A metric characterisation of Daugavet property in Lipschitz-free spaces

Abraham Rueda Zoca
(joint with L. García-Lirola and A. Procházka)

5th Workshop on Functional Analysis

Universidad de Granada
Departamento de Análisis Matemático

Support

My research is supported by the fellowship MECD FPU2016/00015 **(since October 1st 2017)**, by MINECO grant MTM2015-65020-P and by Junta de Andalucía grant FQM-0185.

Collaborators

The Daugavet property (I)

Definition (I. Daugavet (1963))

Let X be a Banach space.

The Daugavet property (I)

Definition (I. Daugavet (1963))

Let X be a Banach space. It is said that X has the Daugavet property if every rank-one operator $T : X \longrightarrow X$ satisfies the Daugavet equation

$$\|T + I\| = 1 + \|T\|.$$

The Daugavet property (I)

Definition (I. Daugavet (1963))

Let X be a Banach space. It is said that X has the Daugavet property if every rank-one operator $T : X \longrightarrow X$ satisfies the Daugavet equation

$$\|T + I\| = 1 + \|T\|.$$

It is obvious that this property passes from a space to its predual.

The Daugavet property (I)

Definition (I. Daugavet (1963))

Let X be a Banach space. It is said that X has the Daugavet property if every rank-one operator $T : X \longrightarrow X$ satisfies the Daugavet equation

$$\|T + I\| = 1 + \|T\|.$$

It is obvious that this property passes from a space to its predual. On the other hand, examples of Banach spaces with the Daugavet property are $C([0,1])$, $L_1([0,1])$, $L_\infty([0,1])$, certain classes of spaces of Lipschitz functions ...

The Daugavet property (I)

Definition (I. Daugavet (1963))

Let X be a Banach space. It is said that X has the Daugavet property if every rank-one operator $T : X \longrightarrow X$ satisfies the Daugavet equation

$$\|T + I\| = 1 + \|T\|.$$

It is obvious that this property passes from a space to its predual. On the other hand, examples of Banach spaces with the Daugavet property are $C([0,1])$, $L_1([0,1])$, $L_\infty([0,1])$, certain classes of spaces of Lipschitz functions ... We will have a closer look to this last example.

Daugavet property (II)

Theorem (V. Kadets, R. Shvidkoy, G. Sirotkin and D. Werner, 2001)

Daugavet property (II)

Theorem (V. Kadets, R. Shvidkoy, G. Sirotkin and D. Werner, 2001)

Let X be a Banach space. The following assertions are equivalent:

1. *X has the Daugavet property.*

Daugavet property (II)

Theorem (V. Kadets, R. Shvidkoy, G. Sirotkin and D. Werner, 2001)

Let X be a Banach space. The following assertions are equivalent:

1. *X has the Daugavet property.*
2. *For every point $x \in S_X$, every slice S of B_X and every $\varepsilon > 0$ there exists another slice $T \subseteq S$ such that*

$$\|x + y\| > 2 - \varepsilon$$

holds for every $y \in T$.

Daugavet property (II)

Theorem (V. Kadets, R. Shvidkoy, G. Sirotkin and D. Werner, 2001)

Let X be a Banach space. The following assertions are equivalent:

1. *X has the Daugavet property.*
2. *For every point $x \in S_X$, every slice S of B_X and every $\varepsilon > 0$ there exists another slice $T \subseteq S$ such that*

$$\|x + y\| > 2 - \varepsilon$$

   holds for every $y \in T$.
3. *For every $x \in S_X$ and every $\varepsilon > 0$ then*

$$B_X := \overline{co}\{y \in B_X : \|y + x\| > 2 - \varepsilon\}.$$

Lipschitz functions and Lipschitz-free spaces

Let M be a metric space with a distinguished point $0 \in M$.

