$C^b(X)$ Lindelöf-Σ

M. López-Pellicer (DMA, IUMPA)

joint work with J.C. Ferrando and S. López-Alfonso

Valencia, October 17-20, 2017

Conference on Non Linear Functional

5th Workshop on Functional Analysis

UNIVERSITAT POLITÈCNICA DE VALÈNCIA

Outline

Outline

Some questions around a Talagrand' theorem.

The well known Talagrand theorem stating for a compact K that $C(K)$ is weakly K-analytic iff $C_p(K)$ is K-analytic suggest the following questions:

- The topology of $C_p(K)$, may be replaced by a weaker topology?
- Have $(C(K), \mathrm{weak})$ and $C_p(K)$ the same K-analytic mappings?
- Talagrand theorem works for a pseudocompact space K?
- There exists a Lindelöf-Σ version of Talagrand theorem for Gul'ko compacts?

In this talk we will present a positive answer of this questions based in angelicity and in some sets known as Rainwater sets.

Notations. Well known facts on βX.

X is a completely regular (Hausdorff) space.
$C^b(X)$, and $C(X)$ if X pseudocompact, are endowed with $\|\cdot\|_\infty$.
We identify

$$X \longmapsto \{\delta_x : x \in X\}$$

and

$$\beta X \text{ (Stone-Čech compactification of } X) \longmapsto \overline{\{\delta_x : x \in X\}}^{\text{weak}^*}$$

$f \mapsto f^\beta$ is a linear isometry from $C^b(X)$ onto $C(\beta X)$.

- If X is compact then $C(X)^* = rca(\mathcal{B}(X))$ (Riesz theorem).
- If X is normal, $C^b(X)^* = rba(\mathcal{B}(X)) = rca(\mathcal{B}(\beta(X)))$.

Elementary properties on υX.

Let $x \in \beta X$. Then $x \in \upsilon X$, the Hewitt realcompactification of X,

- iff each $f \in C(X)$ admits a continuous extension to $X \cup \{x\}$,
- iff $V \cap X \neq \emptyset$ for each βX-zero V containing x,
- iff $V \cap X \neq \emptyset$ for each G_δ-subset V of βX with $x \in V$.

Whence

- X is G_δ-dense in υX and
- $f \mapsto f^\nu$ is a bijection from $C(X)$ onto $C(\nu X)$.

X pseudocompact means

$$C(X) = C^b(X) \Longleftrightarrow \upsilon X = \beta X \Longleftrightarrow X \text{ is } G_\delta\text{-dense in } \beta X.$$

A sequentially continuous map.

If $Y \subset C(X)^*$ separates points in $C(X)$, then σ_Y is the Y pointwise convergence topology on $C(X)$. $C_p(X) := (C(X), \sigma_X)$.

Claim

Let Y be a G_δ dense subset of X and let $f_n \in C(X)$, for each $n \in \mathbb{N}_0$.

$$(f_n)_n \to_{\sigma_Y} f_0 \iff (f_n)_n \to_{\sigma_X} f_0$$

$$f_0 \ \sigma_Y\text{-adherent } (f_n)_n \iff f_0 \ \sigma_X\text{-adherent } (f_n)_n.$$

Hence $C_p(X)$ is angelic iff $(C(X), \sigma_Y)$ is angelic.

- In fact, fix $x \in X$ and let $X_n(x) := \{u \in X : f_n(u) = f_n(x)\}$. There exists $y_x \in Y \cap \bigcap_{n=0}^{\infty} X_n(x)$, hence $f_n(y_x) = f_n(x)$, $n \in \mathbb{N}_0$.

Rainwater sets for a Banach space E.

Definition

A subset Y of the dual closed unit ball B_{E^*} is called a *Rainwater set* for the Banach space E if for every *bounded* sequence $\{x_n\}_{n=1}^{\infty}$ of E

$$x_n \to_{\sigma_Y} x \Longrightarrow x_n \to_{weak} x,$$

equivalently, σ_Y and the weak topology have the same convergent sequences in B_E.

