Some remarks on non-symmetric polarization

Felipe Marceca

IMAS, UBA-CONICET

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Joint work with Daniel Carando

Introduction

Let $P : \mathbb{C}^n \to \mathbb{C}$ be an m-homogeneous polynomial of n variables given by

$$P(x) = \sum_{1 \leq j_1 \leq \ldots \leq j_m \leq n} c_{j_1 \ldots j_m} x_{j_1} \ldots x_{j_m}.$$

Let $L : (\mathbb{C}^n)^m \to \mathbb{C}$ be the unique symmetric m-linear form such that

$$L(x, \ldots, x) = P(x) \quad \forall x \in \mathbb{C}^n.$$

It follows from the polarization formula (P.F.) that

$$\sup_{\left\|x^{(k)}\right\| \leq 1} \left| L\left(x^{(1)}, \ldots, x^{(m)}\right) \right| \leq e^m \sup_{\|x\| \leq 1} |P(x)|,$$

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Clearly, $L_P(x, \ldots, x) = P(x)$ for all $x \in \mathbb{C}^n$.

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Our main goal is to compare

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Theorem (Defant-Schlüters 2017)

There exists a universal constant $c_1 \geq 1$ such that

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for every 1-unconditional norm $\| \cdot \|$ on $\mathbb{C}^n$.

For $\| \cdot \|_p$ with $1 \leq p < 2$, there is a constant $c_2 = c_2(p) \geq 1$ for which

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Idea of the proof

- Define partial symmetrizations $\mathcal{S}_k$ for $1 \leq k \leq m$ such that

$$L_P = \mathcal{S}_1 L_P, \quad \mathcal{S}_2 L_P, \quad \dots, \quad \mathcal{S}_{m-1} L_P, \quad \mathcal{S}_m L_P = L \overset{\text{P.F.}}{\rightsquigarrow} P.$$

- Identify each $\mathcal{S}_k L_P$ with its coefficents matrix and find a matrix $\mathfrak{A}_k$ such that

$$\mathcal{S}_{k-1} L_P = \mathfrak{A}_k * \mathcal{S}_k L_P,$$

where $*$ denotes de coordinatewise product.
- Break $\mathfrak{A}_k$ down into simpler building blocks.
- Estimate how these building blocks change the supremum norm when applied to an m-form.

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Symmetrization

Probabilistic point of view

Let Σ be the permutation group of m elements endowed with the equiprobability measure. We have

$$L\left(x^{(1)}, \ldots, x^{(m)}\right) = E_\sigma\left[\sum_{1 \le j_1 \le \ldots \le j_m \le n} c_{j_1 \ldots j_m} x^{(1)}_{j_{\sigma(1)}} \ldots x^{(m)}_{j_{\sigma(m)}}\right].$$

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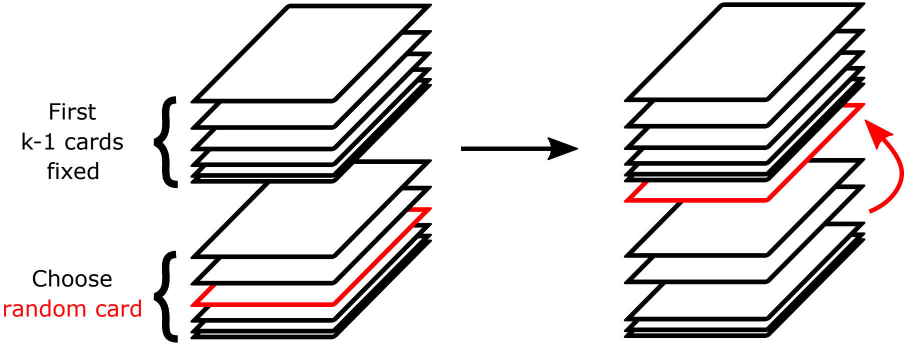
Fisher-Yates shuffle

Step k of the Fisher-Yates shuffle:



Partial shuffles

For $1 \leq k \leq m-1$, let μ_k be the probability distribution on the permutation group Σ associated to performing the first k steps of the Fisher-Yates shuffle.

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where $\sigma \sim \mu_k$.

We get

$$L_P = \mathcal{S}_0 L_P, \quad \mathcal{S}_1 L_P, \quad \ldots, \quad \mathcal{S}_{m-2} L_P, \quad \mathcal{S}_{m-1} L_P = L.$$

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Let $C(n,m)$ be the best constant such that

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We have that $C(n,m) \gtrsim (\log n)^{m/2}$ if $n \gg m$.

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Lower bounds for $C_p(n, m)$

Taking $P(x) = x_1 \dots x_m$ an easy computation gives

$$\sup_{\left\|x^{(k)}\right\|_p \le 1} \left| L_P\left(x^{(1)}, \dots, x^{(m)}\right)\right| = 1 \quad \text{and}$$

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So for $1 \le p < 2$ and $n \ge m$,

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Bad news: hypercontractivity cannot be achieved.