Lipschitz functions and Lipschitz-free spaces

Let M be a metric space with a distinguished point $0 \in M$. We denote

$$Lip(M) := \{f : M \longrightarrow \mathbb{R} \text{ Lipschitz} : f(0) = 0\}.$$

Lipschitz functions and Lipschitz-free spaces

Let M be a metric space with a distinguished point $0 \in M$. We denote

$$Lip(M) := \{f : M \longrightarrow \mathbb{R} \text{ Lipschitz} : f(0) = 0\}.$$

It is a Banach space when endowed with the classical Lipschitz seminorm

$$\|f\| := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)}.$$

Lipschitz functions and Lipschitz-free spaces

Let M be a metric space with a distinguished point $0 \in M$. We denote

$$Lip(M) := \{f : M \longrightarrow \mathbb{R} \text{ Lipschitz} : f(0) = 0\}.$$

It is a Banach space when endowed with the classical Lipschitz seminorm

$$\|f\| := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}.$$

Given $m \in M$ we can define $\delta_m \in Lip(M)^*$ such that $\delta_m(f) = f(m)$ for every $f \in Lip(M)$.

Lipschitz functions and Lipschitz-free spaces

Let M be a metric space with a distinguished point $0 \in M$. We denote

$$Lip(M) := \{f : M \longrightarrow \mathbb{R} \text{ Lipschitz} : f(0) = 0\}.$$

It is a Banach space when endowed with the classical Lipschitz seminorm

$$\|f\| := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}.$$

Given $m \in M$ we can define $\delta_m \in Lip(M)^*$ such that $\delta_m(f) = f(m)$ for every $f \in Lip(M)$. Then if we define $\mathcal{F}(M) := \overline{span}\{\delta_m : m \in M\}$ we have

$$\mathcal{F}(M)^* = Lip(M).$$

Sufficient conditions for the Daugavet property on $Lip(M)$

Sufficient conditions for the Daugavet property on $Lip(M)$

Definition (length space)

Let M be a metric space.

Sufficient conditions for the Daugavet property on $Lip(M)$

Definition (length space)

Let M be a metric space. M is said to be *length* if, for every pair of points $x, y \in M$, it follows that $d(x, y)$ is the infimum of the length rectifiable curves joining them.

Sufficient conditions for the Daugavet property on $Lip(M)$

Definition (length space)

Let M be a metric space. M is said to be *length* if, for every pair of points $x, y \in M$, it follows that $d(x, y)$ is the infimum of the length rectifiable curves joining them.

It is well known that a complete metric space M is length if, and only if, for every pair of points and every $x, y \in M$ and every $\varepsilon > 0$ then there exists $z \in M$ such that

$$d(x,z) \leq \frac{1+\varepsilon}{2} d(x,y) \text{ and } d(y,z) \leq \frac{1+\varepsilon}{2} d(x,y).$$

Sufficient conditions for the Daugavet property on $Lip(M)$

Definition (length space)

Let M be a metric space. M is said to be *length* if, for every pair of points $x, y \in M$, it follows that $d(x, y)$ is the infimum of the length rectifiable curves joining them.

It is well known that a complete metric space M is length if, and only if, for every pair of points and every $x, y \in M$ and every $\varepsilon > 0$ then there exists $z \in M$ such that

$$d(x,z) \leq \frac{1+\varepsilon}{2} d(x,y) \text{ and } d(y,z) \leq \frac{1+\varepsilon}{2} d(x,y).$$

Theorem (Y. Ivakhno, V. Kadets and D. Werner (2007))

If M is length, then $Lip(M)$ has the Daugavet property.

Looking for the converse

Problem

If $Lip(M)$ has the Daugavet property, is M length?

Looking for the converse

Problem

If $Lip(M)$ has the Daugavet property, is M length?

- An affirmative answer would solve our problem of characterising Daugavet property of $Lip(M)$ in terms of M.

Looking for the converse

Problem

If $Lip(M)$ has the Daugavet property, is M length?

- An affirmative answer would solve our problem of characterising Daugavet property of $Lip(M)$ in terms of M.
- In the compact case, Y. Ivakhno, V. Kadets and D. Werner proved the following result:

Theorem

Looking for the converse

Problem

If $Lip(M)$ has the Daugavet property, is M length?

- An affirmative answer would solve our problem of characterising Daugavet property of $Lip(M)$ in terms of M.
- In the compact case, Y. Ivakhno, V. Kadets and D. Werner proved the following result:

Theorem

Let M be a compact metric space. Then the following assertions are equivalent:

Looking for the converse

Problem

If $Lip(M)$ has the Daugavet property, is M length?

