Hypercyclic Algebras

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(Joint work with J.Bès and A. Conejero)

Valencia, October 2016

Definition

Let X be a separable Fréchet space. An operator $T : X \to X$ is **hypercyclic** if there exists $f \in X$ such that

$$\overline{Orb(T, f)} = \overline{\{f, Tf, T^2 f, \dots\}} = X.$$

Such f is called a **hypercyclic vector** (for T). Also, we denote

$$HC(T) := \{ \text{ hypercyclic vectors } \}$$

T is called weakly mixing if

$$T \oplus T : X \times X \to X \times X$$

is hypercyclic.

Herrero 1991, Bourdon 1993, Bés 1999, Wengenroth 2003
T hypercyclic $\Rightarrow HC(T)$ contains a dense T-invariant linear manifold.

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Birkhoff, 1929

For each $0 \neq a \in \mathbb{C}$, the translation operator

$$T_a : H(\mathbb{C}) \to H(\mathbb{C}), \quad f(z) \mapsto f(z+a), \quad \text{is hypercyclic.}$$

MacLane, 1952

The differentiation operator $D : H(\mathbb{C}) \to H(\mathbb{C})$, $f \mapsto f'$, is hypercyclic.

Godefroy and Shapiro, 1991.

(a) $L : H(\mathbb{C}) \to H(\mathbb{C})$ satisfies $LD = DL \Leftrightarrow LT_a = T_aL$ for each $a \in \mathbb{C} \Leftrightarrow$ $L = \Phi(D)$, for some $\Phi \in H(\mathbb{C})$ of exponential type.

(b) If $\Phi \in H(\mathbb{C})$ is non-constant and of exponential type, $\Phi(D)$ is hypercyclic.

($\Phi = \sum_{n \geq 0} a_n z^n \in H(\mathbb{C})$ is of exponential type provided $|\Phi(z)| \leq C\ e^{A|z|}$.

If so, it induces $\Phi(D) : H(\mathbb{C}) \to H(\mathbb{C})$, $f \mapsto \sum_{n \geq 0} a_n D^n f$

Example: $T_a = \Phi(D)$ when $\Phi(z) = e^{az}$.)

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Let $\Phi \in H(\mathbb{C})$ non-constant of exponential type.
What else can we say about $HC(\Phi(D))$?

Petersson '06, Shkarin '10, Menet '13

$HC(\Phi(D))$ contains a closed infinite-dimensional subspace.

Q: Let $T = \Phi(D)$**. Does** $f \in HC(T) \overset{?}{\Rightarrow} f^2 \in HC(T)$**.**

Aron, Conejero, Peris, Seoane-Sepúlveda, 2007

- If $\Phi(z) = e^{az}$, NO! Indeed, if $f \in HC(T_a)$, then

$$0 \neq g \in \overline{\mathrm{Orb}(f^p, T_a)} \Rightarrow \text{ all zeroes of } g \text{ have multiplicity in } p\mathbb{N}.$$

- The set $\{f : \forall n \in \mathbb{N} : f^n \in HC(D)\}$ is residual in $H(\mathbb{C})$!

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D supports a hypercyclic algebra!

(Aron) Does $HC(\Phi(D))$ **contain an algebra, when** Φ **is a polynomial?**

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Bés, Conejero, P. 2016

If Φ is non-constant polynomial with $\Phi(0) = 0$, then $\Phi(D)$ supports a hypercyclic algebra.

Q: Can we eliminate the assumption $\Phi(0) = 0$?

Q: Does $HC(\Phi(D))$ contain an algebra if and only if Φ is a polynomial?

Bés, Conejero, P. 2017

Let Φ be of exponential type so that $\Phi^{-1}(\partial\mathbb{D})$ contains a nontrivial, strictly convex compact arc Γ satisfying

$$conv(\Gamma \cup \{0\}) \setminus (\Gamma \cup \{0\}) \subset \Phi^{-1}(\mathbb{D})$$

then $\Phi(D)$ supports a hypercyclic algebra.

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Bayart-Matheron

Let T be operator on the separable F-algebra X. If for each open U, V, W with $0 \in W$ and each $m \in \mathbb{N}$ there exists $P \in U$ and $q \in \mathbb{N}$ so that

1. $T^q P^j \in W$ for $0 \leq j < m$ and
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Corollary

The following operators support hypercyclic algebras

- $D + I$
- $(a_0 I + a_1 D^k)^n$ for $a_1 \neq 0, |a_0| \leq 1, n, k \in \mathbb{N}$,
- $DT_1 = De^D$
- $T_1 - aI = e^D - aI, 0 < a \leq 1.$
- $\cos(aD)$ and $\sin(aD)$, $a \neq 0$,

Corollary

The operator

$$\frac{d}{dx} : C^\infty(\mathbb{R}, \mathbb{C}) \to C^\infty(\mathbb{R}, \mathbb{C})$$

supports a hypercyclic algebra.

Q: What about translations on $C^\infty(\mathbb{R}, \mathbb{C})$?

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- $DT_1 = De^D$
- $T_1 - aI = e^D - aI, 0 < a \leq 1$.
- $\cos(aD)$ and $\sin(aD), a \neq 0$,

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The operator

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Let T be a hypercyclic multiplicative operator on the F-algebra X over $\mathbb{K}$ T.F.A.E.,

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Back to $D : H(\mathbb{C}) \to H(\mathbb{C})$.

(Aron) Does $HC(D)$ contain a non finitely generated algebra?

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References

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THANK YOU!