Each Rainwater set Y separates the points of E, i.e. $Y_\perp = \{0\}$, because if $x \in Y_\perp$ then

$$((x_n = x)_n \to_{\sigma_Y} 0) \implies ((x_n = x)_n \to_{weak} 0) \Longrightarrow x = 0.$$

James boundaries and Rainwater theorem.

Definition

Let E be a Banach space. A subset J of B_{E^*} is a James boundary for B_{E^*} if for each $x \in E$ there exists $x' \in J$ such that $x'(x) = \|x\|$.

- By Corollary 11 in 1972 Simons' paper *A convergence theorem with boundary*, each James boundary J for B_{E^*} is a Rainwater set for E (trivially the converse is not true).
- In particular, as $\mathrm{Ext}\, B_{E^*}$ is a James boundary for B_{E^*}, then $\mathrm{Ext}\, B_{E^*}$ is a Rainwater set for E.

The fact that $\mathrm{Ext}\, B_{E^*}$ is a Rainwater set appears in 1963 Rainwater's paper *Weak convergence of bounded sequences*. It follows also from Choquet's integral representation theorem.

Grothendieck spaces and Rainwater sets

Definition

A Banach space E is a Grothendieck space if in E^* the weak* and the weak convergent sequences are the same.

A ring $\mathcal{R}$ of subsets of a set Ω has property $\mathcal{G}$ if $\ell^\infty(\mathcal{R})$ is a Grothendieck space.
This happens iff each bounded sequence of $\ell^\infty(\mathcal{R})^* = ba(\mathcal{R})$ which converges pointwise on $\mathcal{R}$ converges weakly.
In other words and denoting by $\mathcal{R}$ the embedding of $\mathcal{R}$ in $ba(\mathcal{R})^*$ we have:

Proposition

$\mathcal{R}$ has property $\mathcal{G}$ if and only if $\mathcal{R}$ is a Rainwater set for $ba(\mathcal{R})$.

Rainwater theorem for $C(X)$.

Theorem (Rainwater's theorem for $C(X)$)

Each compact X is a Rainwater set for $C(X)$.

Proof.

By Arens-Kelly theorem, $\operatorname{Ext} B_{C(X)^*} = \{\pm\delta_x : x \in X\}$.
Hence $\{\pm\delta_x : x \in X\}$, and also X, is a $C(X)$-Rainwater set □

Other proof.

Proof.

By Riesz representation theorem, $C(X)^* = rca(\mathcal{B}(X))$.
If $\{f_n\}_{n=1}^{\infty}$ is $C(X)$-bounded and $f_n \to_{\sigma X} f$, $\forall x \in X$,
then, by Lebesgue dominated convergence theorem,
$\int f_n d\mu \to \int f d\mu$ for every $\mu \in C(X)^*$, i.e., $f_n \to_{weak} f$. □

G_δ-dense subsets of Rainwater sets.

Proposition

Let Y be $C^b(X)$-Rainwater set and let Z be a G_δ-dense subset of a Y. Then Z is a Rainwater set for $C^b(X)$.

Proof.

Let $\{f_n\}_{n=1}^{\infty}$ be bounded in $C^b(X)$ with $f_n \to_{\sigma_Z} f$.
By G_δ-density of Z in Y we get that $f_n \to_{\sigma_Y} f$
As Y is a Rainwater set for $C^b(X)$ then $f_n \to_{\text{weak}} f$.
Therefore Z is a Rainwater set for $C^b(X)$. □

(In brief: σ_Z and σ_Y have the same convergent sequences)
If Z is a dense C-embedded subspace of Y then $Y \subset \upsilon Z$.
Hence Z is G_δ-dense in Y.

Outline

2. Topological properties of some Rainwater sets for $C^b(X)$.
   - $C(X)$ Rainwater sets contained in X.
   - Other Rainwater sets for $C^b(X)$.

$C(X)$ Rainwater subsets of a compact X.

Proposition

For a subset Y of a compact X are equivalent

1. *Y is G_δ-dense in X.*
2. *Y is a James boundary for $B_{C(X)^*}$.*
3. *Y is a Rainwater set for $C(X)$.*

Proof.

