The sup-norm vs. the norm of the coefficients

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Organization of the talk

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- The basics.

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- Comparing the norms.

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- Mixed unconditionality.

The basics

Examples:

- $P_1(z_1, z_2, z_3) = 3z_1^2 - 2z_1z_2 + 7z_2^2 - 5z_3^2 + 4z_1z_3.$

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- $P_1(z_1, z_2, z_3) = 3z_1^2 - 2z_1z_2 + 7z_2^2 - 5z_3^2 + 4z_1z_3$.
- $P_2(z_1, z_2, z_3, z_4) = 5z_1^5 + \frac{3}{2}z_1z_2^4 - z_2^3z_3z_4 + 2z_3z_4^4$.

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In both cases, all the terms have the same degree ($\equiv$ homogeneity degree).

The basics

Definition

An m-homogeneous polynomial in n complex variables is a function

$$P : \mathbb{C}^n \longrightarrow \mathbb{C},$$

that can be written as

$$P(z) = \sum_{1 \le j_1 \le \cdots \le j_m \le n} c_{(j_1,\dots,j_m)}(P)\, z_{j_1} \cdots z_{j_m}$$

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- $z_{\mathbf{j}} = z_{j_1}\cdots z_{j_m} \rightsquigarrow$ *monomials*.
- $\mathcal{J}(m,n) := \{\mathbf{j} = (j_1,\dots,j_m) \in \{1,\dots,n\}^m \ : \ j_1 \le j_2 \le \cdots \le j_m\}$.

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When $m \in \mathbb{N}$ is fixed $\binom{n+m-1}{m} \sim n^m$, as n goes to infinity.

Norms on $\mathcal{P}(^m\mathbb{C}^n)$

- ℓ_p^n stands for $\mathbb{C}^n$ endowed with the norm

$$\|z\|_{\ell_p^n} := \Big(\sum_{j=1}^n |z_j|^p\Big)^{1/p} \text{ if } p < \infty,$$

and

$$\|z\|_{\ell_\infty^n} := \max_j |z_j|.$$

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For $1 \leq r < \infty$,

$$|P|_r := \Big(\sum_{\mathbf{j} \in \mathcal{J}(m,n)} |c_{\mathbf{j}}(P)|^r \Big)^{\frac{1}{r}},$$

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Question

How do they relate each to other?

The problem...

More precisely,

Let $A^m_{p,r}(n)$ and $B^m_{r,p}(n)$ be the smallest constants that fulfill the following inequalities: for every m-homogeneous polynomial P in n complex variables,

$$|P|_r \leq A^m_{p,r}(n)\ \|P\|_{\mathcal{P}(^m\ell_p^n)}, \qquad \|P\|_{\mathcal{P}(^m\ell_p^n)} \leq B^m_{r,p}(n)\ |P|_r.$$

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$$\left\| \left(\mathcal{P}(^m\mathbb{C}^n), \|\cdot\|_{\mathcal{P}(^m\ell_p^n)}\right) \xrightarrow{id} \left(\mathcal{P}(^m\mathbb{C}^n), |\cdot|_r\right) \right\| = A^m_{p,r}(n),$$

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Essentially, we want to relate the summability of the coefficients of a given homogeneous polynomial with its uniform norm for ℓ_p-spaces.

The easy part... $B^m_{r,p}(n)$

If $(a_n)_n$ and $(b_n)_n$ are two sequences of real numbers we write $a_n \ll b_n$ if there exists a constant $C > 0$ (independent of n) such that $a_n \leq C b_n$ for every n and we denote $a_n \sim b_n$ if $a_n \ll b_n$ and $b_n \ll a_n$.

