On the chances to extend a surjective isometry between the unit spheres of two operator algebras

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A tribute to Joe Diestel

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Introduction to the problem

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This result provides an idea about the utility of infinite-dimensional holomorphy in Functional Analysis. Certain clases of complex Banach spaces (like, for example, JB*-triples) can be defined in terms of the holomorphic properties of their open unit balls. Personally, I have spent time and efforts studying the connections of these clases with Functional Analysis.

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Today, I am talking on a problem which actually goes, a priori, against the foundations supporting the holomorphic structure.

Can we replace the open unit ball with a certainly smaller set with a poor topological structure?

More concretely, Can we infer some information about a Banach space X from the metric space given by its unit sphere equipped with the natural distance?

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A Banach space X satisfies the *Mazur-Ulam property* provided that for any Banach space Y, every surjective isometry between $S(X)$ and $S(Y)$ is the restriction of a surjective real linear isometry between the two spaces.

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Caution!!

Tingley's problem remains unsolved even in the simplest case of a surjective isometry between the unit spheres of two Banach spaces of dimension 2. However, some particular positive answers in classical Banach spaces have produced significant mathematical contributions.

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The conjugation on the unit sphere of $\mathbb{C}$ doesn't admit a complex linear extension to the whole $\mathbb{C}$. Like in the Mazur-Ulam theorem, we cannot expect complex linearity for the extension.

Positive answers to Tingley's problem

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[G.G. Ding, *Science in China*'2002, *Sci. China Ser. A*'2003, *Acta. Math. Sin. (Engl. Ser.)*'2004, *Sci. China Ser. A*'2004]

If $f : S(\ell_p(\Gamma, \mathbb{R})) \to S(\ell_p(\Delta, \mathbb{R}))$ is a surjective isometry (with $1 \leq p \leq \infty$), then f can be (uniquely) extended to a real linear surjective isometry from $\ell_p(\Gamma)$ onto $\ell_p(\Delta)$.

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[D. Tan , *Taiwanese J. Math.*'2011, *Nonlinear Anal.*'2011, *Acta. Math. Sin. (Engl. Ser.)*'2012]

Let (Ω, Σ, μ) be a σ-finite measure space and let Y be a real Banach space. Then every surjective isometry $f : S(L^p_{\mathbb{R}}(\Omega, \Sigma, \mu)) \to S(Y)$, with $1 \leq p \leq \infty$, can be uniquely extended to a surjective real linear isometry from $L^p_{\mathbb{R}}((\Omega, \Sigma, \mu))$ onto Y. That is, $L^p_{\mathbb{R}}(\Omega, \Sigma, \mu)$ satisfies the Mazur-Ulam property for every $1 \leq p \leq \infty$.

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Let L_1 and L_2 be locally compact Hausdorff spaces. Then every surjective isometry $f : S(C_0(L_1)) \to S(C_0(L_2))$ admits an extension to a surjective real linear isometry from $C_0(L_1)$ onto $C_0(L_2)$.

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[X. Fang, J. Wang, *Acta. Math. Sin. (Engl. Ser.)*'2006]
[R. Liu, *J. Math. Anal. Appl.,*'2007]

Let K be a compact Hausdorff space, and let E be a real normed space. Then every surjective isometry $f : S(C(K,\mathbb{R})) \to S(E)$ admits an extension to a surjective linear isometry from $C(K,\mathbb{R})$ onto E. That is, $C(K,\mathbb{R})$ satisfies the Mazur-Ulam property.

Non-commutative structures

Let ℓ_2^n denote a complex n-dimensional Hilbert space. The C*-algebra $B(\ell_2^n)$ of all bounded linear operators on ℓ_2^n (equipped with the operator norm) can be identified with $M_n(\mathbb{C})$.

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Let $M_n(\mathbb{C})$ denote the algebra of all complex $n \times n$ matrices and let B be another C*-algebra. Suppose $f : S(M_n(\mathbb{C})) \to S(B)$ is a surjective isometry. Then, there exists a surjective real linear isometry $T : M_n(\mathbb{C}) \to B$ whose restriction to $S(M_n(\mathbb{C}))$ is f.

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After Tanaka's contribution, it seems natural to consider Tingley's problem for surjective isometries between the unit spheres of two operator algebras of the form $K(H)$, $K(H)^* = C_1(H)$, and $K(H)^{**} = B(H)$, where H is an *arbitrary* complex Hilbert space.

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It is known that every element a in $K(H)$ can be (uniquely) written as a (possibly finite) sum

$$a = \sum_{n=1}^{\infty} \lambda_n \, \eta_n \otimes \xi_n,$$

where $(\lambda_n) \subset \mathbb{R}^+$, (ξ_n), (η_n) are orthonormal systems in H. The operator norm can be recovered by the expression $\|a\| = \|a\|_\infty = \sup\{\lambda_n : n \in \mathbb{N}\}$.

