Uniqueness of norm-preserving extensions of functionals on strict ideals

Julia Martsinkevitš

University of Tartu, Estonia

19 October 2017

Conference on Non-Linear Functional Analysis, Valencia

The presentation is based on joint research with Märt Põldvere.

[1] J. Martsinkevitš and M. Põldvere,
Uniqueness of norm-preserving extensions of functionals on the space of compact operators, submitted paper, 2017.

[2] J. Martsinkevitš and M. Põldvere,
On the structure of the dual unit ball of strict u-ideals, submitted paper, 2017.

The notion of an ideal

Let X be a Banach space over the scalar field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Let Y be a closed subspace of X.

The notion of an ideal

Let X be a Banach space over the scalar field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Let Y be a closed subspace of X.

G. Godefroy, N. J. Kalton, and P. D. Saphar (1993):

- Y is an ideal in X if there exists a bounded linear norm-one projection P on X^* with

$$\ker P = Y^\perp := \{x^* \in X^* \colon x^*|_Y = 0\}.$$

The notion of an ideal

Let X be a Banach space over the scalar field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Let Y be a closed subspace of X.

G. Godefroy, N. J. Kalton, and P. D. Saphar (1993):

- Y is an ideal in X if there exists a bounded linear norm-one projection P on X^* with

$$\ker P = Y^{\perp} := \{x^* \in X^* \colon\ x^*|_Y = 0\}.$$

- Y is a strict ideal in X if Y is an ideal in X with respect to a projection P on X^* and the range of P is norming for X, i.e.,

$$\|x\| = \sup_{x^* \in B_{\operatorname{ran} P}} |x^*(x)| \quad \text{for all } x \in X.$$

The notion of an ideal

Let X be a Banach space over the scalar field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Let Y be a closed subspace of X.

G. Godefroy, N. J. Kalton, and P. D. Saphar (1993):

- Y is an ideal in X if there exists a bounded linear norm-one projection P on X^* with

$$\ker P = Y^{\perp} := \{x^* \in X^* \colon\ x^*|_Y = 0\}.$$

- Y is a strict ideal in X if Y is an ideal in X with respect to a projection P on X^* and the range of P is norming for X, i.e.,

$$\|x\| = \sup_{x^* \in B_{\operatorname{ran} P}} |x^*(x)| \quad \text{for all } x \in X.$$

- Y is a u-ideal in X if Y is an ideal in X with respect to a projection P on X^* and $\|I - 2P\| = 1$.

Some background

If P is an ideal projection for Y in X, then

- for every $x^* \in X^*$, the functional $Px^* \in X^*$ is a norm-preserving extension of the restriction $x^*|_Y \in Y^*$;

Some background

If P is an ideal projection for Y in X, then

- for every $x^* \in X^*$, the functional $Px^* \in X^*$ is a norm-preserving extension of the restriction $x^*|_Y \in Y^*$;
- the mapping
$$J_P \colon Y^* \ni y^* \mapsto Px^* \in X^*,$$
where $x^* \in X^*$ is any extension of y^*, is a linear isometry; in particular, $\operatorname{ran} J_P = \operatorname{ran} P$ and $\operatorname{ran} P \cong Y^*$;

Some background

If P is an ideal projection for Y in X, then

- for every $x^* \in X^*$, the functional $Px^* \in X^*$ is a norm-preserving extension of the restriction $x^*|_Y \in Y^*$;
- the mapping
$$J_P \colon Y^* \ni y^* \mapsto Px^* \in X^*,$$
where $x^* \in X^*$ is any extension of y^*, is a linear isometry; in particular, $\operatorname{ran} J_P = \operatorname{ran} P$ and $\operatorname{ran} P \cong Y^*$;
- each $x \in X$ induces a functional $x_P \in Y^{**}$ defined by
$$x_P(y^*) = (J_P y^*)(x), \qquad y^* \in Y^*.$$
If P is strict, the mapping $x \mapsto x_P$ is an isometry and one can identify X with the closed subspace $X_P = \{x_P \in Y^{**} \colon x \in X\}$ of Y^{**}.

