Inversion of nonsmooth maps between Banach spaces

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Theorem (J. Hadamard, 1906; R. Plastock, 1974)

Let X, Y be Banach spaces, and $f : X \to Y$ a $\mathcal{C}^1$ map. Assume that:

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Then f is a diffeomorphism.

Remark

The former integral condition is fulfilled if

$$\sup_{x \in X} \|f'(x)^{-1}\| < \infty \qquad \text{(Hadamard-Levy condition)}.$$

Definition

Let X, Y be Banach spaces, $f : U \subset X \to Y$ a map, U open, and $x \in U$. A set $Jf(x) \subset \mathcal{L}(X, Y)$ is a pseudo-Jacobian of f at x if

$$\forall y^* \in Y^*, v \in X: \; (y^* \circ f)'_+(x; v) \leq \sup\{\langle y^*, T(v)\rangle : \; T \in Jf(x)\},$$

where $\quad (y^* \circ f)'_+(x; v) = \limsup_{t \to 0^+} \frac{(y^* \circ f)(x+tv) - (y^* \circ f)(x)}{t}.$

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The former inequality is equivalent to:

$$(y^* \circ f)'_-(x;\, v) \geq \inf\left\{\langle y^*, \; T(v)\rangle : \; T \in Jf(x)\right\},$$

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$Jf : U \to 2^{\mathcal{L}(X,Y)}$ is a pseudo-Jacobian mapping for f on U if, for all $x \in U$, $Jf(x)$ is a pseudo-Jacobian of f at x.

Example (Weakly differentiable maps)

A map $f : U \subset X \to Y$ is *weakly Gâteaux differentiable* at $x \in U$ if there exists an operator $f'(x) \in \mathcal{L}(X, Y)$ such that

$$\lim_{t \to 0} \left\langle y^*, \frac{f(x+tv) - f(x)}{t} \right\rangle = \langle y^*, f'(x)(v) \rangle, \quad \forall v \in X, y^* \in Y^*.$$

f is weakly Gâteaux differentiable at $x \Rightarrow$ the set $Jf(x) := \{f'(x)\}$ is a pseudo-Jacobian of f at x.

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Example (Gâteaux prederivatives, Ioffe)

Let $f : U \subset X \to Y$ be a map, $x \in U$. A set $Jf(x) \subset \mathcal{L}(X, Y)$ is a *Gâteaux prederivative* of f at x if, for each $\varepsilon > 0$ and each $v \in X$, there is $\delta > 0$ such that

$$f(x+tv) - f(x) \in Jf(x)(tv) + \varepsilon |t| \cdot \overline{B}_Y, \text{ whenever } |t| < \delta.$$

$Jf(x)$ is a pseudo-Jacobian of f at x.

Example (Locally Lipschitz maps)

A map $f : U \subset X \to Y$ is *locally Lipschitz* at $x \in U$ if there exist $L, r > 0$ s.t. $B(x, r) \subset U$ and f is L-Lipschitz on $B(x, r)$.
If f is locally Lipschitz at x, then the set

$$Jf(x) := \mathrm{Lip}\, f(x) \cdot \overline{B}_{\mathcal{L}(X,Y)}$$

is a pseudo-Jacobian of f at x, where

$$\mathrm{Lip}\, f(x) = \inf_{r>0} \sup \left\{ \frac{\|f(u)-f(v)\|}{\|u-v\|} : \ u, v \in B(x, r) \text{ and } u \neq v \right\}.$$

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Example (Clarke subdifferential)

Let $\varphi : U \subset X \to \mathbb{R}$ be a function, locally Lipschitz at $x \in U$. The *Clarke subdifferential* of φ at x is

$$\partial\varphi(x) := \{x^* \in X^* \ : \ x^*(v) \leq \varphi^\circ(x;\ v), \text{ for all } v \in X\},$$

where

$$\varphi^\circ(x;\ v) := \limsup_{\substack{z \to x \\ t \to 0^+}} \frac{\varphi(z+tv)-\varphi(z)}{t}.$$

$\partial\varphi(x)$ is a pseudo-Jacobian of φ at x.

