Distortion of Lipschitz Functions on $c_0(\Gamma)$

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Introduction

Definition

Let X be a Banach space and $f : S_X \to \mathbb{R}$. We say f is **oscillation stable** if for every infinite dimensional subspace $Z \subset X$ and every $\varepsilon > 0$ there exists an infinite dimensional subspace $Y \subset Z$ such that $|f(x) - f(y)| \leq \varepsilon$ for every $x,y \in S_Y$.

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A function $f : S_X \to \mathbb{R}$ is said to be **distorted** if there exists an $\varepsilon > 0$ such that for every infinite dimensional subspace Y of X there exist $x,y \in S_Y$ such that $|f(x) - f(y)| > \varepsilon$.

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Every Lipschitz function $f : S_{c_0} \to \mathbb{R}$ is oscillation stable.

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Theorem (Odell, Schlumprecht)

There is a distorted Lipschitz function on ℓ_1*. For every* $1 < p < \infty$*, there is a distorted equivalent norm on* ℓ_p*.*

Nonseparable case

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Let $(X, \|\cdot\|)$ be a Banach space with a symmetric (possibly uncountable) Schauder basis $\{e_\gamma\}_{\gamma\in\Gamma}$, where Γ is any nonempty set. We say that a function $f : X \to \mathbb{R}$ is **symmetric** if the value $f(x)$ is preserved under any permutation of the coordinates of x.

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Theorem (Hájek, N.)

There is a 1*-Lipschitz symmetric function* $F: S_{c_0(\Gamma)} \to \mathbb{R}$*, taking values in* $[0,1]$*, such that for every nonseparable subspace* $Y \subseteq c_0(\Gamma)$ *there are points* $x, y \in S_Y$ *such that* $|F(x) - F(y)| > \frac{1}{4}$.

Proof

Definition

On $c_{00}(\omega_1)$ define equivalence $x \sim y$ whenever $|\operatorname{supp} x| = |\operatorname{supp} y|$ and there exists a bijection $f : \operatorname{supp} x \to \operatorname{supp} y$ such that $x(\gamma) = y(f(\gamma))$. We call every equivalence class $[x] \in X := c_{00}(\omega_1)/\sim$ a shape.

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Notation

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Lemma (Modified extension formula)

Suppose (M,d) is a metric space and $g : S \to \mathbb{R}$ a K-Lipschitz function on some $S \subseteq M$, taking values only in the interval $[0,1]$. Then the following formula defines a K-Lipschitz function $\overline{g} : M \to \mathbb{R}$, taking values only in $[0,1]$ such that $\overline{g}|_S = g$.

$$\overline{g}(x) = \min\left\{\inf_{y \in S}\{g(y) + Kd(x,y)\}, 1\right\}. \tag{1}$$

Thank you for your attention.