On $\mathbb{Z}_d$-symmetry of spectra of linear operators in Banach spaces

Oleg Reinov

Saint Petersburg State University

M. I. Zelikin's Remark

All the main results below have their beginning in the following remark of M.I. Zelikin (Moscow State University):

Remark

The spectrum of a linear operator $A : \mathbb{R}^k \to \mathbb{R}^k$ *is central-symmetric iff the trace of any odd power of A equals zero:*

$$\text{trace } A^{2n-1} = 0, \ n \in \mathbb{N}.$$

Zelikin's Theorem

To formulate the theorem, we need a definition:

- The spectrum of A is central-symmetric, if together with any eigenvalue $\lambda \neq 0$ it has the eigenvalue $-\lambda$ of the same multiplicity.

  It was proved in a paper by M. I. Zelikin

M. I. Zelikin, A criterion for the symmetry of a spectrum", Dokl. Akad. Nauk 418 (2008), no. 6, 737-740

- **Theorem.** The spectrum of a nuclear operator A acting on a separable Hilbert space is central-symmetric iff $trace\ A^{2n-1} = 0$, $n \in \mathbf{N}$.

Zelikin's Theorem

To formulate the theorem, we need a definition:

- The spectrum of A is central-symmetric, if together with any eigenvalue $\lambda \neq 0$ it has the eigenvalue $-\lambda$ of the same multiplicity.
  It was proved in a paper by M. I. Zelikin

M. I. Zelikin, A criterion for the symmetry of a spectrum", Dokl. Akad. Nauk 418 (2008), no. 6, 737-740

- **Theorem.** The spectrum of a nuclear operator A acting on a separable Hilbert space is central-symmetric iff $trace\, A^{2n-1} = 0,\ n \in \mathbf{N}$.

Mityagin's $\mathbb{Z}_d$-symmetry

Definition

Let T be an operator in X, all non-zero spectral values of which are eigenvalues of finite multiplicity and have no limit point except possibly zero. For a fixed $d = 2, 3, \ldots$ and for the operator T, the spectrum of T is called $\mathbb{Z}_d$-symmetric, if $0 \neq \lambda \in \mathrm{sp}\,(T)$ implies $t\lambda \in \mathrm{sp}\,(T)$ for every $t \in \sqrt[d]{1}$ and of the same nultiplicity.

If $d = 2$, then one has the central symmetry.

Mityagin's Theorem

Theorem

Let X be a Banach space and $T : X \to X$ is a compact operator. Suppose that some power of T is nuclear. The spectrum of T is $\mathbb{Z}_d$-symmetric iff there is an integer $K \geq 0$ such that for every $l > Kd$ the value $\operatorname{trace} T^l$ *is well defined and*

$$\operatorname{trace} T^{kd+r} = 0$$

$$\text{for all } k = K, K+1, K+2, \dots \text{ and } r = 1, 2, \dots, d-1.$$

In the proof, the Riesz theory of compact operators is used.

B. S. Mityagin, *A criterion for the $\mathbb{Z}_d$-symmetry of the spectrum of a compact operator*, J. Operator Theory, **76**:1 (2016), 57–65.

Mityagin's Theorem

Theorem

Let X be a Banach space and $T : X \to X$ is a compact operator. Suppose that some power of T is nuclear. The spectrum of T is $\mathbb{Z}_d$-symmetric iff there is an integer $K \geq 0$ such that for every $l > Kd$ the value trace T^l *is well defined and*

$$\text{trace } T^{kd+r} = 0$$

$$\textit{for all } k = K, K+1, K+2, \dots \textit{ and } r = 1, 2, \dots, d-1.$$

In the proof, the Riesz theory of compact operators is used.

B. S. Mityagin, *A criterion for the $\mathbb{Z}_d$-symmetry of the spectrum of a compact operator*, J. Operator Theory, **76**:1 (2016), 57–65.