- An affirmative answer would solve our problem of characterising Daugavet property of $Lip(M)$ in terms of M.
- In the compact case, Y. Ivakhno, V. Kadets and D. Werner proved the following result:

Theorem

Let M be a compact metric space. Then the following assertions are equivalent:

1. *$Lip(M)$ has the Daugavet property.*

Looking for the converse

Problem

If $Lip(M)$ has the Daugavet property, is M length?

- An affirmative answer would solve our problem of characterising Daugavet property of $Lip(M)$ in terms of M.
- In the compact case, Y. Ivakhno, V. Kadets and D. Werner proved the following result:

Theorem

Let M be a compact metric space. Then the following assertions are equivalent:

1. *$Lip(M)$ has the Daugavet property.*
2. *$\mathcal{F}(M)$ has the Daugavet property.*

Looking for the converse

Problem

If *Lip*(M) has the Daugavet property, is M length?

- An affirmative answer would solve our problem of characterising Daugavet property of *Lip*(M) in terms of M.
- In the compact case, Y. Ivakhno, V. Kadets and D. Werner proved the following result:

Theorem

Let M be a compact metric space. Then the following assertions are equivalent:

1. *Lip*(M) *has the Daugavet property.*
2. $\mathcal{F}(M)$ *has the Daugavet property.*
3. *M is* ***local****, that is, for every $f \in Lip(M)$ and every $\varepsilon > 0$ there exist $u \neq v \in M$ such that $0 < d(u,v) < \varepsilon$ and $\frac{f(u)-f(v)}{d(u,v)} > 1 - \varepsilon$.*

Characterising the Daugavet property on *Lip*(M)

Theorem (L. García-Lirola, A. Procházka and A.R.Z. (preprint))

Let M be a complete metric space. The following assertions are equivalent:

1. *M is length.*

Characterising the Daugavet property on $Lip(M)$

Theorem (L. García-Lirola, A. Procházka and A.R.Z. (preprint))

Let M be a complete metric space. The following assertions are equivalent:

1. *M is length.*
2. *$Lip(M)$ has the Daugavet property.*

Characterising the Daugavet property on $Lip(M)$

Theorem (L. García-Lirola, A. Procházka and A.R.Z. (preprint))

Let M be a complete metric space. The following assertions are equivalent:

1. *M is length.*
2. *$Lip(M)$ has the Daugavet property.*
3. *$\mathcal{F}(M)$ has the Daugavet property.*

Characterising the Daugavet property on $Lip(M)$

Theorem (L. García-Lirola, A. Procházka and A.R.Z. (preprint))

Let M be a complete metric space. The following assertions are equivalent:

1. *M is length.*
2. *$Lip(M)$ has the Daugavet property.*
3. *$\mathcal{F}(M)$ has the Daugavet property.*
4. *M is local.*

$\mathcal{F}(M)$ Daugavet implies M local (I)

Assume that $\mathcal{F}(M)$ has the Daugavet property and pick $f \in S_{Lip(M)}$.

$\mathcal{F}(M)$ Daugavet implies M local (I)

Assume that $\mathcal{F}(M)$ has the Daugavet property and pick $f \in S_{Lip(M)}$. Let us prove that f approximates the Lipschitz norm in arbitrarily close points.

$\mathcal{F}(M)$ Daugavet implies M local (I)

Assume that $\mathcal{F}(M)$ has the Daugavet property and pick $f \in S_{Lip(M)}$. Let us prove that f approximates the Lipschitz norm in arbitrarily close points. To this aim, pick $x \neq y$ such that $\frac{f(x)-f(y)}{d(x,y)} > 1-\varepsilon$ and define

$$f_1 = f, f_2 := d(\cdot, y), f_3 := -d(\cdot, x), f_4(t) := f_{xy}(t) = \frac{d(x,y)}{2}\frac{d(t,y)-d(t,x)}{d(t,y)+d(t,x)}.$$

Define $g := \frac{1}{4}\sum_{i=1}^{4} f_i$.

