On groups of Hölder diffeomorphisms and their regularity

Joint work with Armin Rainer

David Nenning

University of Vienna

Valencia, October 17, 2017

Diffeomorphism groups generated by time-dependent vector fields

Diffeomorphism groups generated by time-dependent vector fields

- $C_0^1(\mathbb{R}^d, \mathbb{R}^d)\ldots$ Space of continuously differentiable functions vanishing at infinity together with their first derivative.

Diffeomorphism groups generated by time-dependent vector fields

- $C_0^1(\mathbb{R}^d, \mathbb{R}^d) \dots$ Space of continuously differentiable functions vanishing at infinity together with their first derivative.
- For $u : I \times \mathbb{R}^d \to \mathbb{R}^d$ such that
  - $u(\cdot, x)$ is measurable for all $x \in \mathbb{R}^d$,
  - $u(t, \cdot) \in C_0^1$ for all $t \in I$,
  - $\int_0^1 \|u(t, \cdot)\|_{C^1}\, dt < \infty$,

  the ODE

  $$\Phi(t) = x + \int_s^t u(r, \Phi(r))\, dr$$

  admits a unique solution for each fixed $s \in I,\ x \in \mathbb{R}^d$; denoted as $\Phi_u(t, s, x) = x + \phi_u(t, s, x)$.

Diffeomorphism groups generated by time-dependent vector fields

- $C_0^1(\mathbb{R}^d, \mathbb{R}^d) \ldots$ Space of continuously differentiable functions vanishing at infinity together with their first derivative.
- For $u : I \times \mathbb{R}^d \to \mathbb{R}^d$ such that
  - $u(\cdot, x)$ is measurable for all $x \in \mathbb{R}^d$,
  - $u(t, \cdot) \in C_0^1$ for all $t \in I$,
  - $\int_0^1 \|u(t, \cdot)\|_{C^1}\, dt < \infty$,

  the ODE
  $$\Phi(t) = x + \int_s^t u(r, \Phi(r))\, dr$$
  admits a unique solution for each fixed $s \in I,\ x \in \mathbb{R}^d$; denoted as $\Phi_u(t, s, x) = x + \phi_u(t, s, x)$.
- $\Phi_u(t, s, \cdot)$ is a C^1-diffeomorphism and $\phi_u(t, s, \cdot) \in C_0^1$.

Diffeomorphism groups generated by time-dependent vector fields

Diffeomorphism groups generated by time-dependent vector fields

For a locally convex space $E \hookrightarrow C_0^1(\mathbb{R}^d, \mathbb{R}^d)$, we set $\mathfrak{X}_E \ldots$ Space of **pointwise time-dependent** E**-vector fields** $u : I \times \mathbb{R}^d \to \mathbb{R}^d$ with

- $u(\cdot, x)$ is measurable for all $x \in \mathbb{R}^d$,
- $u(t, \cdot) \in E$ for all $t \in I$,
- $\int_0^1 p(u(t, \cdot))\, dt < \infty$ for all continuous seminorms p.

Diffeomorphism groups generated by time-dependent vector fields

For a locally convex space $E \hookrightarrow C_0^1(\mathbb{R}^d, \mathbb{R}^d)$, we set $\mathfrak{X}_E \ldots$ Space of **pointwise time-dependent** E**-vector fields** $u : I \times \mathbb{R}^d \to \mathbb{R}^d$ with

- $u(\cdot, x)$ is measurable for all $x \in \mathbb{R}^d$,
- $u(t, \cdot) \in E$ for all $t \in I$,
- $\int_0^1 p(u(t, \cdot))\, dt < \infty$ for all continuous seminorms p.

Definition (Trouvé group)

$\mathcal{G}_E := \left\{ \Phi_u(1, 0, \cdot) : u \in \mathfrak{X}_E \right\}$.