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for every m-homogeneous polynomial $P : \mathbb{C}^n \to \mathbb{C}$.
Equivalently, by the maximum modulus principle we get

$$\|P\|_{C(\mathbb{T}^n)} \leq \|L_P\|_{C(\mathbb{T}^{nm})} \leq C(n, m)\|P\|_{C(\mathbb{T}^n)},$$

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Lower bounds for $C(n, m)$

We know that

$$\sup_{\|x\|_\infty \le 1} |P(x)| \le \sup_{\|x^{(k)}\|_\infty \le 1} \left| L_P\left(x^{(1)}, \dots, x^{(m)}\right) \right| \le C(n, m) \sup_{\|x\|_\infty \le 1} |P(x)|,$$

for every m-homogeneous polynomial $P : \mathbb{C}^n \to \mathbb{C}$.
Equivalently, by the maximum modulus principle we get

$$\|P\|_{C(\mathbb{T}^n)} \le \|L_P\|_{C(\mathbb{T}^{nm})} \le C(n, m)\|P\|_{C(\mathbb{T}^n)},$$

where $\mathbb{T} = \{z \in \mathbb{C} \ : \ |z| = 1\}$.
The monomials of P and L_P are characters of the compact abelian groups $\mathbb{T}^n$ and $\mathbb{T}^{nm}$.

Employing Banach space theory

Theorem (Pełczyński)

Let $(f_j)_{j\in J}$ and $(g_j)_{j\in J}$ be sequences of characters on compact abelian groups S and T. Suppose there are constants $K_1, K_2 > 0$ such that

$$\frac{1}{K_1}\Big\|\sum_{j\in J} c_j f_j\Big\|_{C(S)} \leq \Big\|\sum_{j\in J} c_j g_j\Big\|_{C(T)} \leq K_2\Big\|\sum_{j\in J} c_j f_j\Big\|_{C(S)},$$

for every sequence of scalars $(c_j)_{j\in J} \subseteq \mathbb{C}$.

Then, for every Banach space E and every sequence of vectors $(v_j)_{j\in J} \subseteq E$ we have

$$\frac{1}{K_1K_2}\Big\|\sum_{j\in J} v_j f_j\Big\|_{L^1(S,E)} \leq \Big\|\sum_{j\in J} v_j g_j\Big\|_{L^1(T,E)} \leq K_1K_2\Big\|\sum_{j\in J} v_j f_j\Big\|_{L^1(S,E)}.$$

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Employing Banach space theory

From the inequality

$$\|P\|_{C(\mathbb{T}^n)} \leq \|L_P\|_{C(\mathbb{T}^{nm})} \leq C(n,m)\|P\|_{C(\mathbb{T}^n)},$$

and Pełczyński's theorem we get

$$\Big\| \sum_{1\leq j_1\leq \ldots \leq j_m \leq n} v_j x_{j_1}^{(1)} \ldots x_{j_m}^{(m)} \Big\|_{L^1(\mathbb{T}^{nm},E)} \leq$$

$$\leq C(n,m) \Big\| \sum_{1\leq j_1\leq \ldots \leq j_m \leq n} v_j x_{j_1} \ldots x_{j_m} \Big\|_{L^1(\mathbb{T}^{n},E)},$$

for every Banach space E and every sequence of vectors $(v_j)_{j\in J} \subseteq E$.
Equivalently, for every vector valued m-homogeneous polynomial $P : \mathbb{C}^n \to E$ we have that

$$\|L_P\|_{L^1(\mathbb{T}^{nm},E)} \leq C(n,m)\|P\|_{L^1(\mathbb{T}^n,E)}.$$

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The case $m = 2$

An example provided by Bourgain and included in a paper by McConnell and Taqqu.

An example for $m = 2$ taking $E = \mathcal{L}(\ell_2)$

There is a vector valued 2-homogeneous polynomial $P : \mathbb{C}^n \to \mathcal{L}(\ell_2)$ such that

$$\|P\|_{L^1(\mathbb{T}^n, \mathcal{L}(\ell_2))} \leq \pi \quad \text{and} \quad \|L_P\|_{L^1(\mathbb{T}^{nm}, \mathcal{L}(\ell_2))} \geq \log n - \pi.$$

Therefore,

$$\frac{\log n}{\pi} - 1 \leq C(n, 2) \leq c \log n.$$

The $\log n$ estimate from the example arises from the norm of the main triangle projection.

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Extending the example for $m > 2$

Let $E = \bigotimes_{k=1}^{m/2} \mathcal{L}(\ell_2)$ be the projective tensor product of $m/2$ copies of $\mathcal{L}(\ell_2)$.

One may define a vector valued m-homogeneous polynomial $P : \mathbb{C}^n \to E$ given by taking the tensor product of $m/2$ copies of the 2-homogeneous polynomial in the previous example.

We can obtain

$$\|P\|_{L^1(\mathbb{T}^n,E)} \leq \pi^{m/2} \quad \text{and} \quad \|L_P\|_{L^1(\mathbb{T}^{nm},E)} \gtrsim (\log n)^{m/2} \quad \text{if } n \gg m.$$

Finally, for $n \gg m$ we get

$$(\log n)^{m/2} \lesssim C(n,m) \leq c_1^m m^m (\log n)^{m-1}.$$

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Thank You!