$\mathcal{F}(M)$ Daugavet implies M local (I)

Assume that $\mathcal{F}(M)$ has the Daugavet property and pick $f \in S_{Lip(M)}$. Let us prove that f approximates the Lipschitz norm in arbitrarily close points. To this aim, pick $x \neq y$ such that $\frac{f(x)-f(y)}{d(x,y)} > 1 - \varepsilon$ and define

$$f_1 = f, f_2 := d(\cdot, y), f_3 := -d(\cdot, x), f_4(t) := f_{xy}(t) = \frac{d(x,y)}{2} \frac{d(t,y) - d(t,x)}{d(t,y) + d(t,x)}.$$

Define $g := \frac{1}{4} \sum_{i=1}^{4} f_i$. Since $\mathcal{F}(M)$ has the Daugavet property, we can find $u \neq v \in M$ such that $g(u) - g(v) > \left(1 - \frac{\varepsilon}{4}\right) d(u,v)$ and that $\left\| \frac{\delta_x - \delta_y}{d(x,y)} - \frac{\delta_u - \delta_v}{d(u,v)} \right\| > 2 - \varepsilon$.

$\mathcal{F}(M)$ Daugavet implies M local (I)

Assume that $\mathcal{F}(M)$ has the Daugavet property and pick $f \in S_{Lip(M)}$. Let us prove that f approximates the Lipschitz norm in arbitrarily close points. To this aim, pick $x \neq y$ such that $\frac{f(x)-f(y)}{d(x,y)} > 1-\varepsilon$ and define

$$f_1 = f, f_2 := d(\cdot, y), f_3 := -d(\cdot, x), f_4(t) := f_{xy}(t) = \frac{d(x,y)}{2}\frac{d(t,y)-d(t,x)}{d(t,y)+d(t,x)}.$$

Define $g := \frac{1}{4}\sum_{i=1}^4 f_i$. Since $\mathcal{F}(M)$ has the Daugavet property, we can find $u \neq v \in M$ such that $g(u)-g(v) > \left(1-\frac{\varepsilon}{4}\right) d(u,v)$ and that $\left\Vert \frac{\delta_x-\delta_y}{d(x,y)} - \frac{\delta_u-\delta_v}{d(u,v)}\right\Vert > 2-\varepsilon$. An easy convexity argument yields that $f_i(u)-f_i(v) > (1-\varepsilon)d(u,v)$ and, in particular, $\frac{f(u)-f(v)}{d(u,v)} > 1-\varepsilon$. In order to prove that M is local we have to prove that u and v can be choosen very close.

$\mathcal{F}(M)$ Daugavet implies M local (II)

$$f_1 = f, f_2 := d(\cdot, y), f_3 := -d(\cdot, x), f_4(t) := f_{xy}(t) = \frac{d(x,y)}{2}\frac{d(t,y)-d(t,x)}{d(t,y)+d(t,x)}.$$

Since $f_i(u) - f_i(v) > (1-\varepsilon)d(u,v)$ we get:

1. $i = 2, 3 \Rightarrow \min\{d(y,u) - d(y,v), d(x,v) - d(x,u)\} > (1-\varepsilon)d(u,v)$.

$\mathcal{F}(M)$ Daugavet implies M local (II)

$$f_1 = f, f_2 := d(\cdot, y), f_3 := -d(\cdot, x), f_4(t) := f_{xy}(t) = \frac{d(x,y)}{2}\frac{d(t,y)-d(t,x)}{d(t,y)+d(t,x)}.$$

Since $f_i(u) - f_i(v) > (1-\varepsilon)d(u,v)$ we get:

1. $i = 2,3 \Rightarrow \min\{d(y,u)-d(y,v), d(x,v)-d(x,u)\} > (1-\varepsilon)d(u,v)$.
2. $i = 4 \Rightarrow (1-\varepsilon)\max\{d(x,v)+d(y,v), d(x,u)+d(y,u)\} \leq d(x,y)$.

$\mathcal{F}(M)$ Daugavet implies M local (II)

$$f_1 = f, f_2 := d(\cdot, y), f_3 := -d(\cdot, x), f_4(t) := f_{xy}(t) = \frac{d(x,y)}{2}\frac{d(t,y)-d(t,x)}{d(t,y)+d(t,x)}.$$

Since $f_i(u) - f_i(v) > (1-\varepsilon)d(u,v)$ we get:

1. $i = 2, 3 \Rightarrow \min\{d(y,u) - d(y,v), d(x,v) - d(x,u)\} > (1-\varepsilon)d(u,v)$.
2. $i = 4 \Rightarrow (1-\varepsilon)\max\{d(x,v) + d(y,v), d(x,u) + d(y,u)\} \leq d(x,y)$.
3. $\left\Vert \frac{\delta_x - \delta_y}{d(x,y)} - \frac{\delta_u - \delta_v}{d(u,v)} \right\Vert > 2 - \varepsilon \Rightarrow (1-\varepsilon)(d(x,y) + d(u,v)) \leq \min\{d(x,u) + d(y,v), d(x,v) + d(y,u)\}$.