1 $\Rightarrow$ 2. If $f \in C(X)$ then $\{x \in X : |f(x)| = \|f\|_\infty\}$ is a nonempty G_δ-subset of X, hence it meets Y. 2 $\Rightarrow$ 3 (Simons' corollary). No 1 $\Rightarrow$ No 3. If there exists a nonvoid G_δ-closed $Z \subset X$ with $Z \cap Y = \emptyset$, then $\exists\, f \in C(X)$, $f(X) = [0,1]$, $Z = f^{-1}(\{1\})$ and

$$\{f^n\}_{n=1}^\infty \to_{\sigma_Y} 0 \quad \text{and} \quad \langle f^n, \delta_z \rangle \to 1 \quad \text{if} \quad z \in Z, \quad \text{(3 fails)}$$

$C^b(X)$ Rainwater subsets of X.

Corollary

A subset Y of X is a Rainwater set for $C^b(X)$ if and only if Y is G_δ-dense in X and X is pseudocompact.

Proof.

Y Rainwater set for $C^b(X) \iff$
Y Rainwater set for $C(\beta X) \iff$
Y is G_δ-dense in $\beta X \iff$
Y is G_δ-dense in X and X is G_δ-dense in βX.
Finally X is G_δ-dense in $\beta X \iff X$ is pseudocompact. □

Example

An infinite discrete space I is not Rainwater set for $\ell_\infty(I)$.

Proper G_δ-dense subsets in compacts.

Recall that if $\aleph_0 < \kappa$ the Σ-product $\Sigma\mathbb{R}^\kappa$ is

$$\Sigma\mathbb{R}^\kappa := \{x \in \mathbb{R}^\kappa : |\{i : x_i \neq 0\}| \leq \aleph_0\}.$$

$\Sigma\mathbb{R}^\kappa$ is Fréchet-Urysohn, i.e., if $x \in \overline{A}$ then $x = \lim_n a_n$, $a_n \in A$.
Each compact K embeds in some $[0,1]^\kappa$.
$Y = K \cap \Sigma\mathbb{R}^\kappa$ is sequentially compact, hence Y is dense in K iff Y is G_δ-dense in K, iff Y is a Rainwater set for $C(K)$.
Then K is a Valdivia compact.
K is Corson compact if K is homeomorphic to a subset of $\Sigma\mathbb{R}^\kappa$.

Example

Each Valdivia and non Corson compact K, for instance $[0,1]^\kappa$, with $\aleph_0 < \kappa$, contains a proper Rainwater subset Y for $C(K)$.

Examples: Rainwater sets non pseudocompact.

Example

Let X be the one-point compactification of an uncountable discrete space Y.
The non pseudocompact Y is G_δ-dense in X.
Hence Y is a Rainwater set for $C(X)$.

Example

If G is pseudocompact then $G \times G$ is G_δ-dense in the pseudocompact $G \times \beta G$ (pseudocompact $\times$ compact).
Hence $G \times G$ is a Rainwater set for $C(G \times \beta G)$.
Additionally $G \times G$ may be not pseudocompact (see Engelking).

Norming and Rainwater sets for a Banach space E.

Proposition

Let Y be an E-norming subset of B_{E^*}. If $Y \subseteq Z \subseteq B_{E^*}$ and Y is a Rainwater set for $C^b(Z)$, then Y is a Rainwater set for E.

Proof.

As Y(norming) $\subseteq Z \subseteq B_{E^*}$, the restriction $T : E \to C^b(Z)$, $Tu = u|_Z$, is an embedding (there exists $0 < k \leq 1$ such that

$$k \|u\| \leq \sup_{y \in Y} |u(y)| \leq \|Tu\|_\infty = \sup_{x \in Z} |u(x)| \leq \|u\|).$$

Let $(u_n)_{n=1}^\infty$ E-bounded and $u_n \to_{\sigma Y} 0$. By Y-Rainwaterness $\langle Tu_n, \mu \rangle \to 0$ for all $\mu \in C^b(Z)^*$. Hence $\langle u_n, T^*\mu \rangle \to 0$, if $T^*\mu \in T^*\left(C^b(Z)^*\right) = E^*$ (as T embeds), i.e., $u_n \to_{\text{weak}} 0$. □

$C^b(X)$ Rainwater sets containing X.