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Proposition

We have

$$B^m_{r,p}(n) \sim \begin{cases} 1 & \text{for } r \leq p', \\ n^{m(1-\frac{1}{p}-\frac{1}{r})} & \text{for } r \geq p'. \end{cases}$$

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Some known inequalities... Bohnenblust and E. Hille (1931), Hardy and J. Littlewood (1934), Praciano-Pereira (1981), Dimant-Sevilla (2013).

$$(i) \qquad A^m_{p,\frac{2m}{m+1}}(n) \sim 1 \qquad \text{for } p = \infty.$$

$$(ii) \qquad A^m_{p,\frac{2mp}{mp+p-2m}}(n) \sim 1 \qquad \text{for } 2m \leq p < \infty.$$

$$(iii) \qquad A^m_{p,\frac{p}{p-m}}(n) \sim 1 \qquad \text{for } m \leq p \leq 2m,$$

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Galicer, M., Muro (2016)

$$\begin{cases} A^m_{p,r}(n) \sim 1 & \text{for } (A): [\frac{1}{2} \leq \frac{1}{r} \leq \frac{m+1}{2m} - \frac{1}{p}] \text{ or } [\frac{1}{r} \leq \frac{1}{2} \ \wedge \ \frac{m}{p} \leq 1 - \frac{1}{r}], \\ A^m_{p,r}(n) \sim n^{\frac{m}{p}+\frac{1}{r}-1} & \text{for } (B): [\frac{1}{2m} \leq \frac{1}{p} \leq \frac{1}{m} \ \wedge \ -\frac{m}{p} + 1 \leq \frac{1}{r}], \\ A^m_{p,r}(n) \sim n^{m(\frac{1}{p}+\frac{1}{r}-\frac{1}{2})-\frac{1}{2}} & \text{for } (C): [\frac{m+1}{2m} \leq \frac{1}{r} \ \wedge \ \frac{1}{p} \leq \frac{1}{2}] \text{ or} \\ & \quad [\frac{1}{2} \leq \frac{1}{r} \leq \frac{m+1}{2m} \leq \frac{1}{p} + \frac{1}{r} \wedge \ \frac{1}{p} \leq \frac{1}{2}], \\ A^m_{p,r}(n) \sim n^{\frac{m}{r}+\frac{1}{p}-1} & \text{for } (D): [\frac{1}{2} \leq \frac{1}{p} \ \wedge \ 1 - \frac{1}{p} \leq \frac{1}{r}], \\ A^m_{p,r}(n) \ll n^{\frac{m-1}{r}} & \text{for } (E): [\frac{1}{2} \leq \frac{1}{p} \leq 1 - \frac{1}{r}], \\ A^m_{p,r}(n) \sim n^{\frac{1}{r}} & \text{for } (F): [\frac{m-1}{p} \leq 1 - \frac{1}{r} \ \wedge \ \frac{1}{m} \leq \frac{1}{p} \leq \frac{1}{m-1}]. \end{cases}$$

Moreover, the power of n in (E) cannot be improved.

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WHAT DOES THIS MEAN?

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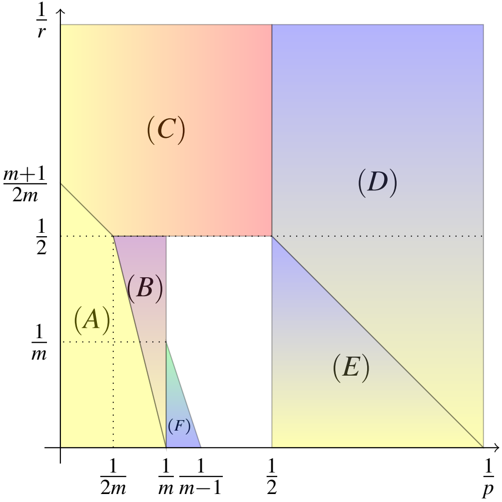

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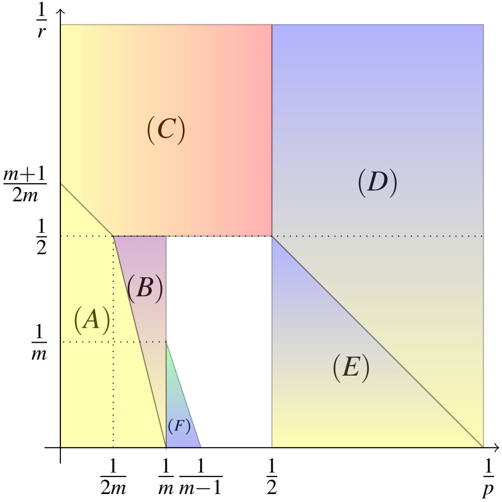

Note that for $m = 2$ the square is filled.

