Banach spaces with weak*-sequential dual ball

Conference on Non-Linear Functional Analysis
Valencia

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October 16th, 2017

This work was supported by Ministerio de Economía y Competitividad and FEDER (project MTM2014-54182-P) and by the research project 19275/PI/14 funded by Fundación Séneca - Agencia de Ciencia y Tecnología de la Región de Murcia within the framework of PCTIRM 2011-2014.

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- *K is said to be* **sequential** *if any sequentially closed subspace is closed.*
- *K is said to have* **countable tightness** *if for every subspace F of K, every point in the closure of F is in the closure of a countable subspace of F.*

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K is FU $\Longrightarrow$ K is sequential

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$$K \text{ is FU} \implies K \text{ is sequential} \implies K \text{ is sequentially compact}$$

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A Banach space with weak-FU dual ball is said to have* **weak*-angelic dual**.

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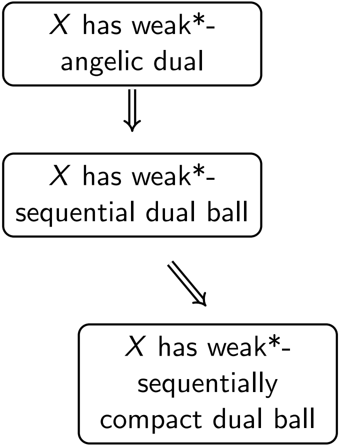
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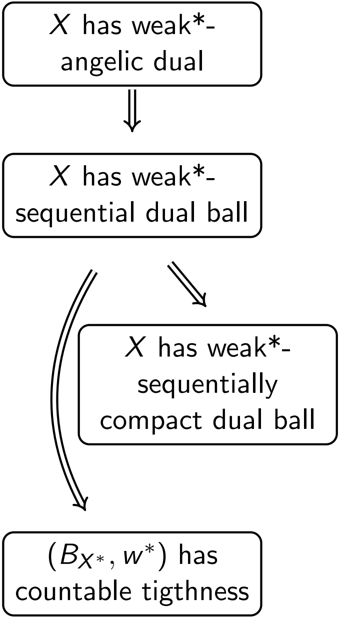
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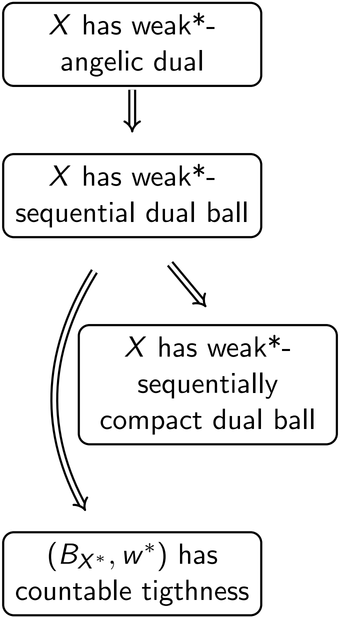