Examples of ideals

(1) Y is an ideal in its bidual Y^{**} with respect to the canonical projection $\pi_Y := j_{Y^*}(j_Y)^*$ where $j_Y\colon\ Y \to Y^{**}$ and $j_{Y^*}\colon\ Y^* \to Y^{***}$ are canonical embeddings;

Examples of ideals

(1) Y is an ideal in its bidual Y^{**} with respect to the canonical projection $\pi_Y := j_{Y^*}(j_Y)^*$ where $j_Y\colon\ Y \to Y^{**}$ and $j_{Y^*}\colon\ Y^* \to Y^{***}$ are canonical embeddings;

(2) c_0 is a strict u-ideal in its bidual.

Our starting point

Proposition 1 (V. Lima and Å. Lima, 2009)

Let Y be a strict u-ideal in X. Then every ideal projection for Y in X is strict.

Our starting point

Proposition 1 (V. Lima and Å. Lima, 2009)

Let Y be a strict u-ideal in X. Then every ideal projection for Y in X is strict.

Denote

- $\tau_P := \sigma(Y^*, X_P)$,
- $\mathcal{C} := \{y^* \in S_{Y^*} : y^*$ has a unique norm-preserving extension to $X\}$.

Our starting point

Proposition 1 (V. Lima and Å. Lima, 2009)

Let Y be a strict u-ideal in X. Then every ideal projection for Y in X is strict.

Denote

- $\tau_P := \sigma(Y^*, X_P)$,
- $\mathcal{C} := \{y^* \in S_{Y^*} : y^* \text{ has a unique norm-preserving extension to } X\}$.

Proposition 2

Let Y be a strict ideal in X with respect to an ideal projection $P \in \mathcal{L}(X^*)$. If $B_{Y^*} = \overline{\text{co}}^{\tau_P}(\mathcal{C})$, then every ideal projection for Y in X is strict.

When $B_{Y^*} = \overline{\text{co}}^{\tau_P}(\mathcal{C})$?

Theorem 3

Let Y be a strict u-ideal in a separable Banach space X with respect to an ideal projection $P \in \mathcal{L}(X^*)$. Then $B_{Y^*} = \overline{\text{co}}^{\tau_P}(\mathcal{C})$.

When $B_{Y^*} = \overline{\text{co}}^{\tau_P}(\mathcal{C})$?

Theorem 3

Let Y be a strict u-ideal in a separable Banach space X with respect to an ideal projection $P \in \mathcal{L}(X^*)$. Then $B_{Y^*} = \overline{\text{co}}^{\tau_P}(\mathcal{C})$.

THE QUESTION:

Whether the aforecited theorem holds if we remove the assumption of separability?

Description of functionals in S_{Y^*} admitting a unique norm-preserving extension to X

Proposition 4

Let Y be a strict ideal in X with respect to a projection $P \in \mathcal{L}(X^*)$, and let $y^* \in S_{Y^*}$. The following assertions are equivalent:

(i) *y^* has a unique norm-preserving extension to X;*

(ii) *y^* is a weak*-to-τ_P-PC of B_{Y^*}, i.e., for any net (y^*_α) in B_{Y^*},*

$$y^*_\alpha \xrightarrow[\alpha]{w^*} y^* \qquad \Longrightarrow \qquad y^*_\alpha \xrightarrow[\alpha]{\tau_P} y^*,$$

i.e., whenever

$$y^*_\alpha(y) \underset{\alpha}{\to} y^*(y) \quad \text{for all } y \in Y,$$

one has

$$J_P y^*_\alpha(x) = x_P(y^*_\alpha) \underset{\alpha}{\to} x_P(y^*) = J_P y^*(x) \quad \text{for all } x \in X.$$

Generalisation of "ordinary" dentability and denting points

Let Z be a Banach space, and let τ be a locally convex topology on Z. Given a $z \in Z$, a seminorm p on Z, and an $\varepsilon > 0$, we define

$$\mathcal{U}_p(z,\varepsilon) := \{v \in Z \colon p(v - z) < \varepsilon\}.$$

Generalisation of “ordinary” dentability and denting points

Let Z be a Banach space, and let τ be a locally convex topology on Z. Given a $z \in Z$, a seminorm p on Z, and an $\varepsilon > 0$, we define

$$\mathcal{U}_p(z,\varepsilon) := \{v \in Z \colon p(v-z) < \varepsilon\}.$$

Suppose that τ_1 and τ_2 are locally convex topologies on Z such that $\tau_1 \subset \tau_2 \subset \tau_{\|\cdot\|}$. Let C be a non-empty bounded subset of Z.

