The linear structure of some dual Lipschitz-free spaces

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UBFC

1. Lipschitz free spaces
2. Natural preduals
3. Application to the Extremal Structure
4. Uniformly discrete and bounded case

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Definition

Lipschitz-free space over M :
$\mathcal{F}(M) := \overline{\text{span}\, \{\delta(x) \ : \ x \in M\}}^{\|\cdot\|} \subset Lip_0(M)^*$.

Proposition (Fundamental factorisation property)

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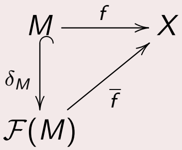

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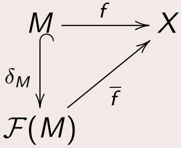

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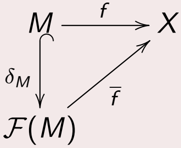

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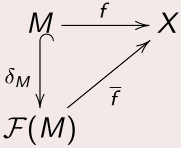

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ii) $M \underset{bi-Lip}{\hookrightarrow} N \Longrightarrow \mathcal{F}(N) \underset{linear}{\hookrightarrow} \mathcal{F}(M)$.

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ii) $\mathrm{lip}_0(\mathbb{N}) = \mathrm{Lip}_0(\mathbb{N})$, and also $\mathrm{lip}_0(D) = \mathrm{Lip}_0(D)$ for any uniformly discrete metric space D.

Definition

We say that a subspace $\mathrm{lip}_0(M)$ 1-separates points uniformly (1-S.P.U.) if $\forall \varepsilon > 0$ $\forall x \neq y \in M$, $\exists f \in \mathrm{lip}_0(M)$ with $\|f\|_L \leq 1 + \varepsilon$ and $|f(x) - f(y)| = d(x, y)$.

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Example (Trivial)

If M is a compact metric space, then every predual (if it exists) is natural.

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Now it is easy to check that τ and ω^* coincides on $\delta(M)$. So $\delta(M)$ is indeed ω^* closed. □

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2. There is a unif. discrete bounded and sep. metric space M such that $\mathcal{F}(M)$ admits both a natural predual and a non natural predual any natural predual.

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Aliaga and Guirao (2017) : Yes to 2) when M is compact, and yes to 1) when M is compact and $\mathrm{lip}_0(M)$ 1-S.P.U.

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Since $\|\gamma\| = 1$, $C = d(x, y)$. □

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Proof.

iii) $\implies$ ii) : Let $x \neq y \in M$ such that $[x, y] = \{x, y\}$.

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Thus $A = \overline{co}(ext(A)) = \{m_{xy}\}$, and so f_{xy} exposes m_{xy}. □

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$\rightarrow$ If no, there is an equivalent ℓ_1 norm that fails the (MAP).

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Luis García-Lirola, Colin Petitjean, Antonin Procházka, Abraham Rueda Zoca, *Extremal structure and Duality of Lipschitz free spaces*, preprint, Available at : https://arxiv.org/abs/1707.09307