Diffeomorphism groups generated by time-dependent vector fields

For a locally convex space $E \hookrightarrow C_0^1(\mathbb{R}^d, \mathbb{R}^d)$, we set $\mathfrak{X}_E \ldots$ Space of **pointwise time-dependent E-vector fields** $u : I \times \mathbb{R}^d \to \mathbb{R}^d$ with

- $u(\cdot, x)$ is measurable for all $x \in \mathbb{R}^d$,
- $u(t, \cdot) \in E$ for all $t \in I$,
- $\int_0^1 p(u(t, \cdot))\, dt < \infty$ for all continuous seminorms p.

Definition (Trouvé group)

$\mathcal{G}_E := \left\{ \Phi_u(1, 0, \cdot) : u \in \mathfrak{X}_E \right\}$.

Lemma

$\mathcal{G}_E$ is a group with respect to composition.

Diffeomorphism groups generated by time-dependent vector fields

For a locally convex space $E \hookrightarrow C^1_0(\mathbb{R}^d, \mathbb{R}^d)$, we set $\mathfrak{X}_E \ldots$ Space of **pointwise time-dependent E-vector fields** $u : I \times \mathbb{R}^d \to \mathbb{R}^d$ with

- $u(\cdot, x)$ is measurable for all $x \in \mathbb{R}^d$,
- $u(t, \cdot) \in E$ for all $t \in I$,
- $\int_0^1 p(u(t,\cdot))\, dt < \infty$ for all continuous seminorms p.

Definition (Trouvé group)

$\mathcal{G}_E := \left\{ \Phi_u(1, 0, \cdot) : u \in \mathfrak{X}_E \right\}.$

Lemma

$\mathcal{G}_E$ is a group with respect to composition.

Definition (ODE-closedness)

E is called ODE-closed iff $\mathcal{G}_E \subseteq \mathrm{Id} + E$.

Hölder Diffeomorphism groups

Hölder Diffeomorphism groups

- $C_0^{n,\beta}(\mathbb{R}^d,\mathbb{R}^d)\ldots$ Space of n times continuously differentiable functions vanishing at infinity together with their derivatives up to order n and β-Hölder continuous n-th derivative with global Hölder constant, i.e. for $f \in C_0^{n,\beta}$, $f^{(k)}(x) \to 0$ as $x \to \infty$ for $0 \le k \le n$, and

$$\|f\|_{n,\beta} := \max\{\|f^{(k)}\|_{L^\infty(\mathbb{R}^d, L_k(\mathbb{R}^d,\mathbb{R}^d))} : 0 \le k \le n; \sup_{x,y\in\mathbb{R}^d} \frac{\|f^{(n)}(x) - f^{(n)}(y)\|_{L_n(\mathbb{R}^d,\mathbb{R}^d)}}{\|x-y\|^\beta}\} < \infty.$$

Hölder Diffeomorphism groups

- $C_0^{n,\beta}(\mathbb{R}^d,\mathbb{R}^d)\dots$Space of n times continuously differentiable functions vanishing at infinity together with their derivatives up to order n and β-Hölder continuous n-th derivative with global Hölder constant, i.e. for $f\in C_0^{n,\beta}$, $f^{(k)}(x)\to 0$ as $x\to\infty$ for $0\le k\le n$, and

$$\|f\|_{n,\beta} := \max\{\|f^{(k)}\|_{L^\infty(\mathbb{R}^d,L_k(\mathbb{R}^d,\mathbb{R}^d))} : 0\le k\le n; \sup_{x,y\in\mathbb{R}^d}\frac{\|f^{(n)}(x)-f^{(n)}(y)\|_{L_n(\mathbb{R}^d,\mathbb{R}^d)}}{\|x-y\|^\beta}\}<\infty.$$

- The set of orientation preserving diffeomorphisms that differ from the identity by a $C_0^{n,\beta}$-function is denoted as

$$\mathrm{Diff}\, C_0^{n,\beta} := \{\Phi\in \mathrm{Id}+C_0^{n,\beta} : \det\Phi'(x)>0\ \forall x\in\mathbb{R}^d\}.$$

It is a Banach manifold modelled on $C_0^{n,\beta}$ with global chart $\Phi\mapsto\Phi-\mathrm{Id}$. $(\mathrm{Diff}\, C_0^{n,\beta})_0$ shall denote the connected component of the identity.