$\mathcal{F}(M)$ Daugavet implies M local (II)

$$f_1 = f, f_2 := d(\cdot, y), f_3 := -d(\cdot, x), f_4(t) := f_{xy}(t) = \frac{d(x,y)}{2}\frac{d(t,y)-d(t,x)}{d(t,y)+d(t,x)}.$$

Since $f_i(u) - f_i(v) > (1-\varepsilon)d(u,v)$ we get:

1. $i = 2, 3 \Rightarrow \min\{d(y,u) - d(y,v), d(x,v) - d(x,u)\} > (1-\varepsilon)d(u,v)$.
2. $i = 4 \Rightarrow (1-\varepsilon)\max\{d(x,v) + d(y,v), d(x,u) + d(y,u)\} \leq d(x,y)$.
3. $\left\|\frac{\delta_x - \delta_y}{d(x,y)} - \frac{\delta_u - \delta_v}{d(u,v)}\right\| > 2 - \varepsilon \Rightarrow (1-\varepsilon)(d(x,y) + d(u,v)) \leq$ $\min\{d(x,u) + d(y,v), d(x,v) + d(y,u)\}$. So

$$\frac{d(x,y)}{1-\varepsilon}$$

$\mathcal{F}(M)$ Daugavet implies M local (II)

$$f_1 = f, f_2 := d(\cdot, y), f_3 := -d(\cdot, x), f_4(t) := f_{xy}(t) = \frac{d(x,y)}{2}\frac{d(t,y) - d(t,x)}{d(t,y) + d(t,x)}.$$

Since $f_i(u) - f_i(v) > (1-\varepsilon)d(u,v)$ we get:

1. $i = 2, 3 \Rightarrow \min\{d(y,u) - d(y,v), d(x,v) - d(x,u)\} > (1-\varepsilon)d(u,v)$.
2. $i = 4 \Rightarrow (1-\varepsilon)\max\{d(x,v) + d(y,v), d(x,u) + d(y,u)\} \leq d(x,y)$.
3. $\left\|\frac{\delta_x - \delta_y}{d(x,y)} - \frac{\delta_u - \delta_v}{d(u,v)}\right\| > 2 - \varepsilon \Rightarrow (1-\varepsilon)(d(x,y) + d(u,v)) \leq \min\{d(x,u) + d(y,v), d(x,v) + d(y,u)\}$. So

$$\frac{d(x,y)}{1-\varepsilon} \overset{(2)}{\geq} d(x,u) + d(y,u)$$

$\mathcal{F}(M)$ Daugavet implies M local (II)

$$f_1 = f, f_2 := d(\cdot, y), f_3 := -d(\cdot, x), f_4(t) := f_{xy}(t) = \frac{d(x,y)}{2}\frac{d(t,y)-d(t,x)}{d(t,y)+d(t,x)}.$$

Since $f_i(u) - f_i(v) > (1-\varepsilon)d(u,v)$ we get:

1. $i = 2,3 \Rightarrow \min\{d(y,u)-d(y,v), d(x,v)-d(x,u)\} > (1-\varepsilon)d(u,v)$.
2. $i = 4 \Rightarrow (1-\varepsilon)\max\{d(x,v)+d(y,v), d(x,u)+d(y,u)\} \leq d(x,y)$.
3. $\left\| \frac{\delta_x-\delta_y}{d(x,y)} - \frac{\delta_u-\delta_v}{d(u,v)} \right\| > 2-\varepsilon \Rightarrow (1-\varepsilon)(d(x,y)+d(u,v)) \leq \min\{d(x,u)+d(y,v), d(x,v)+d(y,u)\}$. So

$$\frac{d(x,y)}{1-\varepsilon} \overset{(2)}{\geq} d(x,u)+d(y,u) \overset{(1)}{\geq} d(x,u)+d(y,v)+(1-\varepsilon)d(u,v)$$

