# The Bartle–Dunford–Schwartz and the Dinculeanu–Singer Theorems Revisited

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# Index

# Index

**Theorem (Bartle–Dunford–Schwartz, 1955).**

For every operator $S \in \mathcal{L}(\mathcal{C}(\Omega), Y)$ there exists a unique vector measure $\mu : \Sigma \to Y^{**}$ of bounded semivariation such that

$$S\varphi = \int_\Omega \varphi \, d\mu \quad \text{for all } \varphi \in \mathcal{C}(\Omega).$$

**Theorem (Bartle–Dunford–Schwartz, 1955).**

Let $Y$ be a Banach space and let $\Omega$ be a compact Hausdorff space. For every $S \in \mathcal{L}(\mathcal{C}(\Omega), Y)$ there exists a weak*-countably additive measure $\mu : \Sigma \to Y^{**}$ such that

(i) $\langle \mu(\cdot), y^* \rangle$ is a regular countably additive Borel measure for each $y^* \in Y^*$;

(ii) the map $Y^* \to \mathcal{C}(\Omega)^*$, $y^* \mapsto \langle \mu(\cdot), y^* \rangle$, is weak*-to-weak* continuous;

(iii) $\langle S\varphi, y^* \rangle = \int_\Omega \varphi \, d(\langle \mu(\cdot), y^* \rangle)$, for each $\varphi \in \mathcal{C}(\Omega)$ and each $y^* \in Y^*$; and

(iv) $\|S\| = \|\mu\|(\Omega)$.

Conversely, any vector measure $\mu : \Sigma \to Y^{**}$ that satisfies (i) and (ii) defines an operator $S \in \mathcal{L}(\mathcal{C}(\Omega), Y)$ by means of (iii), and (iv) follows.

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**Theorem (Dinculeanu–Singer, 1959, 1965).**

For every operator $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$ there exists a unique vector measure $m : \Sigma \to \mathcal{L}(X, Y^{**})$ of bounded 1-semivariation such that

$$Uf = \int_\Omega f \, dm \quad \text{for all } f \in \mathcal{C}(\Omega, X).$$

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For every $x \in X$, we define an operator $S_x \in \mathcal{L}(\mathcal{C}(\Omega), Y)$ by

$$S_x\varphi = (S\varphi)x, \varphi \in \mathcal{C}(\Omega).$$

(B–D–S) $\quad S_x \in \mathcal{L}(\mathcal{C}(\Omega), Y) \quad \leftrightarrow \quad m_x : \Sigma \to Y^{**}$

We define $m : \Sigma \to \mathcal{L}(X, Y^{**})$ by

$$\langle y^*, m(E)x \rangle = \langle y^*, m_x(E) \rangle, \text{for all } x \in X \text{ and } y^* \in Y^*.$$

$$
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**Question.**

What is the relation between $\mu : \Sigma \to \mathcal{L}(X, Y)^{**}$ and $m : \Sigma \to \mathcal{L}(X, Y^{**})$?

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$$J : \mathcal{L}(X, Y) \to \mathcal{L}(X, Y^{**}), \ J(A) = j_Y A, \ A \in \mathcal{L}(X, Y),$$

be the natural isometric embedding, where $j_Y : Y \to Y^{**}$, and

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be the natural projection.

**Corollary**

$$m(E) = PJ^{**}\mu(E) \quad \text{for all } E \in \Sigma.$$

Moreover, if $S$ is weakly compact, then $m = \mu : \Sigma \to \mathcal{L}(X, Y)$.

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**Theorem (Grothendieck compactness principle, 1955).**

$K \subset X$ is **relatively compact** iff there exists $(x_n) \in c_0(X)$ such that

$$K \subset \{\textstyle\sum_n a_n x_n \ : \ (a_n) \in B_{\ell_1}\}.$$

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**Definition (Sinha–Karn, 2002).**

Let $1 \leq p \leq \infty$ and $1/p + 1/p' = 1$ (with $1/\infty = 0$).
$K \subset X$ is **relatively $p$-compact** iff there exists $(x_n) \in \ell_p(X)$ ($(x_n) \in c_0(X)$ if $p = \infty$) such that

$$K \subset p\text{-co}(x_n) := \{\sum_n a_n x_n \,:\, (a_n) \in B_{\ell_{p'}}\}.$$

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- $1 \le p \le q \le \infty \Rightarrow p$-compact sets are $q$-compact.

Of course, $p$-compact sets are compact.