Hölder Diffeomorphism groups

Hölder Diffeomorphism groups

Theorem 1

Diff $C_0^{n,\beta}$ *is a group with respect to composition. Right translations are smooth, but left translations are in general discontinuous.*

Hölder Diffeomorphism groups

Theorem 1

Diff $C_0^{n,\beta}$ *is a group with respect to composition. Right translations are smooth, but left translations are in general discontinuous.*

Theorem 2

$\mathcal{G}_{n,\beta} = (\text{Diff } C_0^{n,\beta})_0$. *In particular* $C_0^{n,\beta}$ *is ODE-closed.*

Proof(sketch): Diff $C_0^{n,\beta}$ is a group

Proof(sketch): Diff $C_0^{n,\beta}$ is a group

- Composition closedness: Classical proofs for composition closedness of $C^{n,\beta}(\mathbb{R}^d, \mathbb{R}^d)$ can be modified for $(\mathrm{Id} + \phi, \mathrm{Id} + \psi) \mapsto (\mathrm{Id} + \phi) \circ (\mathrm{Id} + \psi)$. Right translation (in chart representation) is affine and continuous and therefore smooth.

Proof(sketch): Diff $C_0^{n,\beta}$ is a group

- Composition closedness: Classical proofs for composition closedness of $C^{n,\beta}(\mathbb{R}^d, \mathbb{R}^d)$ can be modified for $(\mathrm{Id} + \phi, \mathrm{Id} + \psi) \mapsto (\mathrm{Id} + \phi) \circ (\mathrm{Id} + \psi)$. Right translation (in chart representation) is affine and continuous and therefore smooth.
- Inversion closedness: Let $\Phi = \mathrm{Id} + \phi$, $\Phi^{-1} = \Psi = \mathrm{Id} + \psi$. Then

$$\psi \circ \Phi = -\phi \in C_0^{n,\beta}.$$

Apply **Faà di Bruno's formula**, i.e.

$$(\psi \circ \Phi)^{(n)}(x) = \mathrm{sym} \sum_{l=1}^{n} \sum_{\gamma \in \Gamma(l,n)} c_\gamma \psi^{(l)}(\Phi(x)) \cdot (\Phi^{(\gamma_1)}(x), \cdots, \Phi^{(\gamma_l)}(x)),$$

where $\Gamma(l, n) := \{\gamma \in \mathbb{N}_{>0}^l : |\gamma| = n\}$, $c_\gamma := \frac{n!}{l!\gamma!}$, and sym denotes the symmetrization of multilinear mappings.

Proof(sketch): Diff $C_0^{n,\beta}$ is a group

Proof(sketch): Diff $C_0^{n,\beta}$ is a group

This yields

$$\begin{aligned}
&\psi^{(n)}(\Phi(x))(\Phi'(x),\cdots,\Phi'(x)) - \psi^{(n)}(\Phi(y))(\Phi'(y),\cdots,\Phi'(y)) \\
= \quad & -(\phi^{(n)}(x) - \phi^{(n)}(y)) \\
& - \operatorname{sym}\Bigg(\sum_{l=1}^{n-1}\sum_{\gamma\in\Gamma(l,n)} c_\gamma \psi^{(l)}(\Phi(x))\cdot(\Phi^{(\gamma_1)}(x),\cdots,\Phi^{(\gamma_l)}(x)) \\
& \quad - \sum_{l=1}^{n-1}\sum_{\gamma\in\Gamma(l,n)} c_\gamma \psi^{(l)}(\Phi(y))\cdot(\Phi^{(\gamma_1)}(y),\cdots,\Phi^{(\gamma_l)}(y))\Bigg),
\end{aligned}$$

which (after some manipulations) yields $\psi \in C_0^{n,\beta}$.