Corollary

Let $X \subset Y \subset B_{C^b(X)^*}$ and $(Y, \mathrm{weak}^*|_Y)$ pseudocompact. Then Y is a Rainwater set for $C^b(X)$.

Proof.

$X \subset Y \subset B_{C^b(X)^*}$ imply that Y is $C^b(X)$-norming
By pseudocompactnes, Y is a $C^b((Y, \mathrm{weak}^*|_Y))$-Rainwater.
Apply preceding Proposition with $Z := Y$. □

Example

If D is a dense subset of $\beta\mathbb{N} \setminus \mathbb{N}$, then $\mathbb{N} \cup D$ is pseudocompact, hence $Y = \mathbb{N} \cup D$ is a Rainwater set for ℓ_∞.

Outline

3. Analytic applications.
   - Rainwater sets and weak K-analyticity in $C^b(X)$.
   - A characterization of Talagrand and Gul'ko compact sets.

Some definitions.

Definition

A topological space X is Lindelöf-Σ if it exists an onto uscc $T : \Sigma(\subset \mathbb{N}^{\mathbb{N}}) \to \mathcal{K}(X)$. If $\Sigma = \mathbb{N}^{\mathbb{N}}$ then X is K-analytic.

T is named a Lindelöf-Σ (resp. K-analytic) map.
Usc means $[(\alpha_n)_n \to_\Sigma \alpha; (x_n \in T(\alpha_n))_n] \implies (x_n)_n \rightsquigarrow x \in T(\alpha)$
Hence, for X angelic, there exists $(x_{n_k})_k \to x \in T(\alpha)$.

K-analytic $\Rightarrow$ Lindelöf-Σ and for a Banach space E,

$$(E, \mathrm{weak}) \text{ Lindelöf-}\Sigma \Rightarrow (B_{E^*}, weak^*|_{B_{E^*}}) \text{ Corson compact}$$

If (E, weak) is Lindelöf-Σ it is said that the Banach space E is a *WCD* space or a Vašák space.
E is *WLD* if $\left(B_{E^*}, \mathrm{weak}^*|_{B_{E^*}}\right)$ is Corson compact.

WLD and pseudocompactness.

Recall that if $x_n \in \beta X$, $x_m \neq x_n$ when $m \neq n \in \mathbb{N}$, then

$$\lim_{n\to\infty} x_n = x \Longrightarrow x \in \upsilon X$$

Lemma

If $C^b(X)$ is WLD then X is pseudocompact, so a Rainwater set.

Proof.

$WLD \iff \left(B_{C^b(X)^*}, weak^*|_{B_{C^b(X)^*}}\right)$ is Corson compact.
$\Longrightarrow \beta X$ is Corson compact$\Longrightarrow \beta X$ is Fréchet-Urysohn.
Then for $x \in \beta X \backslash X$ we have:
$x = \lim_n x_n$, with $x_n \in X$, $x_m \neq x_n$ when $m \neq n$.
Hence $x \in \upsilon X$ and then $\beta X = \upsilon X$. □

Lindelöf-Σ (K-analytic) equivalence.

Let τ_1 and τ_2 be two topologies on a space X. $\tau_1(\text{L-}\Sigma)\ \tau_2$ if τ_1 and τ_2 have the same nonvoid set of Lindelöf-Σ mappings. Let τ_1 Lindelöf-Σ. If τ_1 and τ_2 have the same compact subsets and coincide in the separable subsets then $\tau_1(\text{L-}\Sigma)\ \tau_2$. Hence if τ_1 and τ_2 are angelic with the same convergent sequences then $\tau_1(\text{L-}\Sigma)\ \tau_2$.

Proposition

Let Y be a Rainwater set for the Banach space E. If (E, σ_Y) is Lindelöf Σ-space and angelic, then weak$|_{B(E)}(\text{L-}\Sigma)\ \sigma_Y|_{B(E)}$.

Proof.