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Idea:

Use the probabilistic method to find polynomials with small norm and many non-zero coefficients.

Randomness and polynomials

$(\varepsilon_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)} \rightsquigarrow$ independent Bernoulli random variables with $\mathbb{P}(\varepsilon_{\mathbf{j}} = 1) = \mathbb{P}(\varepsilon_{\mathbf{j}} = -1) = \frac{1}{2}$.

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Boas (2000) - Bayart (2010)

$$\mathbb{E}\left(\|P\|_{\mathcal{P}(^m\ell_p^n)} \right) \ll \begin{cases} n^{1-\frac{1}{p}} & \text{if } 1 \leq p \leq 2, \\ n^{m(\frac{1}{2}-\frac{1}{p})+\frac{1}{2}} & \text{if } 2 \leq p \leq \infty. \end{cases}$$

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There must be $(\theta_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}$ such that the polynomial $P(z) := \sum_{\mathbf{j}\in\mathcal{J}(m,n)} \theta_{\mathbf{j}} z_{\mathbf{j}}$ verifies:

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Thus,

$$n^{m(\frac{1}{r}+\frac{1}{p})-1} = \frac{n^{m/r}}{n^{1-\frac{1}{p}}} \ll \frac{|P|_r}{\|P\|_{\mathcal{P}(^m\ell_p^n)}} \leq A^m_{p,r}(n)$$

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Galicer, Muro, Sevilla (2015)

Let $m \geq 2$ there is $\mathcal{S} \subset \mathcal{J}(m,n)$ with $|\mathcal{S}| \gg n^{m-1}$ and signs $(\theta_{\mathbf{j}})_{\mathbf{j}\in\mathcal{S}}$ such that the m-homogeneous polynomial $P = \sum_{\mathbf{j}\in\mathcal{S}} \theta_{\mathbf{j}} z_{\mathbf{j}}$ satisfies

$$\|P\|_{\mathcal{P}(^m\ell_p^n)} \ll \begin{cases} \log^{\frac{3p-3}{p}}(n) & \text{for } 1 \leq p \leq 2, \\ \log^{\frac{3}{p}}(n) n^{m(\frac{1}{2}-\frac{1}{p})} & \text{for } 2 \leq p < \infty. \end{cases}$$

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Remark

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In particular $[\ell^n_{p_0}, \ell^n_{p_1}]_\theta = \ell^n_p$.

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we can think of...

$$\mathcal{P}({}^m[\ell_{p_0}^n, \ell_{p_1}^n]_\theta) \text{ and } [\mathcal{P}({}^m\ell_{p_0}^n), \mathcal{P}({}^m\ell_{p_1}^n)]_\theta.$$

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For $m = 2$ the answer is affirmative [Defant, Michels (2000); Kouba (2004)].

Let's go back to the graphic...

Remember: $\left\| \left(\mathcal{P}(^m\mathbb{C}^n), \| \cdot \|_{\mathcal{P}(^m\ell_p^n)} \right) \xrightarrow{id} \left(\mathcal{P}(^m\mathbb{C}^n), | \cdot |_r \right) \right\| = A^m_{p,r}(n).$

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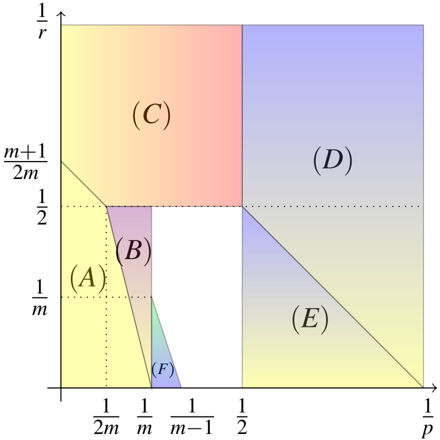