Proof(sketch): $\mathcal{G}_{n,\beta} \subseteq (\mathrm{Diff}\ C_0^{n,\beta})_0$

Proof(sketch): $\mathcal{G}_{n,\beta} \subseteq (\text{Diff } C_0^{n,\beta})_0$

Let u be a pointwise time-dependent $C_0^{n,\beta}$-vector field.

Proof(sketch): $\mathcal{G}_{n,\beta} \subseteq (\mathrm{Diff}\, C_0^{n,\beta})_0$

Let u be a pointwise time-dependent $C_0^{n,\beta}$-vector field.
Recall: $u : I \times \mathbb{R}^d \to \mathbb{R}^d$ such that

- $u(\cdot, x)$ is measurable for all $x \in \mathbb{R}^d$,
- $u(t, \cdot) \in C_0^{n,\beta}$ for all $t \in I$,
- $\int_0^1 \|u(t, \cdot)\|_{n,\beta}\, dt < \infty$.

Proof(sketch): $\mathcal{G}_{n,\beta} \subseteq (\mathrm{Diff}\, C_0^{n,\beta})_0$

Let u be a pointwise time-dependent $C_0^{n,\beta}$-vector field.
Recall: $u : I \times \mathbb{R}^d \to \mathbb{R}^d$ such that

- $u(\cdot, x)$ is measurable for all $x \in \mathbb{R}^d$,
- $u(t, \cdot) \in C_0^{n,\beta}$ for all $t \in I$,
- $\int_0^1 \|u(t,\cdot)\|_{n,\beta}\, dt < \infty$.

Show:

$$t \mapsto \phi_u(t,\cdot) \in C(I, C_0^{n,\beta}),$$

where

$$\Phi_u(t,x) = x + \phi_u(t,x) = x + \int_0^t u(s, \Phi_u(s,x))\, ds.$$

Proof(sketch): $\mathcal{G}_{n,\beta} \subseteq (\mathrm{Diff}\ C_0^{n,\beta})_0$

Proof(sketch): $\mathcal{G}_{n,\beta} \subseteq (\mathrm{Diff}\ C_0^{n,\beta})_0$

It is well known that $\phi_u(t,\cdot) \in C^n$ and there are constants C_1, C_2 such that for all $t \in I$

$$\|\phi_u(t,\cdot)\|_{C^n} \leq C_1 e^{C_2 \int_0^1 \|u(s,\cdot)\|_{C^n}\, ds}.$$

Proof(sketch): $\mathcal{G}_{n,\beta} \subseteq (\text{Diff } C_0^{n,\beta})_0$

It is well known that $\phi_u(t,\cdot) \in C^n$ and there are constants C_1, C_2 such that for all $t \in I$

$$\|\phi_u(t,\cdot)\|_{C^n} \leq C_1 e^{C_2 \int_0^1 \|u(s,\cdot)\|_{C^n} \, ds}.$$

And $\partial_x^n \phi_u(t,x)$ fulfills

$$\partial_x^n \phi_u(t,x) = \int_0^t \partial_x^n(u(s,\Phi_u(s,x)))\, ds. \tag{1}$$

Proof(sketch): $\mathcal{G}_{n,\beta} \subseteq (\mathrm{Diff}\ C_0^{n,\beta})_0$

It is well known that $\phi_u(t,\cdot) \in C^n$ and there are constants C_1, C_2 such that for all $t \in I$

$$\|\phi_u(t,\cdot)\|_{C^n} \leq C_1 e^{C_2 \int_0^1 \|u(s,\cdot)\|_{C^n}\, ds}.$$

And $\partial_x^n \phi_u(t,x)$ fulfills

$$\partial_x^n \phi_u(t,x) = \int_0^t \partial_x^n (u(s,\Phi_u(s,x))\, ds. \tag{1}$$

Consider

$$V_{x,y}^n(t) := \frac{\partial_x^n \phi_u(t,x) - \partial_x^n \phi_u(t,y)}{\|x-y\|^\beta}.$$