The space $C_1(H)$, of trace class operators on H, that is, the subspace of all $a \in K(H)$ such that

$$\|a\|_1 := \left(\sum_{n=1}^{\infty} |\lambda_n(a)| \right) < \infty.$$

$C_1(H)$ is a two-sided ideal of the space $B(H)$, and $(C_1, \|.\|_1)$ is a Banach algebra.

One of the key results in the study of Tingley's problem is the following:

[Cheng and Dong, JMAA'2011, Tanaka, JMAA'2016]

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We recall some notation. Given a Banach space X, and subsets $F \subseteq \mathcal{B}_X$ and $G \subseteq \mathcal{B}_{X^*}$, we define

$$F' = \{a \in \mathcal{B}_{X^*} : a(x) = 1 \ \forall x \in F\}, \quad G_\prime = \{x \in \mathcal{B}_X : a(x) = 1 \ \forall a \in G\}.$$

Then, F' is a weak* closed face of $\mathcal{B}_{X^*}$ and $G_\prime$ is a norm closed face of $\mathcal{B}_X$. The subset F is said to be a *norm-semi-exposed face* of $\mathcal{B}_X$ if $F = (F')_\prime$, while the subset G is called a *weak*-semi-exposed face* of $\mathcal{B}_{X^*}$ if $G = (G_\prime)'$.

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[F.J. Fernández-Polo, J. Garcés, A.M. Peralta, I. Villanueva, *Linear Algebra Appl.*'2017]

Let X and Y be Banach spaces satisfying the following two properties:

(1) Every norm closed face of $\mathcal{B}_X$ (respectively, of $\mathcal{B}_Y$) is norm-semi-exposed;

(2) Every weak* closed proper face of $\mathcal{B}_{X^*}$ (respectively, of $\mathcal{B}_{Y^*}$) is weak*-semi-exposed.

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Let $f : S(X) \to S(Y)$ be a surjective isometry. The following statements hold:

(*a*) Let $\mathcal{F}$ be a convex set in $S(X)$. Then $\mathcal{F}$ is a norm closed face of $\mathcal{B}_X$ if and only if $f(\mathcal{F})$ is a norm closed face of $\mathcal{B}_Y$;

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(*b*) Given $e \in S(X)$, we have that $e \in \partial_e(\mathcal{B}_X)$ if and only if $f(e) \in \partial_e(\mathcal{B}_Y)$. $\square$

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Let us observe that C*-algebras and preduals of von Neumann algebras satisfy the hypothesis of the above theorem.

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Let C be a maximal norm-closed face of $\mathcal{B}_{_{K(H)}}$. Then there exists a unique rank-one partial isometry $e = \eta\otimes\xi \in K(H)$ such that

$$C = e + \{x \in K(H) : \|x\| \leq 1, x = (1-\eta\otimes\eta)x(1-\xi\otimes\xi)\} = e + \left(\mathcal{B}_{_{K(H)}} \cap \{e\}^{\perp}\right).$$

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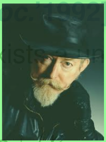

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Since every maximal norm-closed face is the translation (via a minimal partial isometry e) of the closed unit ball of another Banach space (the closed unit ball of the orthogonal complement of e), we are in position to apply a result of P. Mankiewicz which assures the extension of our surjective isometry to an affine function between the corresponding faces.

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[A.M. Peralta, R. Tanaka, preprint 2016]

Let $f : S(K(H)) \to S(K(H'))$ be a surjective isometry. Then there exists a surjective real linear isometry $T : K(H) \to K(H')$ whose restriction to $S(K(H))$ is f.

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[C.M. Edwards and G.T. Ruttimann, *J. London Math. Soc.*'1988]

Every (maximal) proper norm-closed face $\mathcal{F}$ of $\mathcal{B}_{C_1(H)}$ is of the form

$$\mathcal{F} = \{w\}_, = \{x \in C_1(H) = B(H)_* : \|x\|_1 = 1 = x(w)\},$$

for a unique (maximal or complete) partial isometry $w \in B(H)$. Furthermore the mapping $w \mapsto \{w\}_,$ is an order preserving bijection between the lattices of all partial isometries in $B(H)$ and all norm closed faces of $\mathcal{B}_{C_1(H)}$.