Our Generalization of Mityagin's Theorem

Theorem

Let X be a Banach space and $T : X \to X$ is a linear continuous operator. Suppose that some power of T is nuclear. The spectrum of T is $\mathbb{Z}_d$-symmetric iff there is an integer $K \geq 0$ such that for every $l > Kd$ the value $\operatorname{trace} T^l$ is well defined and

$$\operatorname{trace}\, T^{kd+r} = 0$$

for all $k = K, K+1, K+2, \ldots$ *and* $r = 1, 2, \ldots, d-1.$

Oleg Reinov, *Some remarks on spectra of nuclear operators*, SPb. Math. Society Preprint 2016-09, 1-9

In the proof we use the Fredholm theory of A. Grothendieck:

A. Grothendieck, *La théorie de Fredholm*, Bull. Soc. Math. France, **84** (1956), 319–384.

Our Generalization of Mityagin's Theorem

Theorem

Let X be a Banach space and $T : X \to X$ is a linear continuous operator. Suppose that some power of T is nuclear. The spectrum of T is $\mathbb{Z}_d$-symmetric iff there is an integer $K \geq 0$ such that for every $l > Kd$ the value $\operatorname{trace} T^l$ *is well defined and*

$$\operatorname{trace}\ T^{kd+r} = 0$$

$$\textit{for all } k = K, K+1, K+2, \ldots \textit{ and } r = 1, 2, \ldots, d-1.$$

Oleg Reinov, *Some remarks on spectra of nuclear operators*, SPb. Math. Society Preprint 2016-09, 1-9

In the proof we use the Fredholm theory of A. Grothendieck:

A. Grothendieck, *La théorie de Fredholm*, Bull. Soc. Math. France, **84** (1956), 319–384.

Our Generalization of Mityagin's Theorem

Theorem

Let X be a Banach space and $T : X \to X$ is a linear continuous operator. Suppose that some power of T is nuclear. The spectrum of T is $\mathbb{Z}_d$-symmetric iff there is an integer $K \geq 0$ such that for every $l > Kd$ the value $\operatorname{trace} T^l$ is well defined and

$$\operatorname{trace}\ T^{kd+r} = 0$$

$$\textit{for all}\ \ k = K, K+1, K+2, \dots \ \ \textit{and}\ \ r = 1, 2, \dots, d-1.$$

Oleg Reinov, *Some remarks on spectra of nuclear operators*, SPb. Math. Society Preprint 2016-09, 1-9

In the proof we use the Fredholm theory of A. Grothendieck:

A. Grothendieck, *La théorie de Fredholm*, Bull. Soc. Math. France, **84** (1956), 319–384.

Remark

Let Π_p be the ideal of absolutely p-summing operators. Then for some n one has $\Pi_p^n \subset N$. In particular, $\Pi_2^2(C[0,1]) \subset N(C[0,1])$, but not every absolutely 2-summing operator in $C[0,1]$ is compact.

The p-summing operators in such spaces provide instances of operators to which the last theorem may be applied even though Mityagin's Theorem is not always applicable.

General notation

X, Y Banach spaces.
$L(X, Y)$ — linear continuous operators.
For $T : X \to Y$,

$$||T|| = \sup\{||T(x)|| : \ x \in X, \ ||x|| \leq 1\}.$$

$X^* = L(X, \mathbb{C})$.
For $0 < p < \infty$,

$$l^p = \{(a_k) : \ a_k \in \mathbb{C}, \sum_{k=1}^{\infty} |a_k|^p \leq \infty\},$$

$||(a_k)||_{l^p} = \{\sum_{k=1}^{\infty} |a_k|^p\}^{1/p}$.

$$l^\infty = \{(a_k) : \ ||(a_k)||_{l^\infty} = \sup_k |a_k| < \infty\};$$

$$c_0 = \{(a_k) \subset l^\infty; \ a_k \to 0\}.$$

Preliminaries

$$\mathcal{F}(X,Y) = \{T \in L(X,Y) : \ \mathrm{rank}\, T < \infty\}$$

$$T \in \mathcal{F}(X,Y) \implies T(x) = \sum_{k=1}^{n} x_k'(x) y_k,$$

where $x_k' \in X^*$, $y_k \in Y$.
If $T \in \mathcal{F}(X,X)$, then $T(x) = \sum_{k=1}^{n} x_k'(x) x_k$ $(x_k' \in X^*,\ x_k \in X)$
and

$$\text{trace } T := \sum_{k=1}^{n} x_k'(x_k).$$

"Trace" does not depend on a representation of T and

$$\text{trace } T = \sum \textit{eigenvalues}\,(T)$$

(written according their multiplicities).