Proof(sketch): $\mathcal{G}_{n,\beta} \subseteq (\mathrm{Diff}\ C_0^{n,\beta})_0$

It is well known that $\phi_u(t,\cdot) \in C^n$ and there are constants C_1, C_2 such that for all $t \in I$

$$\|\phi_u(t,\cdot)\|_{C^n} \leq C_1 e^{C_2 \int_0^1 \|u(s,\cdot)\|_{C^n}\, ds}.$$

And $\partial_x^n \phi_u(t,x)$ fulfills

$$\partial_x^n \phi_u(t,x) = \int_0^t \partial_x^n (u(s,\Phi_u(s,x)))\, ds. \qquad (1)$$

Consider

$$V_{x,y}^n(t) := \frac{\partial_x^n \phi_u(t,x) - \partial_x^n \phi_u(t,y)}{\|x-y\|^\beta}.$$

- Apply Faà di Bruno's formula to the integrand in (1); yields linear ODE for $V_{x,y}^n$.

Proof(sketch): $\mathcal{G}_{n,\beta} \subseteq (\text{Diff } C_0^{n,\beta})_0$

It is well known that $\phi_u(t,\cdot) \in C^n$ and there are constants C_1, C_2 such that for all $t \in I$

$$\|\phi_u(t,\cdot)\|_{C^n} \leq C_1 e^{C_2 \int_0^1 \|u(s,\cdot)\|_{C^n}\, ds}.$$

And $\partial_x^n \phi_u(t,x)$ fulfills

$$\partial_x^n \phi_u(t,x) = \int_0^t \partial_x^n (u(s, \Phi_u(s,x)))\, ds. \tag{1}$$

Consider

$$V_{x,y}^n(t) := \frac{\partial_x^n \phi_u(t,x) - \partial_x^n \phi_u(t,y)}{\|x-y\|^\beta}.$$

- Apply Faà di Bruno's formula to the integrand in (1); yields linear ODE for $V_{x,y}^n$.
- Gronwall's inequality together with integrability of $t \mapsto \|u(t,\cdot)\|_{n,\beta}$ yields uniform boundedness w.r.t. x, y, t of $V_{x,y}^n(t)$.

Proof(sketch): $\mathcal{G}_{n,\beta} \subseteq (\mathrm{Diff}\ C_0^{n,\beta})_0$

It is well known that $\phi_u(t,\cdot) \in C^n$ and there are constants C_1, C_2 such that for all $t \in I$

$$\|\phi_u(t,\cdot)\|_{C^n} \leq C_1 e^{C_2 \int_0^1 \|u(s,\cdot)\|_{C^n}\, ds}.$$

And $\partial_x^n \phi_u(t,x)$ fulfills

$$\partial_x^n \phi_u(t,x) = \int_0^t \partial_x^n (u(s, \Phi_u(s,x)))\, ds. \tag{1}$$

Consider

$$V_{x,y}^n(t) := \frac{\partial_x^n \phi_u(t,x) - \partial_x^n \phi_u(t,y)}{\|x-y\|^\beta}.$$

- Apply Faà di Bruno's formula to the integrand in (1); yields linear ODE for $V_{x,y}^n$.
- Gronwall's inequality together with integrability of $t \mapsto \|u(t,\cdot)\|_{n,\beta}$ yields uniform boundedness w.r.t. x, y, t of $V_{x,y}^n(t)$.
- Integrability also gives continuity into $C_0^{n,\beta}$.

Proof(sketch): $\mathcal{G}_{n,\beta} \supseteq (\mathrm{Diff}\ C_0^{n,\beta})_0$

Proof(sketch): $\mathcal{G}_{n,\beta} \supseteq (\mathrm{Diff}\ C_0^{n,\beta})_0$

- For $\Phi = \mathrm{Id} + \phi \in (\mathrm{Diff}\ C_0^{n,\beta})_0$ there exists a polygon in $(\mathrm{Diff}\ C_0^{n,\beta})_0$ with vertices $\mathrm{Id} = \Phi_0, \Phi_1 = \mathrm{Id} + \phi_1, \cdots, \Phi_k = \mathrm{Id} + \phi_k = \Phi$.