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Let $f : S(C_1(H)) \to S(C_1(H'))$ be a surjective isometry, where H and H' are complex Hilbert spaces. Then the following statements hold:

(a) A subset $\mathcal{F} \subset S(C_1(H))$ is a proper norm-closed face of $\mathcal{B}_{C_1(H)}$ if and only if $f(\mathcal{F})$ is.

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(c) $\dim(H) = \dim(H')$.

(d) For each $e_0 \in \partial_e(\mathcal{B}_{C_1(H)})$ we have $f(ie_0) = if(e_0)$ or $f(ie_0) = -if(e_0)$;

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(*d*) For each $e_0 \in \partial_e(\mathcal{B}_{C_1(H)})$ we have $f(ie_0) = if(e_0)$ or $f(ie_0) = -if(e_0)$;

(*e*) For each $e_0 \in \partial_e(\mathcal{B}_{C_1(H)})$ if $f(ie_0) = if(e_0)$ (respectively, $f(ie_0) = -if(e_0)$) then $f(\lambda e_0) = \lambda f(e_0)$ (respectively, $f(\lambda e_0) = \overline{\lambda} f(e_0)$) for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

More geometric formulae are exploited to compute the distance between pure atoms and between finite real linear combinations of pure atoms. More laborious techniques are applied to obtain our answer to Tingely's problem for trace class operators.

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[F.J. Fernández-Polo, J. Garcés, A.M. Peralta, I. Villanueva, *Linear Algebra Appl.*'2017]

Let $f: S(C_1(H)) \to S(C_1(H))$ be a surjective isometry, where H is an arbitrary complex Hilbert space. Then there exists a surjective complex linear or conjugate linear isometry $T: C_1(H) \to C_1(H)$ satisfying $f(x) = T(x)$ for every $x \in S(C_1(H))$.

The case of $B(H)$ is completely different and requires completely different techniques and ideas.

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The first handicap is that maximal norm closed proper faces of the closed unit ball of a C*-algebra A are determined by minimal partial isometries in it's second dual A^{**}.

[C.A. Akemann and G.K. Pedersen, *Proc. Lond. Math. Soc.*'1992]

Let A be a C*-algebra. The maximal norm closed proper faces of the closed unit ball of A have the form

$$F_v = (v + (1 - vv^*)\mathcal{B}_{A^{**}}(1 - v^* v)) \cap \mathcal{B}_A = \{x \in \mathcal{B}_A : \ xv^* = vv^*\},$$

for some minimal partial isometry v in A^{**}.

When the partial isometry e is already in A we can obtain more information about $f(e)$.

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[F.J. Fernández-Polo, A.M. Peralta, Trans. Amer. Math. Soc. to appear]

Let A and B be C*-algebras, and suppose that $f : S(A) \to S(B)$ is a surjective isometry. Suppose e is a minimal partial isometry in A. Then 1 is isolated in the spectrum of $|f(e)| = (f(e)^* f(e))^{\frac{1}{2}}$.

When the partial isometry e is already in A we can obtain more information about $f(e)$.

[F.J. Fernández-Polo, A.M. Peralta, Trans. Amer. Math. Soc. to appear]

Let A be a C*-algebra, and let H be a complex Hilbert space. Suppose that $f : S(A) \to S(B(H))$ is a surjective isometry, and let e be a minimal partial isometry in A. Then $f(e)$ is a minimal partial isometry in $B(H)$. Moreover, there exists a surjective real linear isometry

$$T_e : (1 - ee^*)A(1 - e^*e) \to (1 - f(e)f(e)^*)B(H)(1 - f(e)^*f(e))$$

such that

$$f(e + x) = f(e) + T_e(x), \text{ for all } x \text{ in } \mathcal{B}_{(1-ee^*)A(1-e^*e)}.$$

In particular the restriction of f to the face $F_e = e + (1 - ee^*)\mathcal{B}_A(1 - e^*e)$ is a real affine function.

A careful treatment on the image by f of a finite convex combinations of mutually orthogonal (minimal) partial isometries, combined with the about geometric results is the remaining ingredient in our solution to Tingley's problem in the case of $B(H)$ spaces.

A careful treatment on the image by f of a finite convex combinations of mutually orthogonal (minimal) partial isometries, combined with the about geometric results is the remaining ingredient in our solution to Tingley's problem in the case of $B(H)$ spaces.

[F.J. Fernández-Polo, A.M. Peralta, Trans. Amer. Math. Soc. to appear]

Let H and K be complex Hilbert spaces. Suppose that $f : S(B(K)) \to S(B(H))$ is a surjective isometry. Then there exists a surjective complex linear or conjugate linear surjective isometry $T : B(K) \to B(H)$ satisfying $f(x) = T(x)$, for every $x \in S(B(K))$.

That was all I had in mind for today.
Thanks for your time!

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