Nuclear representations

Also, a finite rank $T \in L(X, X)$. Consider a *nuclear* representation

$$Tx = \sum_{k=1}^{\infty} x'_k(x) x_k, \ \sum_{k=1}^{\infty} ||x'_k|| \, ||x_k|| < \infty$$

and

$$\alpha := \sum_{k=1}^{\infty} x'_k(x_k).$$

- **Question:** $\alpha = \text{trace } T$?
- Generally, NO.

Enflo P. , A counterexample to the approximation property in Banach spaces, Acta Math., Volume 130, 1973, 309–317

Nuclear representations

Also, a finite rank $T \in L(X, X)$. Consider a *nuclear* representation

$$Tx = \sum_{k=1}^{\infty} x'_k(x) x_k, \ \sum_{k=1}^{\infty} ||x'_k|| \, ||x_k|| < \infty$$

and

$$\alpha := \sum_{k=1}^{\infty} x'_k(x_k).$$

- **Question:** $\alpha = \text{trace } T$?
- Generally, NO.

Enflo P. , A counterexample to the approximation property in Banach spaces, Acta Math., Volume 130, 1973, 309–317

Nuclear representations

Also, a finite rank $T \in L(X, X)$. Consider a *nuclear* representation

$$Tx = \sum_{k=1}^{\infty} x'_k(x) x_k, \ \sum_{k=1}^{\infty} ||x'_k|| \, ||x_k|| < \infty$$

and

$$\alpha := \sum_{k=1}^{\infty} x'_k(x_k).$$

- **Question:** $\alpha =$ trace T?
- Generally, NO.

Enflo P. , A counterexample to the approximation property in Banach spaces, Acta Math., Volume 130, 1973, 309–317

Nuclear operators

Definition

$T : X \to Y$ is nuclear, if

$$\exists \; (x'_k) \subset X^*, (y_k) \subset Y : \; \sum_{k=1}^{\infty} ||x'_k|| \, ||y_k|| < \infty,$$

$$T(x) = \sum_{k=1}^{\infty} x'_k(x) y_k, \;\; \forall \; x \in X.$$

Remark: If T is nuclear, then $T : X \to c_0 \overset{\Delta}{\to} l_1 \to Y$. $\Delta \in l^1$.

Nuclear operators

Definition

$T : X \to Y$ is nuclear, if

$$\exists\ (x'_k) \subset X^*, (y_k) \subset Y :\ \sum_{k=1}^{\infty} ||x'_k||\,||y_k|| < \infty,$$

$$T(x) = \sum_{k=1}^{\infty} x'_k(x) y_k,\ \ \forall\ x \in X.$$

Remark: If T is nuclear, then $T : X \to c_0 \xrightarrow{\Delta} l_1 \to Y$. $\Delta \in l^1$.

s-Nuclear operators

Generally:

Definition

$T : X \to Y$ is s-nuclear $(0 < s \leq 1)$, *if*

$$\exists\ (x'_k) \subset X^*, (y_k) \subset Y:\ \sum_{k=1}^{\infty} ||x'_k||^s\, ||y_k||^s < \infty,$$

$$T(x) = \sum_{k=1}^{\infty} x'_k(x) y_k,\ \ \forall\ x \in X.$$

Remark: If T is s-nuclear, then $T : X \to c_0 \overset{\Delta}{\to} l_1 \to X,\ \ \Delta \in l^s.$

s-Nuclear operators

Generally:

Definition

$T : X \to Y$ is s-nuclear $(0 < s \leq 1)$, if

$$\exists\ (x'_k) \subset X^*, (y_k) \subset Y : \sum_{k=1}^{\infty} ||x'_k||^s\, ||y_k||^s < \infty,$$

$$T(x) = \sum_{k=1}^{\infty} x'_k(x) y_k, \ \ \forall\ x \in X.$$

Remark: If T is s-nuclear, then $T : X \to c_0 \overset{\Delta}{\to} l_1 \to X, \ \Delta \in l^s$.