Proof(sketch): $\mathcal{G}_{n,\beta} \supseteq (\text{Diff } C_0^{n,\beta})_0$

- For $\Phi = \text{Id} + \phi \in (\text{Diff } C_0^{n,\beta})_0$ there exists a polygon in $(\text{Diff } C_0^{n,\beta})_0$ with vertices $\text{Id} = \Phi_0, \Phi_1 = \text{Id} + \phi_1, \cdots, \Phi_k = \text{Id} + \phi_k = \Phi$.
- $\Phi_1 = \Phi_{u_1}(1, \cdot)$ with $u_1(t,x) = \phi_1 \circ ((1-t)\,\text{Id} + t\Phi_1)^{-1}(x)$; and u_1 is pointwise time-dependent Hölder vector field $\Rightarrow \Phi_1 \in \mathcal{G}_{n,\beta}$.

Proof(sketch): $\mathcal{G}_{n,\beta} \supseteq (\mathrm{Diff}\, C_0^{n,\beta})_0$

- For $\Phi = \mathrm{Id} + \phi \in (\mathrm{Diff}\, C_0^{n,\beta})_0$ there exists a polygon in $(\mathrm{Diff}\, C_0^{n,\beta})_0$ with vertices $\mathrm{Id} = \Phi_0, \Phi_1 = \mathrm{Id} + \phi_1, \cdots, \Phi_k = \mathrm{Id} + \phi_k = \Phi$.
- $\Phi_1 = \Phi_{u_1}(1, \cdot)$ with $u_1(t,x) = \phi_1 \circ ((1-t)\,\mathrm{Id} + t\Phi_1)^{-1}(x)$; and u_1 is pointwise time-dependent Hölder vector field $\Rightarrow \Phi_1 \in \mathcal{G}_{n,\beta}$.
- Same argument gives $\Phi_2 \circ \Phi_1^{-1} \in \mathcal{G}_{n,\beta}$. Since $\mathcal{G}_{n,\beta}$ is a group, $\Phi_2 \in \mathcal{G}_{n,\beta}$. Iterate argument another $k-2$ times; yields $\Phi_k = \Phi \in \mathcal{G}_{n,\beta}$.

Further results

Further results

Definition (Intermediate Hölder spaces)

- $C_0^{n,\beta-}(\mathbb{R}^d,\mathbb{R}^d) := \bigcap_{\alpha\in(0,\beta)} C_0^{n,\alpha}(\mathbb{R}^d,\mathbb{R}^d)$,
- $C_0^{n,\beta+}(\mathbb{R}^d,\mathbb{R}^d) := \bigcup_{\alpha\in(\beta,1)} C_0^{n,\alpha}(\mathbb{R}^d,\mathbb{R}^d)$,

endowed with natural projective and inductive locally convex topologies.

Further results

Definition (Intermediate Hölder spaces)

- $C_0^{n,\beta-}(\mathbb{R}^d,\mathbb{R}^d) := \bigcap_{\alpha\in(0,\beta)} C_0^{n,\alpha}(\mathbb{R}^d,\mathbb{R}^d)$,
- $C_0^{n,\beta+}(\mathbb{R}^d,\mathbb{R}^d) := \bigcup_{\alpha\in(\beta,1)} C_0^{n,\alpha}(\mathbb{R}^d,\mathbb{R}^d)$,

endowed with natural projective and inductive locally convex topologies.

Theorem

- Diff $C_0^{n,\beta-}$ *is a topological group,*
- Diff $C_0^{n,\beta+}$, *group operations map smooth curves to continuous curves,*
- $\mathcal{G}_{n,\beta\pm} = (\mathrm{Diff}\ C_0^{n,\beta\pm})_0$.

Thanks for your attention!