Both topologies are angelic with the same convergent sequences in $B(E)$. □

$C^b(X)$ weakly Lindelöf Σ.

Theorem

Let X be completely regular. The following are equivalent:

1. *$C^b(X)$ is weakly Lindelöf Σ-space (= WCD).*
2. *There exists a Rainwater set Y for $C^b(X)$ such that $(C^b(X), \sigma_Y)$ is both Lindelöf Σ-space and angelic.*
3. $\sigma_Y|_{B_{C^b(X)}}(\mathrm{L}\text{-}\Sigma)\mathrm{weak}|_{B_{C^b(X)}}$.

Proof.

$1 \Rightarrow 2$ Take $Y := X$.
$2 \Rightarrow 3$ Apply Proposition.
$3 \Rightarrow 1$ From 3 and $C^b(X) = \bigcup_{n=1}^{\infty} nB_{C^b(X)}$ follows that $(C^b(X), \mathrm{weak})$ is Lindelöf Σ-space. □

A particular case.

Corollary

If Y is a Rainwater set for $C^b(X)$ and (Y, weak^*) is pseudocompact then the following are equivalent:

1. $C^b(X)$ *is weakly K-analytic (resp. WCD).*
2. $(C^b(X), \sigma_Y)$ *is K-analytic (resp. Lindelöf Σ-space).*
3. $\sigma_Y|_{B_{C^b(X)}}$ (K-analytic) (*resp.* (L-Σ)) $\mathrm{weak}|_{B_{C^b(X)}}$.

In particular, this corollary applies when X is pseudocompact.

Proof.

Pseudocompactness imply that $C_p((Y, \mathrm{weak}^*))$ is angelic.
Then apply last theorem. □

Example.

Example

The Banach spaces $C([0,\omega_1])$ and $C([0,\omega_1[)$ are not *WLD*, hence $C_p([0,\omega_1])$ and $C_p([0,\omega_1[)$ are not Lindelöf Σ-space.

Proof.

- $[0,\omega_1]$ is not Fréchet-Urysohn, because the non closed $[0,\omega_1[$ is sequentially closed.
- Hence $C([0,\omega_1])$, and by isometry $C([0,\omega_1[)$, are not *WLD*.
- Therefore $C([0,\omega_1])$ and $C([0,\omega_1[)$ are not *WCD*.
- $[0,\omega_1[$ is pseudocompact.
- By Corollary $C_p([0,\omega_1])$ and $C_p([0,\omega_1[)$ are not Lindelöf Σ-space.

□

Talagrand and Gul'ko compact sets.

For a compact space K we have

1. K is Talagrand compact if $C_p(K)$ is K-analytic.
2. K is Gul'ko compact if $C_p(K)$ is Lindelöf-Σ,

then

$$C_p(K) \text{ Lindelöf-}\Sigma \Longrightarrow K \text{ Corson compact} \quad (\not\Leftarrow \text{ Sokolov })$$

and remind that

$$K \text{ Corson compact} \Longrightarrow C_p(K) \text{ Lindelöf } (\not\Leftarrow \text{ Gul'ko, Alster, Pol })$$

G_δ density and analytic properties.

Theorem

Let Y be a G_δ-dense subset of a pseudocompact X.

1. $C(X)$ *is weakly K-analytic (resp. WCD)* $\Leftrightarrow$
2. $C_p(X)$ *is K-analytic (resp. Lindelöf Σ-space)* $\Leftrightarrow$
3. $(C(X), \sigma_Y)$ *is K-analytic (resp. Lindelöf Σ-space)* $\Leftrightarrow$
4. $\sigma_Y|_{B_{C(X)}}$ (K-analytic) (*resp.* (L-Σ)) weak$|_{B_{C(X)}}$.

Proof.

$3 \Rightarrow 4$ Apply Claim to get angelicity of $(C(X), \sigma_Y)$.
As Y is Rainwater the topologies σ_Y and weak have the same convergent sequences in $B_{C(X)}$.
$4 \Rightarrow 1$ $C(X) = \bigcup_{n=1}^{\infty} nB_{C(X)}$. The rest is obvious. □

A remark on Talagrand and Gul'ko compact sets.