$\mathcal{F}(M)$ Daugavet implies M local (II)

$$f_1 = f, f_2 := d(\cdot, y), f_3 := -d(\cdot, x), f_4(t) := f_{xy}(t) = \frac{d(x,y)}{2}\frac{d(t,y)-d(t,x)}{d(t,y)+d(t,x)}.$$

Since $f_i(u) - f_i(v) > (1-\varepsilon)d(u,v)$ we get:

1. $i = 2,3 \Rightarrow \min\{d(y,u)-d(y,v), d(x,v)-d(x,u)\} > (1-\varepsilon)d(u,v)$.
2. $i = 4 \Rightarrow (1-\varepsilon)\max\{d(x,v)+d(y,v), d(x,u)+d(y,u)\} \leq d(x,y)$.
3. $\left\Vert \frac{\delta_x-\delta_y}{d(x,y)} - \frac{\delta_u-\delta_v}{d(u,v)}\right\Vert > 2-\varepsilon \Rightarrow (1-\varepsilon)(d(x,y)+d(u,v)) \leq$ $\min\{d(x,u)+d(y,v), d(x,v)+d(y,u)\}$. So

$$\frac{d(x,y)}{1-\varepsilon} \overset{(2)}{\geq} d(x,u)+d(y,u) \overset{(1)}{\geq} d(x,u)+d(y,v)+(1-\varepsilon)d(u,v)$$

$$\overset{(3)}{\geq} (1-\varepsilon)(d(x,y)+d(u,v))+(1-\varepsilon)d(u,v).$$

$\mathcal{F}(M)$ Daugavet implies M local (II)

$$f_1 = f, f_2 := d(\cdot, y), f_3 := -d(\cdot, x), f_4(t) := f_{xy}(t) = \frac{d(x,y)}{2}\frac{d(t,y)-d(t,x)}{d(t,y)+d(t,x)}.$$

Since $f_i(u) - f_i(v) > (1-\varepsilon)d(u,v)$ we get:

1. $i = 2,3 \Rightarrow \min\{d(y,u)-d(y,v), d(x,v)-d(x,u)\} > (1-\varepsilon)d(u,v)$.
2. $i = 4 \Rightarrow (1-\varepsilon)\max\{d(x,v)+d(y,v), d(x,u)+d(y,u)\} \leq d(x,y)$.
3. $\left\|\frac{\delta_x-\delta_y}{d(x,y)} - \frac{\delta_u-\delta_v}{d(u,v)}\right\| > 2-\varepsilon \Rightarrow (1-\varepsilon)(d(x,y)+d(u,v)) \leq$ $\min\{d(x,u)+d(y,v), d(x,v)+d(y,u)\}$. So

$$\frac{d(x,y)}{1-\varepsilon} \overset{(2)}{\geq} d(x,u)+d(y,u) \overset{(1)}{\geq} d(x,u)+d(y,v)+(1-\varepsilon)d(u,v)$$

$$\overset{(3)}{\geq} (1-\varepsilon)(d(x,y)+d(u,v)) + (1-\varepsilon)d(u,v).$$

This implies $d(u,v) < \frac{\varepsilon}{(1-\varepsilon)^2} d(x,y)$.

$\mathcal{F}(M)$ Daugavet implies M local (II)

$$f_1 = f, f_2 := d(\cdot, y), f_3 := -d(\cdot, x), f_4(t) := f_{xy}(t) = \frac{d(x,y)}{2}\frac{d(t,y)-d(t,x)}{d(t,y)+d(t,x)}.$$

Since $f_i(u) - f_i(v) > (1-\varepsilon)d(u,v)$ we get:

1. $i = 2,3 \Rightarrow \min\{d(y,u)-d(y,v), d(x,v)-d(x,u)\} > (1-\varepsilon)d(u,v)$.
2. $i = 4 \Rightarrow (1-\varepsilon)\max\{d(x,v)+d(y,v), d(x,u)+d(y,u)\} \leq d(x,y)$.
3. $\left\Vert \frac{\delta_x-\delta_y}{d(x,y)} - \frac{\delta_u-\delta_v}{d(u,v)}\right\Vert > 2-\varepsilon \Rightarrow (1-\varepsilon)(d(x,y)+d(u,v)) \leq$ $\min\{d(x,u)+d(y,v), d(x,v)+d(y,u)\}$. So

$$\frac{d(x,y)}{1-\varepsilon} \overset{(2)}{\geq} d(x,u)+d(y,u) \overset{(1)}{\geq} d(x,u)+d(y,v)+(1-\varepsilon)d(u,v)$$

$$\overset{(3)}{\geq} (1-\varepsilon)(d(x,y)+d(u,v)) + (1-\varepsilon)d(u,v).$$

This implies $d(u,v) < \frac{\varepsilon}{(1-\varepsilon)^2}d(x,y)$. An inductive argument does the trick.

