcal{L}(H^p(\mathbb{T}), H^q(\mathbb{T}))$$

has a trivial kernel in $H^p(\mathbb{T})$ *or a dense image in* $H^q(\mathbb{T})$.

I have never seen this result in the literature explicitly stated.
May I suppose that it is new?