Corollary

Let Y be a G_δ-dense subset of a compact space K. The following are equivalent:

1. *$C(K)$ is weakly K-analytic (resp. WCD).*
2. *K Talagrand compact (Gul'ko compact).*
3. *$(C(K), \sigma_Y)$ is K-analytic (resp. Lindelöf Σ-space).*
4. $\sigma_Y|_{B_{C(K)}}$ (K-analytic) (*resp.* (L-Σ)) weak$|_{B_{C(K)}}$.

Proof.

Apply directly the preceding theorem because

- K is Talagrand compact if $C_p(K)$ is K-analytic and
- K is Gul'ko compact if $C_p(K)$ is Lindelöf-Σ.

Example.

Let $\{x_0\}$ be a non G_δ-subset of a pseudocompact set X. As $Y := X \backslash \{x_0\}$ is G_δ-dense in X, the last theorem implies:

Remark

1. $C(X)$ *is weakly* K*-analytic* (*resp. WCD*) $\Leftrightarrow$
2. $C_p(X)$ *is* K*-analytic* (*resp. Lindelöf* Σ*-space*) $\Leftrightarrow$
3. $(C(X), \sigma_Y)$ *is* K*-analytic* (*resp. Lindelöf* Σ*-space*) $\Leftrightarrow$
4. $\sigma_Y|_{B_{C(X)}}\text{(K-analytic)}$ (*resp.* $(\text{L-}\Sigma)$) $\text{weak}|_{B_{C(X)}}$.

Example

Let $X := [0, \omega_1]$ and $x_0 = \omega_1$.
$\left(C([0, \omega_1]), \sigma_{[0,\omega_1[}\right)$ is not Lindelöf Σ-space, by remark and the non *WCD* of $C([0, \omega_1])$.

References I

- A. R. Alimov, *The Rainwater-Simons weak convergence theorem for the Brown associated norm*, Eurasian Math. J. **5** (2014), 126-131.
- R. F. Arens and J. L. Kelley, *Characterizations of the space of continuous functions over a compact Hausdorff space*, Trans. Amer. Math. Soc. **62** (1947), 499-508.
- B. Cascales and J. Orihuela, *Angelicity and the boundary problem*, Mathematatika **45** (1988), 105-112.
- R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989.

References II

- J.C. Ferrando, J. Kąkol and M. López-Pellicer, *On spaces $C^b(X)$ weakly K-analytic*, Math. Nachr., Accepted: 27 March 2017. DOI: 10.1002/mana.201600406.
- J.-D. Hardtke, *Rainwater-Simons type convergence theorems for generalized convergence methods*. Acta Comment. Univ. Tartu. Math. **14** (2010), 65-74.
- J. Kąkol, W. Kubiś and M. López-Pellicer, *Descriptive Topology in Selected Topics of Functional Analysis*, Springer, Developments in Math. **24**, New York Dordrecht Heidelberg, 2011.
- O. Kalenda, *Valdivia compact spaces in topology and in Banach space theory*, Extracta Math. **15** (2000), 1-85.

References III

- O. Kalenda, *A characterization of Valdivia compact spaces,* Collect. Math. **51** (2000), 59-81.
- D. M. King and S. A. Morris, *The Stone-Čech compactification and weakly Fréchet spaces*, Bull. Austral. Math. Soc. **42** (1990), 349-352.
- S. Moll and L. M. Sánchez Ruiz, *A note on a theorem of Talagrand*, Topol. Appl. **153** (2006), 2905-2907.
- O. Nygaard, *A remark on Rainwater's theorem*, Annal. Math. Inform. **32** (2005), 125-127.
- J. Rainwater, *Weak convergence of bounded sequences*, Proc. Amer. Math. Soc. **14** (1963), 999-999.

References IV

- S. Simons, *A convergence theorem with boundary*, Pacific J. Math. **40** (1972), 703-708.
- M. Talagrand, *Espaces de Banach faiblement K-analytiques*, Annals of Math. **110** (1979), 407-438.

THANK YOU VERY MUCH BY YOUR ATTENTION!!!!!!