- *X has* **property (C)** *of Corson if and only if every point in the closure of C is in the weak*-closure of a countable subset of C for every convex set C in B_{X^*} (Pol's characterization).*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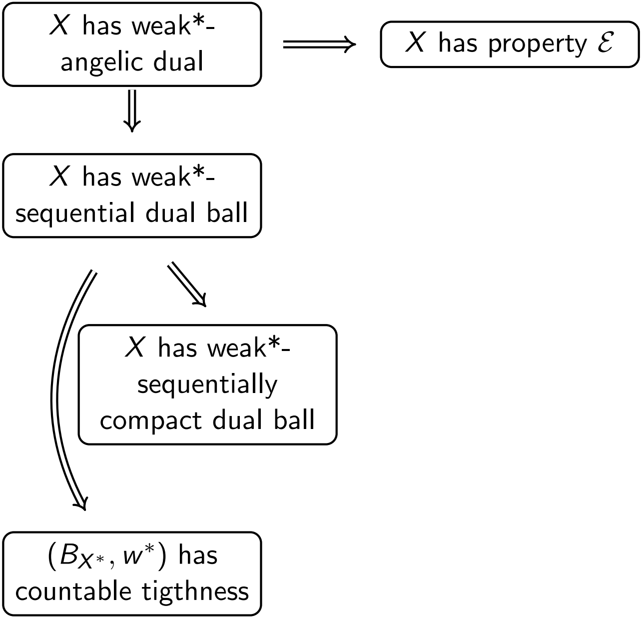
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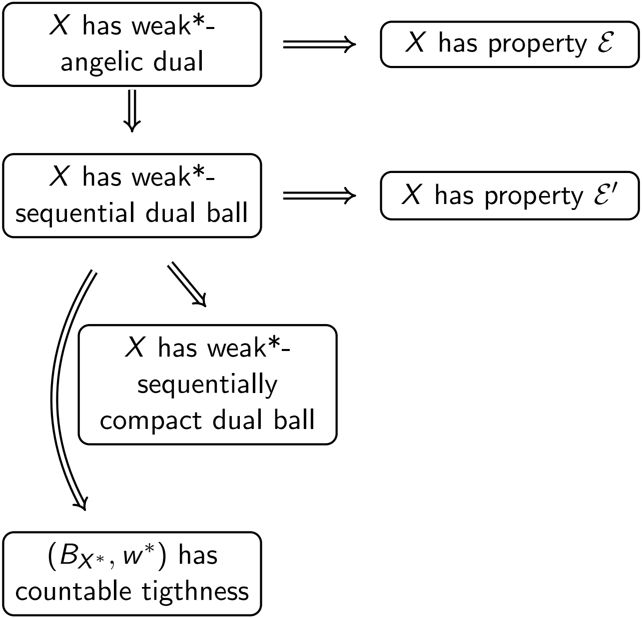
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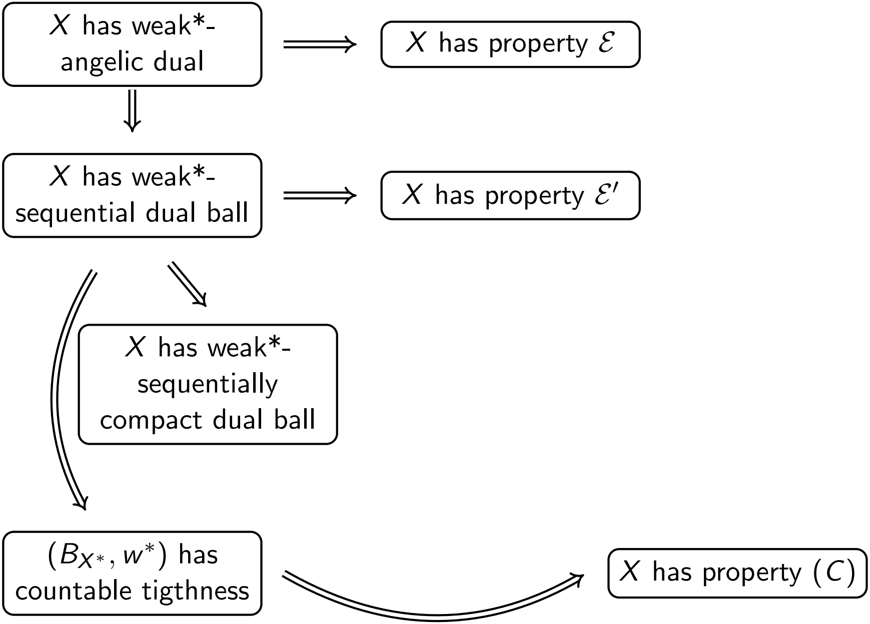
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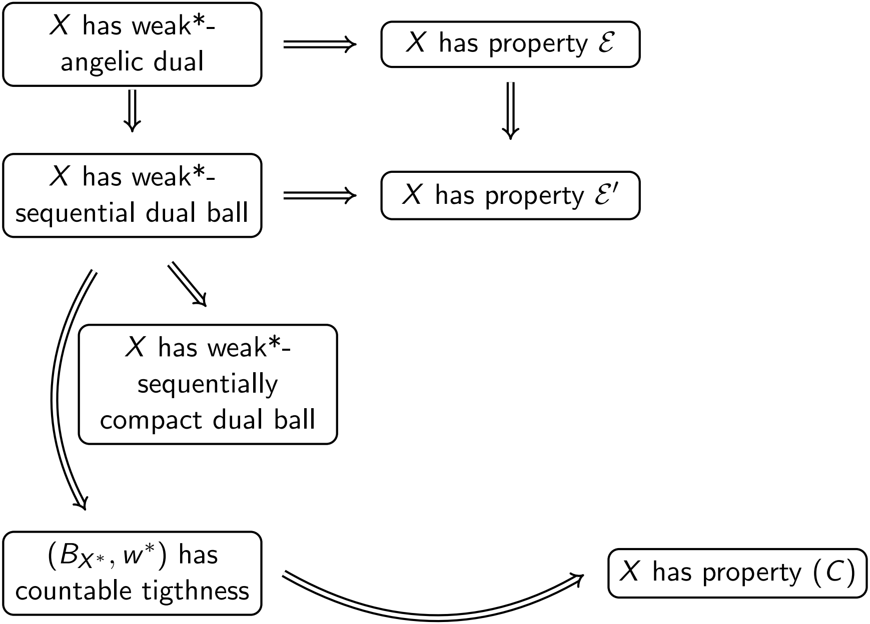
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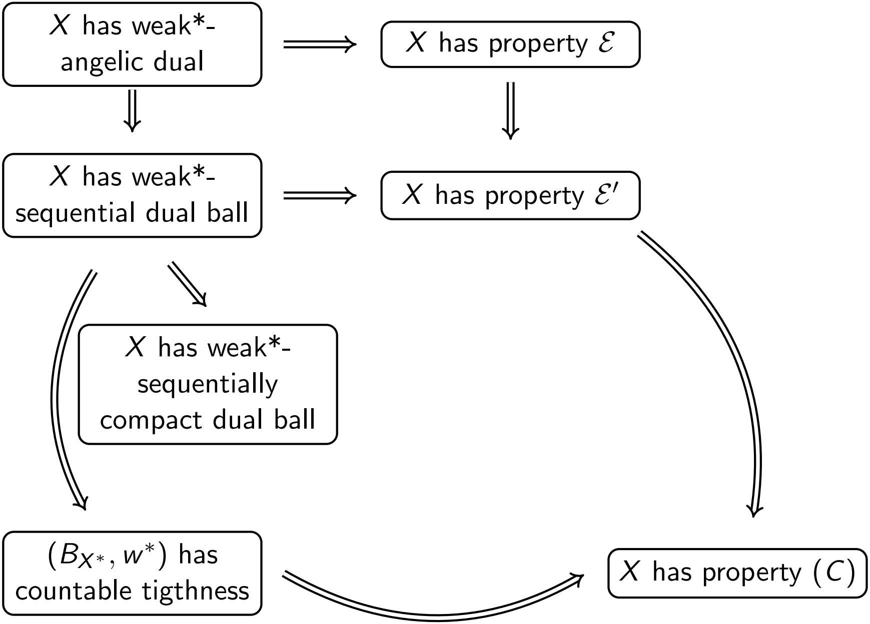
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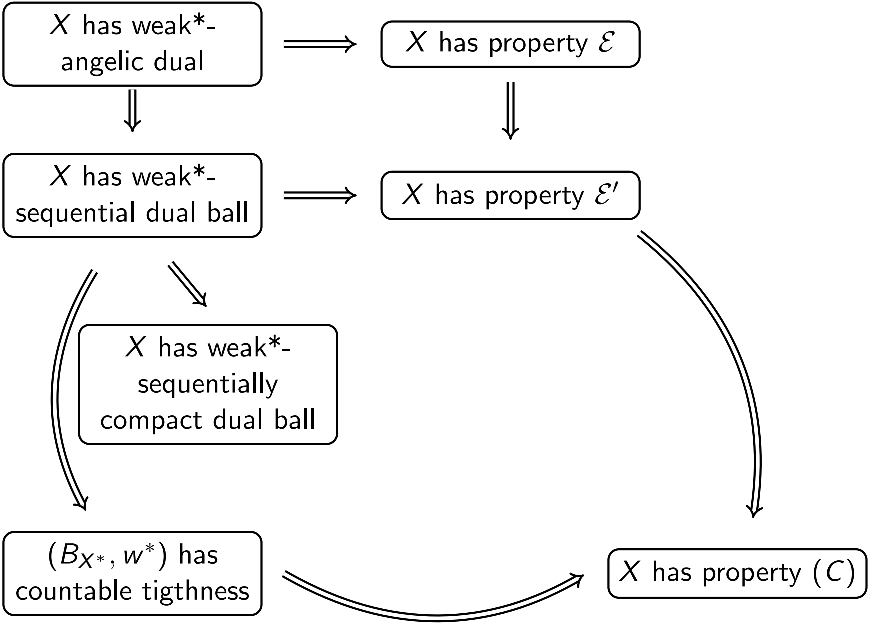
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Question (A. Plichko, 2014)