Nuclear operators: Trace and AP

Definition

Let $T \in L(X, X)$ *be nuclear with*

$$T(x) = \sum_{k=1}^{\infty} x'_k(x) x_k, \ \ \forall \, x \in X.$$

If $\sum_{k=1}^{\infty} x'_k(x_k)$ **does not depend** *on a representation, then it is the (nuclear) trace of* T. *Notation:* trace T.

Definition

If every nuclear $T : X \to X$ *has a trace, then* X *has the AP.*

Nuclear operators: Trace and AP

Definition

Let $T \in L(X, X)$ *be nuclear with*

$$T(x) = \sum_{k=1}^{\infty} x_k'(x) x_k, \ \ \forall\, x \in X.$$

If $\sum_{k=1}^{\infty} x_k'(x_k)$ **does not depend** *on a representation, then it is the (nuclear) trace of* T*. Notation:* $\operatorname{trace} T$.

Definition

If every nuclear $T : X \to X$ *has a trace, then* X *has the AP.*

Grothendieck's *AP*

Grothendieck's Definition:

Definition

X has the AP if id_X is in the closure of $\mathcal{F}(X,X)$ in the topology of compact convergence:

$$\forall\ \varepsilon > 0,\ \forall \text{ compact } K \subset X\ \exists\ R \in \mathcal{F}(X,X):\ \sup_{x\in K} ||Rx - x|| < \varepsilon.$$

A. Grothendieck: *Produits tensoriels topologiques et éspaces nucléaires*, Mem. Amer. Math. Soc., **16**(1955).

Enflo P. , A counterexample to the approximation property in Banach spaces, Acta Math., Volume 130, 1973, 309–317

Grothendieck's Definition:

Definition

X has the AP if id_X is in the closure of $\mathcal{F}(X, X)$ in the topology of compact convergence:

$$\forall\ \varepsilon > 0,\ \forall \text{ compact } K \subset X\ \exists\ R \in \mathcal{F}(X, X) : \sup_{x \in K} ||Rx - x|| < \varepsilon.$$

A. Grothendieck: *Produits tensoriels topologiques et éspaces nucléaires*, Mem. Amer. Math. Soc., **16**(1955).

Enflo P. , A counterexample to the approximation property in Banach spaces, Acta Math., Volume 130, 1973, 309–317

Examples

- $AP : C(K), L_p(\mu), A, L_\infty/H^\infty$ *etc;*
- $\forall\ p \in [1, \infty] \setminus \{2\}\ \exists\ X \subset l_p : \quad X \notin AP$;
- $L(H) \notin AP$, H^∞ — *not known;*

A. Grothendieck:

Theorem

The following are equivalent:
1) Every Banach space has the approximation property.
2) If a nuclear operator $U : c_0 \to c_0$ is such that $\text{trace } U = 1$, then $U^2 \neq 0$.

By Enflo:

Theorem

There exists a nuclear operator $U : c_0 \to c_0$ such that $\text{trace } U = 1$ and $U^2 = 0$.

A characterization of AP

A. Grothendieck:

Theorem

The following are equivalent:
1) Every Banach space has the approximation property.
2) If a nuclear operator $U : c_0 \to c_0$ *is such that* $\operatorname{trace} U = 1$*, then* $U^2 \neq 0$.

By Enflo:

Theorem

There exists a nuclear operator $U : c_0 \to c_0$ *such that* $\operatorname{trace} U = 1$ *and* $U^2 = 0$.