Theorem (Mean value property)

Let $f : U \subset X \to Y$ be continuous, where U is open and convex. If Jf is a pseudo-Jacobian mapping for f on U, then

$$f(v) - f(u) \in \overline{\mathrm{co}}\big(Jf([u, v])\big)\,(v - u), \quad \text{for all} \quad u, v \in U.$$

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Corollary

Let $f : U \subset X \to Y$ be continuous. Then, f is locally Lipschitz on U iff f admits a pseudo-Jacobian mapping $Jf : U \to 2^{\mathcal{L}(X,Y)}$ which is locally bounded on U, i.e., for all $x \in U$ there is $r > 0$ such that

$$Jf\big(B(x,\, r)\big) = \{T : \ T \in Jf(u),\, u \in B(x,\, r)\}$$

is a bounded subset of $\mathcal{L}(X, Y)$.

Notation

In the sequel consider continuous maps $f : U \subset X \to Y$ and $g : V \subset Y \to Z$ such that $f(U) \subset V$, and a point $x \in U$.

Proposition

Let $Jg : V \to 2^{\mathcal{L}(Y,Z)}$ be a pseudo-Jacobian for g on V. If

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then the set $g'\,(f(x)) \circ Jf(x)$ is a pseudo-Jacobian of $g \circ f$ at x.

Notation

Given a pseudo-Jacobian mapping Jf for a map $f : U \subset X \to Y$ and two points $x \in U$ and $y \in Y$ with $y \neq f(x)$ we write

$$A_{x,y}(f) = \partial d_y\left(f(x)\right) \circ \overline{co}\left(Jf(x)\right),$$

where

$$d_y(v) = \|v - y\|,\ v \in Y.$$

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Definition (The chain rule condition)

Let $f : U \subset X \to Y$ be continuous. A pseudo-Jacobian mapping Jf for f on U satisfies the chain rule condition if for every $x \in U$ and every $y \in Y$ with $y \neq f(x)$:

1. $A_{x,y}(f)$ is a w^*-closed and convex subset of X^*, and
2. $A_{x,y}(f)$ is a pseudo-Jacobian of $d_y \circ f$ at x.

Proposition

If $f : U \subset X \to Y$ is continuous and Gâteaux differentiable on all of U, then the pseudo-Jacobian

$$Jf(x) = \{f'(x)\}, \quad x \in U$$

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1. Jf is locally bounded on U, and
2. For each $x \in U$, $Jf(x)$ is WOT-compact.

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Then, Jf satisfies the chain rule condition.

Definition

Let $f : U \subset X \to Y$ be a map, and let Jf be a pseudo-Jacobian mapping for f. The regularity index of Jf at a point $x \in U$ is

$$\alpha_{Jf}(x) := \inf\{/\!/ T /\!/ : T \in \mathrm{co}\, Jf(x)\},$$

where $/\!/ T /\!/ = \inf\{\|Tx\| : \|x\| = 1\}$.

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Example

If $f : U \subset X \to Y$ is weakly Gâteaux differentiable on all of U and

$$Jf = \{f'(x)\}, \quad x \in U,$$

then Jf is regular at $x \in U$ if $f'(x)$ is an isomorphism. Moreover,

$$\alpha_{Jf}(x) = \left\| f'(x)^{-1} \right\|^{-1}.$$

Theorem (Main result)

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Then, for each open ball $B(x_0, \delta) \subset U$ we have

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Then, for each open ball $B(x_0, \delta) \subset U$ we have

$$B(f(x_0); \delta\alpha) \subset f\big(B(x_0; \delta)\big).$$

Furthermore, the set $V := f(U)$ is open in Y, $f : U \to V$ is an homeomorphism, whose inverse is α^{-1}-Lipschitz on V.

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Let $f : X \to Y$ be a continuous and Gâteaux differentiable map. Suppose that $f'(x)$ is an isomorphism for each $x \in X$, and

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Then, f is a global homeomorphism from X onto Y, whose inverse is K-Lipschitz on Y.

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2. For each $x_0 \in X$ and each $\delta > 0$, we have

$$B\big(f(x_0), \rho\big) \subset f\big(B(x_0, \delta)\big), \text{ where } \rho = \int_0^\delta \inf_{\|x - x_0\| \le t} \alpha_{Jf}(x)\,\mathrm{d}t.$$

Corollary

Let $f : X \to Y$ be a locally Lipschitz map between reflexive Banach spaces and Jf be a locally bounded pseudo-Jacobian mapping for f on X. Assume that:

1. For each $x \in X$, $Jf(x)$ is convex and WOT-compact,

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Corollary (Lipschitz perturbations of smooth maps)

Let X, Y be reflexive spaces endowed with a Fréchet-smooth norm. Consider a map $f : X \to Y$ of the form $f = g + h$, where:

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