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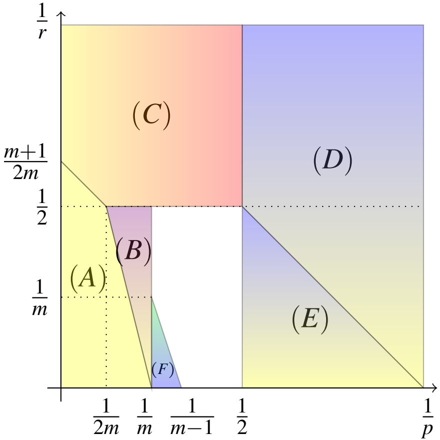

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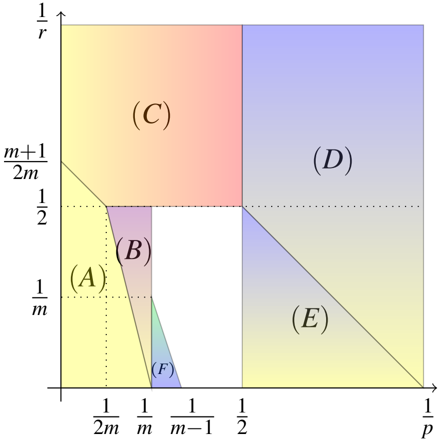

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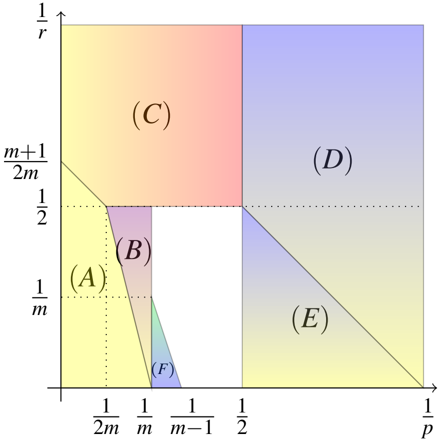

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CONTRADICTION.

Mixed unconditionality in polynomial spaces

Let $(P_i)_{i\in\Lambda}$ be a Schauder basis of $\mathcal{P}(^m\mathbb{C}^n)$. For $1 \leq p, q \leq \infty$ let $\chi_{p,q}((P_i)_{i\in\Lambda})$ be the best constant $C > 0$ such that

$$\|\sum_{i\in\Lambda} \theta_i c_i P_i\|_{\mathcal{P}(^m\ell_q^n)} \leq C\|\sum_{i\in\Lambda} c_i P_i\|_{\mathcal{P}(^m\ell_p^n)},$$

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The (p, q)-mixed unconditional constant of $\mathcal{P}({}^m\mathbb{C}^n)$ is defined as

$$\chi_{p,q}(\mathcal{P}({}^m\mathbb{C}^n)) := \inf\{\chi_{p,q}((P_i)_{i\in\Lambda}) : (P_i)_{i\in\Lambda} \text{ basis for } \mathcal{P}({}^m\mathbb{C}^n)\}.$$

Mixed unconditionality in polynomial spaces

Galicer, M., Muro (2016)

$$\begin{cases} \chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n)) \sim 1 & \text{for } (I), \\ \chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n)) \sim n^{m(\frac{1}{p}-\frac{1}{q}+\frac{1}{2})-\frac{1}{2}} & \text{for } (II), \\ \chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n)) \sim n^{(m-1)(1-\frac{1}{q})+\frac{1}{p}-\frac{1}{q}} & \text{for } (III), \\ \chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n)) \sim_\varepsilon n^{(m-1)(1-\frac{1}{q})+\frac{1}{p}-\frac{1}{q}} & \text{for } (III'). \end{cases}$$

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for every $\varepsilon > 0$.