Bad nuclear operators in l^1

Can be obtain from Davie's

Davie A.M., The approximation problem for Banach spaces, Bull. London Math. Soc., Vol 5, 1973, 261–266

Theorem

There exists a nuclear operator T in l^1 :
(i) T *is s-nuclear for every* $s \in (2/3, 1]$.
(ii) trace $T = 1$.
(iii) $T^2 = 0$.

A proof can be found in

A. Pietsch, Operator ideals, North-Holland, 1978.

Positive results

On the other hand:

A. Grothendieck:

Theorem

If T is 2/3-nuclear (in any X), then trace T *is well defined. Moreover, if* trace $T \neq 0$, *then* $T^2 \neq 0$.

V. B. Lidskiĭ:

Theorem

If $T : l^2 \to l^2$ is 1-nuclear, then trace T *is well defined. Moreover, if* trace $T \neq 0$, *then* $T^2 \neq 0$.

Can be found in

V. B. Lidskiĭ, *Nonselfadjoint operators having a trace*, Dokl. Akad. Nauk SSSR, **125**(1959), 485–487.

or in A. Pitsch's book.

Positive results

On the other hand:

A. Grothendieck:

Theorem

If T is 2/3-nuclear (in any X), then trace T *is well defined. Moreover, if* trace $T \neq 0$, *then* $T^2 \neq 0$.

V. B. Lidskiĭ:

Theorem

If $T : l^2 \to l^2$ is 1-nuclear, then trace T *is well defined. Moreover, if* trace $T \neq 0$, *then* $T^2 \neq 0$.

Can be found in

V. B. Lidskiĭ, *Nonselfadjoint operators having a trace*, Dokl. Akad. Nauk SSSR, **125**(1959), 485–487.

or in A. Pitsch's book.

Our aim

Thus, the cases of nuclear operators in c_0, l^1 and l^2 were considered above.

We are going to consider the cases where $1 < p < \infty$ and *to get an optimal results* (also in case of c_0).

Our main theorems:

Theorem

Let Y is a subspace of a quotient (or a quotient of a subspace) of some $L_p(\mu)$-space, $1 \leq p \leq \infty$ and $1/r = 1 + |1/2 - 1/p|$. If $T : Y \to Y$ is r-nuclear, then trace T *is well defined. For a fixed $d = 2, 3, \ldots,$ the spectrum of T is $\mathbb{Z}_d$-symmetric iff*

$$\text{trace } T^{kd+j} = 0 \text{ for all } k = 0, 1, 2, \ldots \text{ and } j = 1, 2, \ldots, d-1.$$

In particular, if trace $T \neq 0$, *then* $T^2 \neq 0$.

Theorem is optimal with respect to p and r :

Theorem

Let $p \in [1, \infty], p \neq 2$, $1/r = 1 + |1/2 - 1/p|$. There exists a nuclear operator V in l^p (in c_0 for $p = \infty$) such that
1) *V is s-nuclear for each $s \in (r, 1]$;*
2) *V is not r-nuclear;*
3) trace $V = 1$ *and* $V^2 = 0$.

Note that for $p = \infty$ we have $r = 2/3$ and for $p = 2$ we have $r = 1$.

Some more generalizations

The quasi-normed operator ideals $N^{[r,p]}(X,Y)$:
$T \in L(X,Y)$, which admit a representation

$$T = \sum_{i=1}^{\infty} x_i' \otimes y_i,$$

where $(x_i')_{i=1}^{\infty} \in \ell_{p'}^{w}(X^*)$ and $(x_i)_{i=1}^{\infty} \in \ell_r(Y)$.
$N_{[r,p]}$ is the dual operator ideals.

Third main result: Zelikin's Theorem for $N^{[r,p]}$

Theorem

Let $0 < r \leq 1$, $1 \leq p \leq 2$, $1/r = 1/2 + 1/p$ and $d = 2, 3, \ldots$. For any Banach space X and every operator $T \in N_{[r,p']}(X)$ (or $T \in N^{[r,p']}(Z)$) we have: The spectrum of an operator $T \in N_\alpha(X)$ is $\mathbb{Z}_d$-symmetric if and only if

$$\text{trace } T^{kd+j} = 0 \text{ for all } k = 0, 1, 2, \ldots \text{ and } j = 1, 2, \ldots, d-1.$$

In particular, if trace $T \neq 0$, *then* $T^2 \neq 0$.