- The set C is (τ_1,τ_2)-dentable if, whenever p is a τ_2-continuous seminorm on Z and $\varepsilon > 0$, there is an $x \in C$ such that

$$x \notin \overline{\mathrm{co}}^{\tau_1}\big(C \setminus \mathcal{U}_p(x,\varepsilon)\big). \tag{1}$$

Generalisation of "ordinary" dentability and denting points

Let Z be a Banach space, and let τ be a locally convex topology on Z. Given a $z \in Z$, a seminorm p on Z, and an $\varepsilon > 0$, we define

$$\mathcal{U}_p(z,\varepsilon) := \{v \in Z\colon p(v-z) < \varepsilon\}.$$

Suppose that τ_1 and τ_2 are locally convex topologies on Z such that $\tau_1 \subset \tau_2 \subset \tau_{\|\cdot\|}$. Let C be a non-empty bounded subset of Z.

- The set C is (τ_1,τ_2)-dentable if, whenever p is a τ_2-continuous seminorm on Z and $\varepsilon > 0$, there is an $x \in C$ such that

$$x \notin \overline{\mathrm{co}}^{\tau_1}\big(C \setminus \mathcal{U}_p(x,\varepsilon)\big). \tag{1}$$

- A point $x \in C$ is a (τ_1,τ_2)-denting point of C if, whenever p is a τ_2-continuous seminorm on Z and $\varepsilon > 0$, one has (1).

Dentability in locally convex spaces with two comparable topologies has been studied by M. Fundo (1997, 1999).

Proof of Theorem 3

Theorem 3 follows from Propositions 5 and 6.

Proposition 5

Let Y be a strict ideal in X with respect to a projection $P \in \mathcal{L}(X^*)$, and let $y^* \in B_{Y^*}$. The following assertions are equivalent:

(i) *y^* is an extreme point of B_{Y^*} having a unique norm-preserving extension to X;*

(ii) *y^* is both an extreme point and a weak*-to-τ_P-PC of B_{Y^*};*

(iii) *y^* is a (weak^*, τ_P)-denting point of B_{Y^*}.*

Proof of Theorem 3

Theorem 3 follows from Propositions 5 and 6.

Proposition 5

Let Y be a strict ideal in X with respect to a projection $P \in \mathcal{L}(X^*)$, and let $y^* \in B_{Y^*}$. The following assertions are equivalent:

(i) *y^* is an extreme point of B_{Y^*} having a unique norm-preserving extension to X;*

(ii) *y^* is both an extreme point and a weak*-to-τ_P-PC of B_{Y^*};*

(iii) *y^* is a $(weak^*, \tau_P)$-denting point of B_{Y^*}.*

Proposition 6

Let Y be a strict u-ideal in a separable Banach space X with respect to an ideal projection $P \in \mathcal{L}(X^*)$. Then B_{Y^*} is the τ_P-closed convex hull of its $(weak^*, \tau_P)$-denting points.

Theorem 3

Let Y be a strict u-ideal in a separable Banach space X with respect to an ideal projection $P \in \mathcal{L}(X^*)$. Then $B_{Y^*} = \overline{\text{co}}^{\tau_P}(\mathcal{C})$.

(τ_P, τ_P)-dentability of bounded sets in dual spaces of strict ideals in separable spaces

Theorem 7

Let X be a separable Banach space. Then every non-empty bounded subset of the dual space X^* is $(weak^*, weak^*)$-dentable.

Proof of Theorem 7 relies on standard martingale techniques.

(τ_P, τ_P)-dentability of bounded sets in dual spaces of strict ideals in separable spaces

Theorem 7

Let X be a separable Banach space. Then every non-empty bounded subset of the dual space X^* is $(weak^*, weak^*)$-dentable.

Proof of Theorem 7 relies on standard martingale techniques.

Corollary 8

Let Y be a strict ideal in a separable Banach space X with respect to an ideal projection $P \in \mathcal{L}(X^*)$. Then every non-empty bounded subset of Y^* is (τ_P, τ_P)-dentable.

Thank you for attention!