weak-sequential dual ball* $\Rightarrow$ *weak*-angelic dual?*

Theorem

If X is a Banach space with weak-sequentially compact dual ball and $Y \subset X$ is a subspace such that Y and X/Y have weak*-sequential dual ball, then X has weak*-sequential dual ball.*

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Therefore, a topological space T is sequential with sequential order ≤ 1 if and only if it is FU.

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Let X be a Banach space with weak-sequentially compact dual ball. Let $Y \subset X$ be a subspace with weak*-sequential dual ball with sequential order $\leq \gamma_1$ and such that X/Y has weak*-sequential dual ball with sequential order $\leq \gamma_2$. Then X has weak*-sequential dual ball with sequential order $\leq \gamma_1 + \gamma_2$.*

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- *If* $(x_n)_{n\in\mathbb{N}}$ *is a sequence in a Banach space, we say that* $(y_k)_{k\in\mathbb{N}}$ *is a* **convex block subsequence** *of* $(x_n)_{n\in\mathbb{N}}$ *if there is a sequence* $(I_k)_{k\in\mathbb{N}}$ *of finite subsets of* $\mathbb{N}$ *with* $\max(I_k) < \min(I_{k+1})$ *and a sequence* $a_n \in [0,1]$ *with* $\sum_{n\in I_k} a_n = 1$ *for every* $k \in \mathbb{N}$ *such that* $y_k = \sum_{n\in I_k} a_n x_n$.

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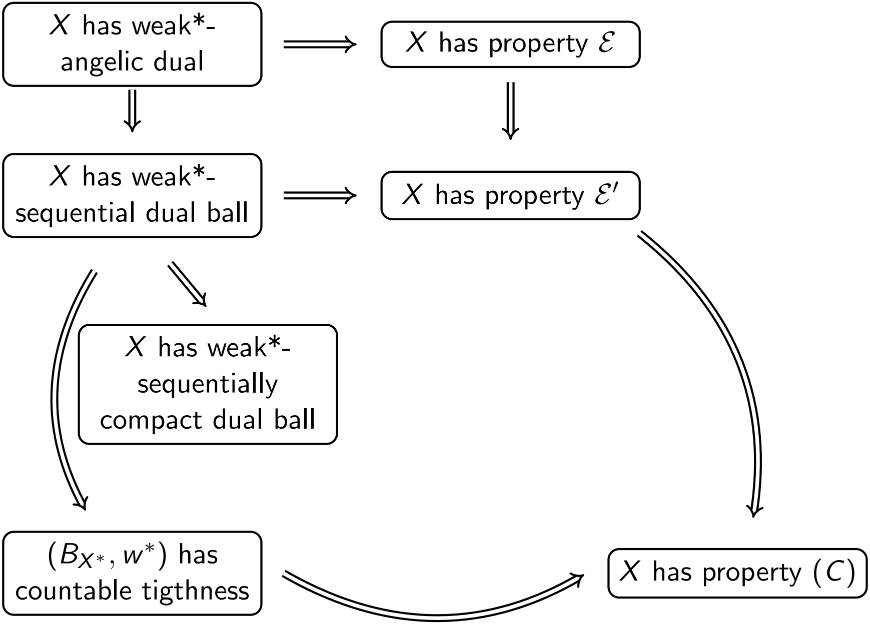
Every Banach space containing no isomorphic copies of ℓ_1 has weak*-convex block compact dual ball (Bourgain, 1979). Therefore, every WPG Banach space (i.e. every Banach space with a linearly dense weakly precompact set) also has weak*-convex block compact dual ball.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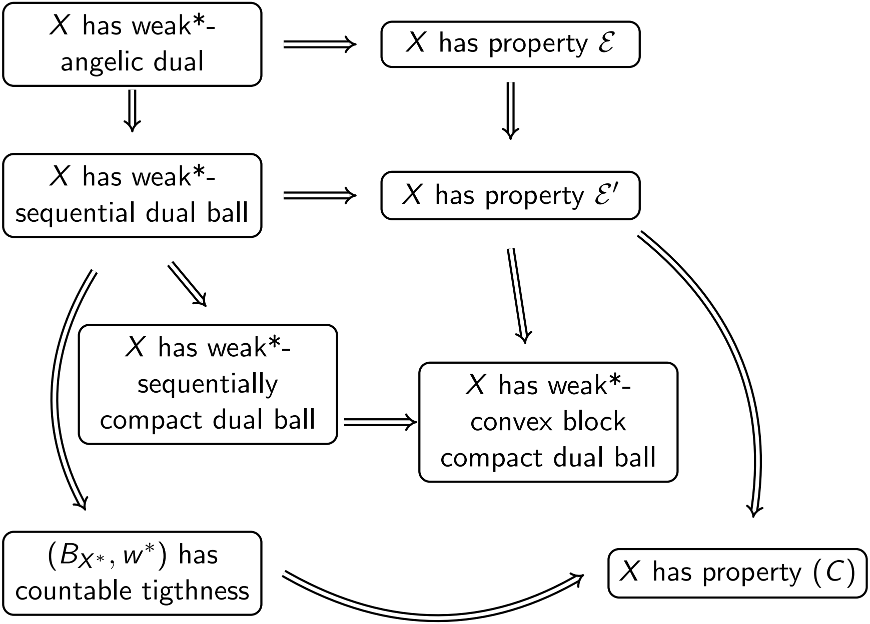
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Thus, property $\mathcal{E}$ is property $\mathcal{E}(1)$ and property $\mathcal{E}'$ is property $\mathcal{E}(\omega_1)$.