Local implies length

Local implies length

Assume that M is not length. Then there is a pair of points $x \neq y$ and $\delta > 0$ such that, for $r = \frac{d(x,y)}{2}$, we get $d(B(x,(1+\delta)r), B(y,(1+\delta)r)) \geq \delta r$.

Local implies length

Assume that M is not length. Then there is a pair of points $x \neq y$ and $\delta > 0$ such that, for $r = \frac{d(x,y)}{2}$, we get $d(B(x,(1+\delta)r), B(y,(1+\delta)r)) \geq \delta r$. Define $f_1(t) := \max\left\{r - \frac{d(x,t)}{1+\delta}, 0\right\}$, $f_2(t) := \min\left\{-r + \frac{d(y,t)}{1+\delta}, 0\right\}$. Note that $\|f_i\| \leq \frac{1}{1+\delta}$ so $f := f_1 + f_2$ is Lipschitz and $\|f\| \geq 1$.

Local implies length

Assume that M is not length. Then there is a pair of points $x \neq y$ and $\delta > 0$ such that, for $r = \frac{d(x,y)}{2}$, we get $d(B(x,(1+\delta)r), B(y,(1+\delta)r)) \geq \delta r$. Define $f_1(t) := \max\left\{r - \frac{d(x,t)}{1+\delta}, 0\right\}$, $f_2(t) := \min\left\{-r + \frac{d(y,t)}{1+\delta}, 0\right\}$. Note that $\|f_i\| \leq \frac{1}{1+\delta}$ so $f := f_1 + f_2$ is Lipschitz and $\|f\| \geq 1$. Notice also that, by construction, $\{z : f_1(z) \neq 0\} \subseteq B(x,(1+\delta)r)$ and $\{z : f_2(z) \neq 0\} \subseteq B(y,(1+\delta)r)$.

Local implies length

Assume that M is not length. Then there is a pair of points $x \neq y$ and $\delta > 0$ such that, for $r = \frac{d(x,y)}{2}$, we get $d(B(x,(1+\delta)r), B(y,(1+\delta)r)) \geq \delta r$. Define $f_1(t) := \max\left\{r - \frac{d(x,t)}{1+\delta}, 0\right\}$, $f_2(t) := \min\left\{-r + \frac{d(y,t)}{1+\delta}, 0\right\}$. Note that $\|f_i\| \leq \frac{1}{1+\delta}$ so $f := f_1 + f_2$ is Lipschitz and $\|f\| \geq 1$. Notice also that, by construction, $\{z : f_1(z) \neq 0\} \subseteq B(x,(1+\delta)r)$ and $\{z : f_2(z) \neq 0\} \subseteq B(y,(1+\delta)r)$. This implies that if $\frac{f(u)-f(v)}{d(u,v)} > \frac{1}{1+\delta}$ then necessarily $u \in B(x,(1+\delta)r)$ and $v \in B(y,(1+\delta)r)$, so $d(u,v) \geq \delta r$. This implies that M is not local, so we are done.

References

- L. García-Lirola, A. Procházka and A. Rueda Zoca, *A characterisation of the Daugavet property in spaces of Lipschitz functions*, preprint. Available at arXiv:1705.05145v3.
- Y. Ivakhno, V. Kadets and D. Werner, *The Daugavet property for spaces of Lipschitz functions*, Math. Scand. **101** (2007), 261-279.
- V. Kadets, R. Shvidkoy, G. Sirotkin and D. Werner, *Banach spaces with the Daugavet property*, Trans. of the AMS **352** (2001), 855-873.

Thank you


THANK YOU FOR YOUR ATTENTION
PLEASE CLAP AND DO NOT MAKE TOUGH QUESTIONS