## Definition: $p$-continuous $X$-valued functions (Muñoz–Oja–Piñeiro, 2015).

Let $X$ be a Banach space and let $\Omega$ be a compact Hausdorff space. Let $1 \leq p \leq \infty$.

$$\mathcal{C}_p(\Omega, X) = \{f \in \mathcal{C}(\Omega, X) \,:\, f(\Omega) \text{ is } p\text{-compact } \}$$

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**Theorem (Muñoz–Oja–Piñeiro, 2015).**

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- Given $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$, we consider the *associated operator* $U^\# \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$, defined by
$$(U^\# \varphi)x = U(\varphi x) \quad \text{for all } \varphi \in \mathcal{C}(\Omega), x \in X.$$

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- **Thus, both measures coincide.**

# Index

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| $\int_\Omega \varphi \, dm$ | | |
| $\|m\|(\Omega) < \infty$ | | |

$\|m\|(\Omega) = \sup \left\| \sum_{E_i \in \Pi} \varepsilon_i m(E_i) \right\|$
$\Pi = (E_i)_{i=1}^n$ finite partitions of $\Omega$ and all finite systems $(\varepsilon_i)_{i=1}^n$ with $|\varepsilon_i| \leq 1$, $1 \leq i \leq n$

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| $\|m\|(\Omega) < \infty$ | $\|m\|_{p'}(\Omega) < \infty$ | $\|m\|_{G-D}(\Omega) < \infty$ |

$$\|m\|_{G-D}(\Omega) = \sup_{y^* \in B_{Y^*}} |m_{y^*}|(\Omega), \text{ with } m_{y^*} := (m(\cdot))^* y^* : \Sigma \to X^*$$

| $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X,Y))$ | $U \in \mathcal{L}(\mathcal{C}_p(\Omega,X),Y)$ | $U \in \mathcal{L}(\mathcal{C}(\Omega,X),Y)$ |
|---|---|---|
| $m : \Sigma \to \mathcal{L}(X, Y^{**})$ | | |
| $\int_\Omega \varphi\, dm$ | $\int_\Omega f\, dm$ | $\int_\Omega f\, dm$ |
| $\|m\|(\Omega) < \infty$ | $\|m\|_{p'}(\Omega) < \infty$ | $\|m\|_{G-D}(\Omega) < \infty$ |

$$\|m\|_{G-D}(\Omega) = \sup_{y^* \in B_{Y^*}} |m_{y^*}|(\Omega), \text{ with } m_{y^*} := (m(\cdot))^* y^* : \Sigma \to X^*$$

For every $y^* \in Y^*$, let $m_{y^*} := (m(\cdot))^* y^* : \Sigma \to X^*$.
Consider the integral operator $I_{y^*} \in \mathcal{L}(\mathcal{C}(\Omega), X^*)$ defined by

$$I_{y^*}\varphi = \int_\Omega \varphi\, dm_{y^*}.$$

| $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ | $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ | $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$ |
|---|---|---|
| $m : \Sigma \to \mathcal{L}(X, Y^{**})$ | | |
| $\int_\Omega \varphi \, dm$ | $\int_\Omega f \, dm$ | $\int_\Omega f \, dm$ |
| $\|m\|(\Omega) < \infty$ | $\|m\|_{p'}(\Omega) < \infty$ | $\|m\|_{G-D}(\Omega) < \infty$ |

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**Definition ($q$-semivariation).**

$1 \leq q \leq \infty$

$$\|m\|_q(\Omega) = \sup_{y^* \in B_{Y^*}} \|I_{y^*}\|_{\mathcal{P}_q}$$

| $S \in \mathcal{L}(\mathcal{C}(\Omega), \mathcal{L}(X, Y))$ | $U \in \mathcal{L}(\mathcal{C}_p(\Omega, X), Y)$ | $U \in \mathcal{L}(\mathcal{C}(\Omega, X), Y)$ |
|---|---|---|
| $m : \Sigma \to \mathcal{L}(X, Y^{**})$ | | |
| $\int_\Omega \varphi \, dm$ | $\int_\Omega f \, dm$ | $\int_\Omega f \, dm$ |
| $\|m\|_\infty(\Omega) < \infty$ | $\|m\|_{p'}(\Omega) < \infty$ | $\|m\|_{G-D}(\Omega) < \infty$ |

$$\|m\|_{G-D}(\Omega) = \sup_{y^* \in B_{Y^*}} |m_{y^*}|(\Omega), \text{ with } m_{y^*} := (m(\cdot))^* y^* : \Sigma \to X^*$$

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|---|---|---|
| $m : \Sigma \to \mathcal{L}(X, Y^{**})$ | | |
| $\int_\Omega \varphi \, dm$ | $\int_\Omega f \, dm$ | $\int_\Omega f \, dm$ |
| $\|m\|_\infty(\Omega) < \infty$ | $\|m\|_{p'}(\Omega) < \infty$ | $\|m\|_1(\Omega) < \infty$ |

$\|m\|_1(\Omega) = \sup_{y^* \in B_{Y^*}} |m_{y^*}|(\Omega)$, with $m_{y^*} := (m(\cdot))^* y^* : \Sigma \to X^*$

For every $y^* \in Y^*$, let $m_{y^*} := (m(\cdot))^* y^* : \Sigma \to X^*$.

Consider the integral operator $I_{y^*} \in \mathcal{L}(\mathcal{C}(\Omega), X^*)$ defined by

$$I_{y^*}\varphi = \int_\Omega \varphi \, dm_{y^*}.$$

**Definition ($q$-semivariation).**

$1 \le q \le \infty$

$$\|m\|_q(\Omega) = \sup_{y^* \in B_{Y^*}} \|I_{y^*}\|_{\mathcal{P}_q}$$

# The Bartle–Dunford–Schwartz and the Dinculeanu–Singer Theorems Revisited

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