Theorem

Let X *be a Banach space with weak*-convex block compact dual ball. Let* $Y \subset X$ *be a subspace with property* $\mathcal{E}(\gamma_1)$ *such that* X/Y *has property* $\mathcal{E}(\gamma_2)$*. Then* X *has property* $\mathcal{E}(\gamma_1 + \gamma_2)$.

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Let X be a Banach space with weak-sequentially compact dual ball. Let $Y \subset X$ be a subspace with weak*-sequential dual ball with sequential order $\leq \gamma_1$ and such that X/Y has weak*-sequential dual ball with sequential order $\leq \gamma_2$. Then X has weak*-sequential dual ball with sequential order $\leq \gamma_1 + \gamma_2$.*

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$$\left\| x + \sum_{1 \leq i \leq k} a_i \chi_{N_{r_i}} \right\| = \max \left\{ \left\| x + \sum_{1 \leq i \leq k} a_i \chi_{N_{r_i}} \right\|_\infty, \left(\sum_{1 \leq i \leq k} |a_i|^2 \right)^{\frac{1}{2}} \right\}.$$

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If we just consider the supremum norm in the definition then we obtain the space JL_0.

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Corollary

Since JL_2/c_0 is isomorphic to $\ell_2(\Gamma)$ and JL_2 has weak-sequentially compact dual ball,*

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Since JL_2/c_0 is isomorphic to $\ell_2(\Gamma)$ and JL_2 has weak-sequentially compact dual ball, JL_2 has weak*-sequential dual ball with sequential order ≤ 2. Since JL_2 does not have weak*-angelic dual, we conclude that*

$$\text{weak*-sequential dual ball} \not\Rightarrow \text{weak*-angelic dual}$$

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Theorem

Let γ be a countable ordinal, X_γ a Banach space and $(X_\alpha)_{\alpha \leq \gamma}$ an increasing sequence of subspaces of X_γ such that:

1. *X_0 has weak*-sequential dual ball with sequential order $\leq \theta$;*
2. *each quotient $X_{\alpha+1}/X_\alpha$ has weak*-angelic dual;*
3. *$X_\alpha = \overline{\bigcup_{\beta<\alpha} X_\beta}$ if α is a limit ordinal;*
4. *X_γ has weak*-sequentially compact dual ball.*

Then each X_α has weak-sequential dual ball with sequential order $\leq \theta + \alpha$ if $\alpha < \omega$ and sequential order $\leq \theta + \alpha + 1$ if $\alpha \geq \omega$.*

Theorem

If K is an infinite scattered compact space with $ht(K) < \omega_1$, then $\mathcal{C}(K)$ has weak-sequential dual ball with sequential order $\leq ht(K)$ if $ht(K) < \omega$ and with sequential order $\leq ht(K) + 1$ if $ht(K) \geq \omega$.*

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Sketch of the proof. For every $\alpha \leq \gamma$, take $X_\alpha = \{f \in \mathcal{C}(K) : f(t) = 0 \text{ for every } t \in K^{(\alpha)}\}$, where $\{K^{(\alpha)} : \alpha \leq \gamma\}$ are the Cantor-Bendixson derivatives of K and $\gamma = ht(K)$.

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Theorem (A.I. Baškirov)

Under CH there exist scattered compact spaces of any sequential order and such that the sequential order and the scattering height coincide whenever the sequential order is a successor ordinal.

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Corollary

Under CH there are Banach spaces with weak-sequential dual balls of any sequential order $< \omega$ and Banach spaces with arbitrarily large countable sequential order.*

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