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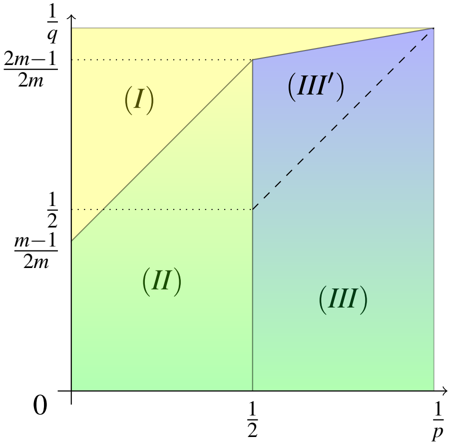

Ideas of the Proof

We have the following relation:

$$\chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n)) \leq \chi_{p,q}\big((z_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}\big) \leq 2^m \chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n)).$$

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- Lower bounds $\rightsquigarrow$ Extreme polynomials: Boas - Bayart

Monomial Unconditionality

For $n, m \in \mathbb{N}$ let $\chi_M(\mathcal{P}(^m Y_n), \mathcal{P}(^m X_n))$ be the best constant $C > 0$ such that

$$\sup_{z \in B_{X_n}} \left| \sum_{\mathbf{j} \in \mathcal{J}(m,n)} \theta_{\mathbf{j}} c_{\mathbf{j}} z_{\mathbf{j}} \right| \leq C \sup_{z \in B_{Y_n}} \left| \sum_{\mathbf{j} \in \mathcal{J}(m,n)} c_{\mathbf{j}} z_{\mathbf{j}} \right|,$$

for every $(c_{\mathbf{j}})_{\mathbf{j} \in \mathcal{J}(m,n)} \subset \mathbb{C}$ and every choice of complex numbers $(\theta_{\mathbf{j}})_{\mathbf{j} \in \mathcal{J}(m,n)}$ of modulus one.

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Remark

When $X_n = \ell_q^n$ and $Y_n = \ell_p^n$ we recover the notion of (p, q)-mixed unconditional constant for the monomial basis

$$\chi_M(\mathcal{P}(^m Y_n), \mathcal{P}(^m X_n)) = \chi_{p,q}((z_{\mathbf{j}})_{\mathbf{j} \in \mathcal{J}(m,n)}).$$

Monomial Convergence

The following is usually called the domain of monomial convergence for $\mathcal{P}({}^m\ell_p)$,

$$\operatorname{dom}(\mathcal{P}({}^m\ell_p)) := \Big\{ z \in \ell_\infty : \sum_{\mathbf{j}\in\mathcal{J}(m)} |c_{\mathbf{j}}(P) z_{\mathbf{j}}| < \infty \text{ for every } P \in \mathcal{P}({}^m\ell_p) \Big\},$$

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Defant, Maestre, Prengel (2009)

Given Y a Banach sequence spaces, the following are equivalent

i) $Y \subset \mathrm{dom}(\mathcal{P}(^m\ell_p))$,

ii) $\chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^mY_n)) \sim 1$.

Monomial Convergence

Defant, Maestre, Prengel (2009)

$\ell_{q_m} \subset \mathrm{dom}(\mathcal{P}(^m\ell_p))$ where

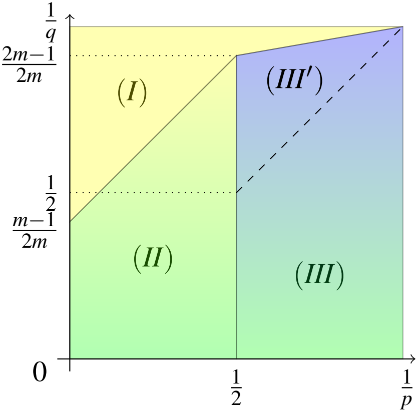
$$\frac{1}{q_m} = \frac{1}{p} + \frac{m-1}{2m}.$$

Bayart, Defant, Schlüters (2015)

For $p \leq 2$, $\ell_{q_m-\varepsilon} \subset mon(\mathcal{P}(^m\ell_p))$ for every $\varepsilon > 0$ where $\frac{1}{q_m} = \frac{m-1}{m} + \frac{1}{mp}$.

Galicer, M., Muro (2016)

$d(w_\lambda, q_m) \subset mon(\mathcal{P}(^m\ell_p))$.



Thanks!!!