Partial cases: $N_{[1,2]}$ and $N^{[1,2]}$.

We obtain Zelikin's theorem, if we put $d = 2$, $r = 1, p = 2$ and $X = H$ (a Hilbert space), since $N_1(H) = S_1(H) = N_{[1,2]}(H) = N^{[1,2]}(H)$.

Theorem

Let $0 < r \leq 1$, $1 \leq p \leq 2$, $1/r = 1/2 + 1/p$ and $d = 2, 3, \ldots$. For any Banach space X and every operator $T \in N_{[r,p']}(X)$ (or $T \in N^{[r,p']}(Z)$) we have: The spectrum of an operator $T \in N_\alpha(X)$ is $\mathbb{Z}_d$-symmetric if and only if

$$\operatorname{trace} T^{kd+j} = 0 \ \textit{ for all } \ k = 0, 1, 2, \ldots \ \textit{ and } \ j = 1, 2, \ldots, d-1.$$

In particular, if $\operatorname{trace} T \neq 0$, *then* $T^2 \neq 0$.

Partial cases: $N_{[1,2]}$ and $N^{[1,2]}$.

We obtain Zelikin's theorem, if we put $d = 2$, $r = 1, p = 2$ and $X = H$ (a Hilbert space), since
$N_1(H) = S_1(H) = N_{[1,2]}(H) = N^{[1,2]}(H)$.

RAPPEL:
A Banach space X is said to be *asymptotically Hilbertian* provided there is a constant K so that for every m there exists n so that X satisfies: there is an n-codimensional subspace X_m of X so that every m-dimensional subspace of X m is K-isomorphic to l_2^m.

Examples

$(\sum_k l_{q_k}^{n_k})_{l_2}$ $(q_k \to 2$ *and* $n_k \to \infty)$

Reminding: asymptotically Hilbertian spaces

RAPPEL:
A Banach space X is said to be *asymptotically Hilbertian* provided there is a constant K so that for every m there exists n so that X satisfies: there is an n-codimensional subspace X_m of X so that every m-dimensional subspace of X m is K-isomorphic to l_2^m.

Examples

$(\sum_k l_{q_k}^{n_k})_{l_2}$ $(q_k \to 2 \text{ and } n_k \to \infty)$

Reminding: asymptotically Hilbertian spaces

It is known:

Theorem

There exist asymptotically Hilbertian spaces without AP.

P. G. Casazza, C. L. García, W. B. Johnson, An example of an asymptotically Hilbertian space which fails the approximation property, Proceedings of the American Mathematical Society, Vol. 129, 10, 2001, 3017-3024.

O. I. Reinov, Banach spaces without approximation property, Functional Analysis and Its Applications, vol 16, issue 4, 1982, 315-317.

O. I. Reinov, *On linear operators with s-nuclear adjoints,* $0 < s \leq 1$, J. Math. Anal. Appl. **415** (2014), 816-824.

Sharpness for $N^{[1,2]}$

Theorem

There exist an asymptotically Hilbertian space $Y_2 := (\sum_k l_{q_k}^{n_k})_{l_2}$ ($q_k \to 2$ *and* $n_k \to \infty$) *and a nuclear operator* U *in this space so that*

1) $U \in N^{[1,2+\varepsilon]}(Y_2)$ *for each* $\varepsilon > 0$.
2) U *is not in* $N^{[1,2]}(Y_2)$.
3) $\text{trace } U = 1$ *and* $U^2 = 0$.

The corresponding statements hold for the adjoint operator U^* *(with* $N_{[1,2+\varepsilon]}$ *instead of* $N^{[1,2+\varepsilon]}$*).*

.

Thank you for your attention!